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MATHEMATICAL THEORY OF ELASTICITY

Mathematical Theory of Elasticity

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SECOND EDITION

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PREFACE

The theory of elasticity, in its broad aspects, deals with a study of the behavior of those substances that possess the property of recovering their size and shape when the forces producing deformations are removed. In common with other branches of applied mathematics, the growth of this theory proceeded from a synthesis of special ideas and techniques devised to solve concrete problems. This resulted in a patchwork of theories treating isolated classes of problems, determined largely by the geometry of bodies under consideration. The embedding of such diverse theories in a unified structure, and the construction of the analytical tools for calculating stresses and deformations in a strained elastic body, are among the dominant concerns of the mathematical theory of elasticity.

This book represents an attempt to present several aspects of the theory of elasticity from a unified point of view and to indicate, along with the familiar methods of solution of the field equations of elasticity, some newer general methods of solution of the two-dimensional problems.

The first edition of this book, published in 1946, had its origin in a course of lectures I gave in 1941 and 1942 in the Program of Advanced Instruction and Research in Mechanics conducted by the Graduate School of Brown University. In those lectures I stressed the contributions to the theory by the Russian school of elasticians and, in particular, the relatively little-known work of great elegance and importance by N. I. Muskhelishvili. I planned to supplement that book by a companion volume dealing with effective methods of attack on the two-dimensional and anisotropic problems of elasticity. The developments in the intervening years, however, were so rapid that I was urged to publish instead a single volume containing an up-to-date treatment of material presented in the first edition and supplement it with new topics, in order to give a rounded idea of the current state of the subject.

The present edition differs from its predecessor by extensive additions and changes. Most of the material appearing in the last three chapters had no counterpart in the first edition. Throughout I have tried to give a clear indication of the frontiers of the developments, and I have constantly kept in mind those readers whose principal concern is with the practical application of the theory. While no volume of this size may claim to an exhaustive list of references to research literature, I have

selected such references with care so as to give an accurate picture of the present state of the topics treated in this book.

I deliberately omitted any discussion of the theory of shells, because a palatable treatment of the shell theory cannot be made in the space of fewer than 300 pages. The best available treatment of this subject, in my opinion, is given in a Russian monograph by A. L. Goldenveiser, Theory of Thin Elastic Shells, Moscow (1953), 544 pp.

The first three chapters, despite their brevity, contain a comprehensive treatment of the underlying theory of mechanics of deformable media. These chapters are essential to the understanding of the remaining chapters, which can be read independently of one another. Chapter 4 gives an up-to-date treatment of extension, torsion, and flexure of beams, including the deformation of homogeneous and nonhomogeneous beams by loads distributed on their lateral surfaces. Chapter 5 is concerned with two important categories of plane problems of elasticity. It contains an account of the general modes of attack on such problems with the aid of the theory of functions of complex variables. Although a clear indication of the use of such methods in the problems of transverse deflection of thin plates is made, illustrations are chosen mainly from problems on plates in the states of plane strain or generalized plane stress.

Chapter 6, dealing with the three-dimensional problems, is brief for the simple reason that effective general techniques for the analytic solution of such problems still remain to be developed. The most promising approach in this connection appears to be (as in Chap. 5) via the use of general solutions of Navier's equations in terms of harmonic functions. The chapter contains a formulation of thermoelastic problems and an introductory account of the theory of vibrations and propagation of waves in elastic media.

Chapter 7 on Variational Methods contains a treatment of the energy theorems in elasticity and their bearing on the variational methods of solution of elastostatic problems. I have tried to present the variational techniques of Ritz, Galerkin, Trefftz, Kantorovich, and others in a unified way, without resorting to function space methods so as to make matters meaningful to a wider circle of readers. This chapter includes a discussion of the method of finite differences and relaxation, which are frequently used when analytic methods fail.

This volume owes much to the recent contribution to elasticity made by Russian scholars. Suitable acknowledgment to sources is made throughout this volume, but my chief debt is to Academician N. I. Muskhelishvili, whose unparalleled monograph, "Some Basic Problems of the Mathematical Theory of Elasticity," Moscow (1954), originally published in 1933 and now in the fourth edition, has left an indelible imprint.

A large part of the material in this volume was prepared in the course

of the investigations and lectures I gave during my tenure as a Guggenheim Fellow during the academic year 1952–1953. I am pleased to have this opportunity to acknowledge my gratitude to the Guggenheim Memorial Foundation, whose grant enabled me to discuss this book with my colleagues in England and on the Continent. I also wish to repeat an acknowledgment, made in the Preface to the first edition, to the Wisconsin Alumni Research Foundation for a grant-in-aid that facilitated the publication of the predecessor of this volume.

I am indebted to Dr. George E. Forsythe, Research Mathematician at the University of California at Los Angeles, for material on Relaxation Methods in Sec. 125, and to Robert K. Froyd, Research Assistant at the University of California at Los Angeles, for his help in proofreading and preparing the index matter.

I. S. SOKOLNIKOFF

CONTENTS

HISTORICAL SKETCH

The theory of elasticity is concerned with the study of the response of elastic bodies to the action of forces. A body is called *elastic* if it possesses the property of recovering its original shape when the forces causing deformations are removed. The elastic property of material media is shared by all substances provided that the deformations do not exceed certain limits determined by the constitutive characteristics of the body. The elastic property is characterized mathematically by certain functional relationships connecting forces and deformations. Among such relationships a linear law stemming from a generalization of Hooke's law[1] is of fundamental importance. Hooke's law states, in effect, that the extensions of springlike bodies, produced by the tensile forces, are proportional to the forces. An identical law was discovered independently by Mariotte[2] in 1680 and used by him to investigate the strength of cantiliver beams. Mariotte concluded that a cantilever beam resists flexure because some of its longitudinal fibers are extended and others are contracted. Although Mariotte's assumption regarding force distribution in fibers was correct, his investigations did not include the study of the shape assumed by the beam's axis. Such a study[3] was made in 1705 by Jacob Bernoulli, who combined elementary equilibrium considerations with Hooke's law to obtain the differential equation of the *elastica*, that is, the curve assumed by the deformed axis of the beam. His equation implies that the curvature of the elastica at each point is proportional to the bending moment acting in the section through the point. It readily follows from this result that the work done in bending the beam is proportional to the integral of the square of the curvature taken along the elastica. Daniel Bernoulli (1700–1782), a strong proponent of the minimum principles that were in the process of formulation at that time, suggested in a letter[4] to Euler that the equation of elastica should emerge on minimizing the integral representing the work done in bending the beam. In this manner Euler deduced Jacob Bernoulli's equation and integrated it for a number of special cases.[5]

[1] Robert Hooke, De potentia restitutiva (1678).

[2] E. Mariotte, Traité de mouvement des eaux (1686).

[3] J. Bernoulli, Collected Works (1744), vol. 2, p. 976.

[4] The twenty-sixth letter of D. Bernoulli to Euler, October, 1742, in P. H. Fuss' Correspondence mathématique et physique (1843), vol. 2.

[5] L. Euler, the addendum to "De curvis elasticis," in Metodus inveniendi lineas curvas maximi minimive proprietate gaudentes (1744). This paper and Euler's

Euler's work in elasticity was confined principally to the study of elastica, and it was not concerned with the distribution of stresses in the cross sections of the beam. The study of the force distribution in long beams subjected to tension and bending was initiated by Coulomb[1] (1736–1806), who was also the first to study the resistance of thin wires to torsion.[2]

During the 150-year period following the discovery of Hooke's law[3] (ca. 1660), the growth of the science of elasticity proceeded from a synthesis of solutions of special problems. This gave in the early nineteenth century a fragmentary theory of flexure of beams, an incomplete theory of torsion, the rudiments of the theory of stability of columns, and a few isolated results on bending and vibration of plates.

The first attempt to deduce general equations of equilibrium and vibration of elastic solids was made by Navier (1785–1836) in a remarkable memoir[4] read on May 14, 1821. This date marks the birth of the mathematical theory of elasticity. Starting with the picture of molecular interaction in which the forces act along the lines joining two particles and are proportional to the change in distance between them, Navier deduced a set of three macroscopic differential equations for the components of displacement in the interior of an isotropic elastic solid. The form of these equations is correct, but, because of the oversimplified picture of molecular interaction, the Navier equations contain only one elastic constant. Navier also obtained the equilibrium equations on the surface of the solid (the boundary conditions) with the aid of Lagrange's principle of virtual work.

Navier's work attracted the attention of A. Cauchy (1789–1857), who, proceeding from different assumptions, gave a formulation of the linear theory of elasticity that remains virtually unchanged to the present day.

Instead of starting with some specific law of molecular interaction, Cauchy shows that the state of stress at an interior point of the deformable body is completely determined by a set of nine functions. When the

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further researches, published in *Acta Academiae Petropolitanae* (1778), provide a foundation for the theory of elastic stability. The work was extended by J. L. Lagrange (1736–1813), who made several basic contributions to the theory of elastic columns.

[1] C. A. Coulomb, Mémoires par divers savants (1784), (1787), pp. 229–269.

[2] *Mémoires de l'académie des sciences*, Paris (1784).

[3] This law was sharpened by Thomas Young in 1807. Young's most important contribution to elasticity is in the clear formulation of the modulus of elasticity in tension. Although Young made an important observation that, in the torsion of rods by couples, the applied torque is balanced by a distribution of the shearing forces in the cross section of the rod, and that the resulting deformation is proportional to the shearing forces, he failed to introduce the shear modulus.

[4] C. L. M. H. Navier, *Mémoires de l'académie des sciences*, Paris, vol. 7 (1827). See also *Bulletin de la société philomathique*, Paris (1823), p. 177.

body is in equilibrium, these nine functions are shown to satisfy three simple partial differential equations and their number reduces to six because of certain symmetry relations. The state of deformation is likewise determined by six functions, which are simply related to the components of the displacement vector, when the displacements are small. Now, when the body is elastic and only small deformations are contemplated, one is justified in assuming that the set of functions characterizing the state of stress is related linearly to the set characterizing the deformation. This assumption represents a far-reaching generalization of Hooke's law. When the body is elastically isotropic, the linear relationship, just mentioned, turns out to contain only two elastic constants. On eliminating the functions characterizing the state of stress from Cauchy's equations, one is led to a set of three differential equations of the same structure as Navier's equations, but which contain two elastic constants instead of one. These important results were presented by Cauchy[1] to the Paris Academy in 1822.

At a later date[2] Cauchy used a special law of molecular interaction to generalize his results to the anisotropic media. The resulting stress-strain relations, for the most general type of anisotropy, turn out to contain 15 elastic constants, instead of 21, because of the restrictive conditions on the arrangement of particles imposed on his model by Cauchy. The controversy between the proponents of Cauchy's "rariconstant theory" and the supporters of the "multiconstant theory" raged for many years. It abated only with the acceptance of George Green's (1793–1841) revolutionary principle of conservation of elastic energy. Green proposed to deduce the fundamental equations of elasticity by following the pattern laid down by Lagrange in *Mécanique analytique*. To do this, he introduced the concept of strain energy and deduced,[3] in 1837, the basic equations of elasticity from the principle of virtual work. The number of elastic constants necessary to characterize the most general elastic medium (when the deformation is small) turns out to be 21, because of the connection of the quadratic form representing the strain energy with stress-strain relations. The existence of Green's energy function, when the body is in an isothermal state, has been argued by Lord Kelvin,[4] and similar arguments have been advanced to establish its existence for the adiabatic state.

The principle of the conservation of elastic energy has led to the

[1] An abstract was published in *Bulletin de la société philomathique*, Paris (1823), and details in three papers in the 1827 and 1828 volumes of the *Exercises de mathématique*.

[2] See two papers in the *Exercises de mathématique*, vol. 3 (1828*a*), p. 213, and vol. 3 (1828*b*), p. 328.

[3] George Green, *Transactions of the Cambridge Philosophical Society*, vol. 7 (1839), p. 121, or Mathematical Papers (1871), p. 245.

[4] William Thomson, *Quarterly Journal of Mathematics*, vol. 5 (1855).

formulation of the basic problems of elasticity as certain minimum principles.[1]

The contributions of Navier, Cauchy, and Green were concerned not so much with the solution of specific types of boundary-value problems as with the formulation of foundations and general theories. In the domain of problems concerned with the torsion and flexure of cylinders monumental contributions have been made by Barré de Saint-Venant[2] (1797–1886).

Important developments in the kinematic theory of thin rods and in the study of the deflection of plates were initiated by G. Kirchhoff[3] (1824–1887).

The developments during this century were concerned principally with the problems of the existence of solutions and the integration of several broad categories of the boundary-value problems. A definitive treatment of the fundamental problems in plane elasticity (primarily by the school of Russian mathematicians influenced by N. I. Muskhelishvili) was given and significant strides made in the theory of shells and in the construction of nonlinear theories of elasticity.[4]

[1] See Chap. 7 of this book.

[2] These are treated in Chap. 4 of this book. Saint-Venant's work on torsion and bending of prisms is contained in two extensive memoirs published in *Mémoires de l' académie des sciences des savants étrangers*, vol. 14 (1855), pp. 233–560, and in *Journal de mathématiques de Liouville*, ser. 2, vol. 1 (1856), pp. 89–189. Many original contributions, representing about one-third of the volume, are also contained in Saint-Venant's translation of A. Clebsch's Theorie der Elasticität fester Körper (1862), which was published in 1883 under the title Théorie de l'élasticité des corps solides.

[3] G. Kirchhoff, Vorlesungen über matematische Physik (1897). An account of the current state of thin rods is contained in a monograph by E. P. Popoff, Non-linear Static Problems of Thin Rods (1948) (in Russian). A survey of the recent work in the theory of plates is contained in a paper by G. Dzanelidze, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 12 (1948), pp. 109–124, which was translated by the American Mathematical Society, *Translation* 6 (1950).

[4] A synoptic account of the contribution of Russian mathematicians in the domain of two-dimensional problems was prepared by D. I. Sherman in the collection Mechanics in SSSR for Thirty Years (1950), pp. 192–225 (in Russian). See also Chap. 5 of this book. A critical summary of developments in nonlinear elasticity was published by C. A. Truesdell, *Journal of Rational Mechanics*, vol. 1 (1952), pp. 125–300, vol. 2 (1953), pp. 593–616; and a systematic account will be found in the English translation of the 1948 edition of V. V. Novozhilov's Russian monograph entitled Foundations of Nonlinear Theory of Elasticity (1953). The theory of shells is still in the formative, patchy state characterized by conflicting approximations. A readable account of a fairly general theory of shells is contained on pp. 375–437 of Theoretical Elasticity, by A. E. Green and W. Zerna (1954). A comprehensive treatment of the shell theory is given in the Theory of Thin Elastic Shells, by A. L. Goldenveiser (1953) (in Russian).

CHAPTER 1

ANALYSIS OF STRAIN

1. Deformation. The first two chapters of this book are not specifically concerned with elastic media. In a great many problems the atomistic structure of matter can be disregarded and the body replaced by a continuous mathematical model whose geometrical points are identified with material points of the body. The study of such models is in the province of the mechanics of continuous media, which covers a vast range of problems in elasticity, hydrodynamics, aerodynamics, plasticity, and electrodynamics.

When the relative position of points in a continuous body is altered, we say that the body is *strained.* The change in the relative position of points is a *deformation,* and the study of deformations is the province of the *analysis of strain.*

Although all material bodies are to some extent deformable, it is useful to introduce an abstraction of a *nondeformable,* or *rigid, body.* A rigid body is an ideal body such that the distance between every pair of its points remains invariant throughout the history of the body. The behavior of rigid bodies subjected to the action of forces is investigated in the mechanics of rigid bodies, where it is shown that the possible displacements in a rigid body consist of translations and rotations. Such displacements are termed *rigid displacements,* and although they are of minor concern in the analysis of strain, it is important to learn how to characterize them analytically.

Let the body τ, occupying in the undeformed state some region R, be referred to an orthogonal set of cartesian axes $O\text{-}X_1X_2X_3$ (Fig. 1) fixed in space. The coordinates of typical point P of τ in the unstrained state are (x_1, x_2, x_3). In the strained state the points of τ will occupy some region R', and we denote the coordinates of the same material point P by (x_1', x_2', x_3'). We shall be concerned only with continuous deformations of R into R' and shall write the equations characterizing the deformation in the form,

$$x_i' = x_i'(x_1, x_2, x_3) \equiv x_i'(x), \qquad (i = 1, 2, 3). \tag{1.1}$$

We shall further suppose that Eqs. (1.1) have a single-valued inverse

$$x_i = x_i(x_1', x_2', x_3') \equiv x_i(x'), \qquad (i = 1, 2, 3), \tag{1.2}$$

so that the transformation of points from R into R' is one-to-one. This restriction is based solely on our desire to deal with the single-valued displacements.[1] To ensure the existence of the single-valued inverse, it would suffice to assume that the functions $x_i'(x)$ are of class C^1 in R and have a nonvanishing Jacobian in that region.[2] We shall assume that this is so.

Part of the transformation defined by Eqs. (1.1) may represent rigid body motions (that is, translations and rotations) of the body as a whole. This part of the deformation leaves unchanged the length of every vector joining a pair of points within the body and is of no interest in the analysis of strain. The remaining part of the transformation (1.1) will be called *pure deformation*. It will be important to learn how to distinguish between pure deformations and rigid body motions when the latter are present in the equation of transformation (1.1). To this end we shall consider first the simplest case of (1.1), that in which the functions appearing therein are linear functions of the coordinates x_1, x_2, x_3. The Eqs. (1.1) in which the functions are linear define what is called an *affine transformation*.

2. Affine Transformations. The properties of the general linear transformation of points,

$$\begin{aligned} x_1' &= \alpha_{10} + (1 + \alpha_{11})x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\ x_2' &= \alpha_{20} + \alpha_{21}x_1 + (1 + \alpha_{22})x_2 + \alpha_{23}x_3, \\ x_3' &= \alpha_{30} + \alpha_{31}x_1 + \alpha_{32}x_2 + (1 + \alpha_{33})x_3, \end{aligned}$$

or, written more compactly,[3]

$$x_i' = \alpha_{i0} + (\delta_{ij} + \alpha_{ij})x_j, \qquad (i, j = 1, 2, 3), \tag{2.1}$$

[1] Although, at first glance, the single-valued character of displacements appears to be demanded by the physics of the situation, it proves convenient to consider multivalued displacements in several important physical problems. Multivalued displacements in two-dimensional elastostatic problems were first considered by A. Timpe, *Zeitschrift für Mathematik und Physik*, vol. 52 (1905), pp. 348–383, and, in a more general way, by V. Volterra, *Annales de l'école normale supérieure*, vol. 24 (1907), pp. 401–517. An account of Volterra's "theory of dislocation," concerned with the deformation of initially strained multiply connected regions, is given by A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity (1927), pp. 221–228. A brief, but highly illuminating, account of multivalued displacements arising in thermoelastic problems is contained in N. I. Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953), pp. 157–165.

[2] See, for example, I. S. Sokolnikoff, Advanced Calculus (1939), p. 422. A function $F(x_1, x_2, x_3)$ is said to be of the class C^n in the region R if it is continuous and has continuous partial derivatives with respect to x_1, x_2, and x_3 up to and including those of order n.

[3] A repeated subscript indicates summation as the index that is repeated takes the values 1, 2, 3. Thus

$$\alpha_{1j}x_j = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3.$$

The symbol δ_{ij}, the Kronecker delta, is defined to have the value one if i equals j,

where the coefficients a_{ij} are constants, are well known. Since it is desirable to demand the existence of an inverse, Eqs. (2.1) must be solvable for the variables x_1, x_2, x_3 as functions of x'_1, x'_2, x'_3. It follows that the determinant $|\delta_{ij} + \alpha_{ij}|$ of the coefficients of the unknowns entering into the right-hand member of (2.1) must not vanish. It is obvious that the inverse transformation

$$(2.2) \qquad x_i = \beta_{i0} + (\delta_{ij} + \beta_{ij})x'_j, \qquad (i, j = 1, 2, 3),$$

is likewise linear.

It is easy to see from (2.1) and (2.2) that an affine transformation carries planes into planes, and hence a rectilinear segment joining the

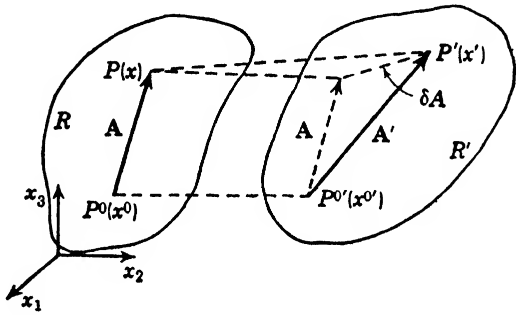

Fig 1

points $P^0(x_1^0, x_2^0, x_3^0)$ and $P(x_1, x_2, x_3)$ is transformed into a rectilinear segment joining the corresponding points $P^{0\prime}(x_1^{0\prime}, x_2^{0\prime}, x_3^{0\prime})$ and $P'(x'_1, x'_2, x'_3)$ (Fig. 1). This follows from the fact that the rectilinear segment $\overline{P^0P}$ can be thought of as joining two points P^0 and P on the intersection of two planes S_1 and S_2; under the transformation (2.1) points P^0 and P go over into points $P^{0\prime}$ and P', which lie on the intersection of the planes S'_1 and S'_2, into which the planes S_1 and S_2 are carried by the transformation.

We shall denote the unit base vectors, directed along the coordinate axes x_1, x_2, and x_3, by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively. Thus, a vector $\mathbf{A}$ whose components along the coordinate axes are A_1, A_2, A_3 can be written as

$$\mathbf{A} = \mathbf{e}_1A_1 + \mathbf{e}_2A_2 + \mathbf{e}_3A_3 \equiv \mathbf{e}_iA_i, \qquad (i = 1, 2, 3).$$

Since the vector $\mathbf{A} = \mathbf{e}_iA_i$ is uniquely determined once its components A_i $(i = 1, 2, 3)$ are prescribed, we can represent the vector $\mathbf{A}$ by the symbol A_i. Under the transformation (2.1) the vector $A_i = x_i - x_i^0$, joining the points $P^0(x^0)$ and $P(x)$, is carried into another vector

$$A'_i = x'_i - x_i^{0\prime}.$$

zero if i differs from j. The reason for writing the coefficients of x_1, x_2, and x_3 in the first, second, and third lines as $1 + \alpha_{11}$, $1 + \alpha_{22}$, $1 + \alpha_{33}$ will appear later.

In general, vectors A_i and A'_i differ in direction and magnitude. From (2.1), which we write in the form

$$x'_i = \alpha_{i0} + x_i + \alpha_{ij}x_j,$$

we have

$$\begin{aligned} A'_i = x'_i - x_i^{0\prime} &= (\alpha_{i0} + x_i + \alpha_{ij}x_j) - (\alpha_{i0} + x_i^0 + \alpha_{ij}x_j^0) \\ &= (x_i - x_i^0) + \alpha_{ij}(x_j - x_j^0) = A_i + \alpha_{ij}A_j, \end{aligned}$$

or

$$\delta A_i \equiv A'_i - A_i = \alpha_{ij}A_j, \qquad (i, j = 1, 2, 3). \tag{2.3}$$

It is clear from (2.3) that two vectors A_i and B_i whose components are equal transform into two vectors A'_i and B'_i whose components are again equal. Also two parallel vectors obviously transform into parallel vectors. Hence, two equal and similarly oriented rectilinear polygons located in different parts of the region R will be transformed into two equal and similarly oriented polygons in the transformed region R'. Thus, the different parts of the body τ, when the latter is subjected to the transformation (2.1), experience the same deformation independently of the position of the parts of the body. For this reason, the deformation characterized by (2.1) is called a *homogeneous deformation*.

Consider the transformation (2.1), and let the variables x'_i be subjected to another affine transformation,

$$x''_k = \gamma_{k0} + (\delta_{ki} + \gamma_{ki})x'_i. \tag{2.4}$$

Recalling the definition of the Kronecker delta, we can write (2.4) as

$$x''_k = \gamma_{k0} + x'_k + \gamma_{ki}x'_i.$$

Let A''_k be the transform of the vector A'_k; then

$$\begin{aligned} A''_k \equiv x''_k - x_k^{0\prime\prime} &= (\gamma_{k0} + x'_k + \gamma_{ki}x'_i) - (\gamma_{k0} + x_k^{0\prime} + \gamma_{ki}x_i^{0\prime}) \\ &= (x'_k - x_k^{0\prime}) + \gamma_{ki}(x'_i - x_i^{0\prime}) = A'_k + \gamma_{ki}A'_i, \end{aligned}$$

or

$$\delta A'_k \equiv A''_k - A'_k = \gamma_{ki}A'_i, \qquad (i, k = 1, 2, 3). \tag{2.5}$$

The product of the two successive affine transformations (2.1) and (2.4) is equivalent to the single transformation obtained by substituting in (2.4) the values of x'_i in terms of x_j from (2.1). Thus one has

$$\begin{aligned} x''_k &= \gamma_{k0} + (\delta_{ki} + \gamma_{ki})[\alpha_{i0} + (\delta_{ij} + \alpha_{ij})x_j] \\ &= \alpha_{k0} + \gamma_{k0} + (\delta_{kj} + \alpha_{kj} + \gamma_{kj})x_j \\ &\quad + \alpha_{i0}\gamma_{ki} + \alpha_{ij}\gamma_{ki}x_j. \end{aligned}$$

Now if the coefficients α_{ij} and γ_{ij} are so small that one is justified in neglecting their products in comparison with the coefficients themselves, then

$$x''_k = \alpha_{k0} + \gamma_{k0} + x_k + (\alpha_{kj} + \gamma_{kj})x_j.$$

The product transformation likewise carries the point (x_1^0, x_2^0, x_3^0) to the point $(x_1^{0\prime\prime}, x_2^{0\prime\prime}, x_3^{0\prime\prime})$ where

$$x_k^{0\prime\prime} = \alpha_{k0} + \gamma_{k0} + x_k^0 + (\alpha_{kj} + \gamma_{kj})x_j^0.$$

The vector $A_k = x_k - x_k^0$ is thus transformed into the vector

$$\begin{aligned} A_k^{\prime\prime} = x_k^{\prime\prime} - x_k^{0\prime\prime} &= (x_k - x_k^0) + (\alpha_{kj} + \gamma_{kj})(x_j - x_j^0) \\ &= A_k + (\alpha_{kj} + \gamma_{kj})A_j, \end{aligned}$$

or

$$\delta A_k \equiv A_k^{\prime\prime} - A_k = (\alpha_{kj} + \gamma_{kj})A_j, \qquad (j, k = 1, 2, 3). \tag{2.6}$$

Thus, if one neglects products of the α_{ij} and γ_{ij}, then the coefficients in the resultant transformation (2.6) are obtained by adding the corresponding coefficients α_{ij} and γ_{ij} in the component transformations (2.3) and (2.5). In this event, it is said that the product transformation (2.6) is obtained by superposition of the original transformations. It is clear from the structure of the formulas (2.6) that the resultant transformation is independent of the order in which the transformations are performed. One of the chief sources of the difficulty that confronts one in the study of finite as distinguished from infinitesimal deformations arises from the fact that the principle of superposition of effects and the independence of the order of transformations are no longer valid.

A transformation of the type (2.1), in which the coefficients are so small that their products can be neglected in comparison with the linear terms, is called an *infinitesimal affine transformation.*

3. Infinitesimal Affine Deformations. In this section we shall be concerned with the problem of separating the infinitesimal affine transformation defined by Eq. (2.3),

$$\delta A_i \equiv A_i^{\prime} - A_i = \alpha_{ij}A_j, \tag{3.1}$$

into two component transformations: one of these corresponds to a rigid body motion; the other, which we have termed pure deformation, will be investigated in detail in the next section. We seek first the conditions on the coefficients α_{ij} if the deformation is to be one of rigid body motion (that is, one consisting of rotation and translation) alone.

A rigid body motion is characterized by the fact that the length

$$A = |\mathbf{A}| = \sqrt{A_iA_i}$$

of any vector $\mathbf{A}$ is unchanged by the transformation. If we replace the A_i in this formula by $A_i + \delta A_i$ and denote the change in length A by δA, we get

$$A\,\delta A = A_i\,\delta A_i \tag{3.2}$$

plus terms of higher order in δA_i, which are neglected, since we are concerned with the infinitesimal affine transformation. When the expres-

sions for δA_i given by (3.1) are inserted in (3.2), one finds that

$$A\,\delta A = \alpha_{ij} A_i A_j,$$

or when written out in full,

$$A\,\delta A = \alpha_{11}A_1^2 + \alpha_{22}A_2^2 + \alpha_{33}A_3^2 + (\alpha_{12} + \alpha_{21})A_1A_2 + (\alpha_{23} + \alpha_{32})A_2A_3 + (\alpha_{31} + \alpha_{13})A_3A_1.$$

Since for a rigid body transformation δA vanishes for all values of A_1, A_2, A_3, we must have

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0,$$
$$\alpha_{12} + \alpha_{21} = \alpha_{23} + \alpha_{32} = \alpha_{31} + \alpha_{13} = 0.$$

Hence a necessary and sufficient condition that the infinitesimal transformation (3.1) represent a rigid body motion is

(3.3) $$\alpha_{ij} = -\alpha_{ji}, \qquad (i, j = 1, 2, 3).$$

In this case, the set of quantities α_{ij} is said to be *skew-symmetric.* When the coefficients α_{ij} are skew-symmetric, the transformation (3.1) takes the form

$$\begin{aligned} \delta A_1 &= \qquad\qquad - \alpha_{21}A_2 + \alpha_{13}A_3, \\ \delta A_2 &= \alpha_{21}A_1 \qquad\qquad - \alpha_{32}A_3, \\ \delta A_3 &= -\alpha_{13}A_1 + \alpha_{32}A_2 \end{aligned}$$

This transformation can be written as the vector product of the infinitesimal rotation vector $\boldsymbol{\omega} = \mathbf{e}_i\omega_i$ and the vector $\mathbf{A}$, namely[1]

$$\delta\mathbf{A} = \boldsymbol{\omega} \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ x_1 - x_1^0 & x_2 - x_2^0 & x_3 - x_3^0 \end{vmatrix},$$

if we take

(3.4) $$\begin{cases} \omega_1 \equiv \alpha_{32} = -\alpha_{23} = \tfrac{1}{2}(\alpha_{32} - \alpha_{23}), \\ \omega_2 \equiv \alpha_{13} = -\alpha_{31} = \tfrac{1}{2}(\alpha_{13} - \alpha_{31}), \\ \omega_3 \equiv \alpha_{21} = -\alpha_{12} = \tfrac{1}{2}(\alpha_{21} - \alpha_{12}). \end{cases}$$

The equations representing the rigid body motion can be obtained by observing that $A_i = x_i - x_i^0$ and that

$$\begin{aligned} \delta A_i &= A_i' - A_i = (x_i' - x_i^{0\prime}) - (x_i - x_i^0) \\ &= (x_i' - x_i) - (x_i^{0\prime} - x_i^0) = \delta x_i - \delta x_i^0 \end{aligned}$$

or

$$\delta x_i = \delta x_i^0 + \delta A_i = \delta x_i^0 + (\boldsymbol{\omega} \times \mathbf{A})_i.$$

[1] We recall that when a rigid body rotates with the angular velocity $\boldsymbol{\Omega}$, the linear velocity $\mathbf{v}$ is $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{A}$ and $\delta\mathbf{A} = \boldsymbol{\Omega} \times \mathbf{A}\,\delta t = \boldsymbol{\omega} \times \mathbf{A}$, where $\boldsymbol{\omega} = \boldsymbol{\Omega}\,\delta t$ is the infinitesimal angle of rotation.

Then the rigid body portion of the infinitesimal affine transformation (2.1) can be written as

$$(3.5)\qquad \begin{cases} \delta x_1 = \delta x_1^0 & - \omega_3(x_2 - x_2^0) + \omega_2(x_3 - x_3^0), \\ \delta x_2 = \delta x_2^0 + \omega_3(x_1 - x_1^0) & - \omega_1(x_3 - x_3^0), \\ \delta x_3 = \delta x_3^0 - \omega_2(x_1 - x_1^0) + \omega_1(x_2 - x_2^0). \end{cases}$$

The quantities $\delta x_i^0 \equiv x_i^{0\prime} - x_i^0$ are the components of the displacement vector representing the translation of the point $P^0(x^0)$ (see Fig. 1), while the remaining terms of (3.5) represent rotation about the point P^0.

At the beginning of this section, we proposed the problem of separating the infinitesimal affine transformation $\delta A_i = \alpha_{ij}A_j$ into two component transformations, one of which is to represent rigid body motion alone; we have seen that this rigid body motion corresponds to a transformation in which the coefficients are skew-symmetric; that is, $\alpha_{ij} = -\alpha_{ji}$. Now any set of quantities α_{ij} may be decomposed into a symmetric and a skew-symmetric set in one, and only one, way.[1] We can thus write

$$\alpha_{ij} = \tfrac{1}{2}(\alpha_{ij} + \alpha_{ji}) + \tfrac{1}{2}(\alpha_{ij} - \alpha_{ji}).$$

Then Eq. (3.1) can be written as

$$\delta A_i = \alpha_{ij}A_j = [\tfrac{1}{2}(\alpha_{ij} + \alpha_{ji}) + \tfrac{1}{2}(\alpha_{ij} - \alpha_{ji})]A_j,$$

or

$$(3.6)\qquad \delta A_i = e_{ij}A_j + \omega_{ij}A_j,$$

where

$$\begin{aligned} e_{ij} &= \quad e_{ji} \equiv \tfrac{1}{2}(\alpha_{ij} + \alpha_{ji}), \\ \omega_{ij} &= -\omega_{ji} \equiv \tfrac{1}{2}(\alpha_{ij} - \alpha_{ji}). \end{aligned}$$

The skew-symmetric coefficients ω_{ij} correspond to a rigid body motion, and from (3.4) it can be seen that they are connected with the components of rotation, ω_1, ω_2, ω_3, by the relations

$$\omega_{32} = \omega_1, \qquad \omega_{13} = \omega_2, \qquad \omega_{21} = \omega_3.$$

It is clear from Eqs. (3.6) for the transformation of the components of a vector that an infinitesimal affine transformation of the vector A_i can be decomposed into transformation $\delta A_i = \omega_{ij}A_j$, representing rigid body motion, and into transformation

$$(3.7)\qquad \delta A_i = e_{ij}A_j,$$

representing pure deformation.

The symmetric coefficients e_{ij} are called *components of the strain tensor*, and they characterize pure deformation. We shall investigate the properties of the strain tensor in the next section.

[1] See Prob. 1 at the end of this chapter.

4. A Geometrical Interpretation of the Components of Strain. The geometrical significance of the components of strain e_{ij} entering into (3.7) can be readily determined by inserting the expressions (3.7) in the formula (3.2), which then takes the form $A\,\delta A = A_i\,\delta A_i = e_{ij}A_iA_j$, or

$$\frac{\delta A}{A} = \frac{e_{ij}A_iA_j}{A^2}. \tag{4.1}$$

If initially the vector $\mathbf{A}$ is parallel to the x_1-axis, so that $A = A_1$ and $A_2 = A_3 = 0$, then it follows from (4.1) that

$$\frac{\delta A}{A} = e_{11}. \tag{4.2}$$

Thus, the component e_{11} of the strain tensor represents the *extension*, or *change in length per unit length*, of a vector originally parallel to the x_1-axis.

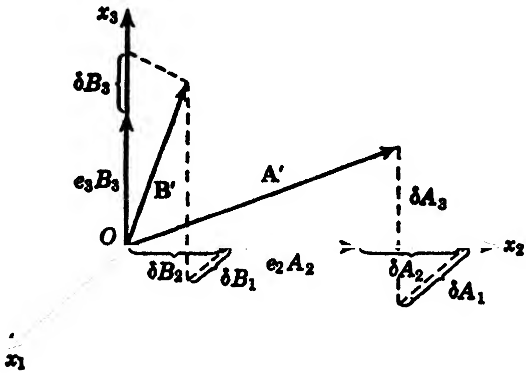

FIG. 2

Hence, if all components of the strain tensor with the exception of e_{11} vanish, then all unit vectors parallel to the x_1-axis will be extended by an amount e_{11} if this strain component is positive and contracted by the same amount if e_{11} is negative. In this event, one has a homogeneous deformation of material in the direction of the x_1-axis. A cube of material whose edges before deformation are l units long will become a rectangular parallelepiped whose dimensions in the x_1-direction are $l(1 + e_{11})$ units and whose dimensions in the directions of the x_2- and x_3-axes are unaltered. A similar significance can be ascribed to the components e_{22} and e_{33}.

In order to interpret geometrically such strain components as e_{23}, consider two vectors $\mathbf{A} = \mathbf{e}_2A_2$ and $\mathbf{B} = \mathbf{e}_3B_3$ (Fig. 2), initially directed along the x_2- and x_3-axes respectively. Upon deformation, these vectors become

$$\begin{aligned}\mathbf{A}' &= \mathbf{e}_1\,\delta A_1 + \mathbf{e}_2(A_2 + \delta A_2) + \mathbf{e}_3\,\delta A_3,\\ \mathbf{B}' &= \mathbf{e}_1\,\delta B_1 + \mathbf{e}_2\,\delta B_2 \qquad\qquad + \mathbf{e}_3(B_3 + \delta B_3).\end{aligned}$$

We denote the angle between $\mathbf{A}'$ and $\mathbf{B}'$ by θ and consider the change $\alpha_{23} = (\pi/2) - \theta$ in the right angle between $\mathbf{A}$ and $\mathbf{B}$. From the definition

of the scalar product of $\mathbf{A}'$ and $\mathbf{B}'$, we have

$$A'B'\cos\theta \equiv \mathbf{A}'\cdot\mathbf{B}' = \delta A_1\,\delta B_1 + (A_2 + \delta A_2)\,\delta B_2 + (B_3 + \delta B_3)\,\delta A_3 \doteq A_2\,\delta B_2 + B_3\,\delta A_3,$$

if we neglect the products of the changes in the components of the vectors $\mathbf{A}$ and $\mathbf{B}$. To the same approximation, we have

$$\cos\theta = \frac{\mathbf{A}'\cdot\mathbf{B}'}{A'B'} \tag{4.3}$$

$$= \frac{A_2\,\delta B_2 + B_3\,\delta A_3}{\sqrt{(\delta A_1)^2 + (A_2+\delta A_2)^2 + (\delta A_3)^2}\sqrt{(\delta B_1)^2 + (\delta B_2)^2 + (B_3+\delta B_3)^2}}$$
$$\doteq (A_2\,\delta B_2 + B_3\,\delta A_3)(A_2+\delta A_2)^{-1}(B_3+\delta B_3)^{-1}$$
$$\doteq \frac{A_2\,\delta B_2 + B_3\,\delta A_3}{A_2B_3} = \frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2}.$$

Since all increments in the components of $\mathbf{A}$ and $\mathbf{B}$ have been neglected except δA_3 and δB_2, the deformation can be represented as shown in Fig. 3. If we remember that

$$A_1 = A_3 = B_1 = B_2 = 0,$$

then Eqs. (3.7) yield

$$\delta B_2 = e_{23}B_3, \qquad \delta A_3 = e_{23}A_2. \tag{4.4}$$

From (4.3) we have

$$\cos\theta = \cos\left(\frac{\pi}{2} - \alpha_{23}\right) = \sin\alpha_{23}$$
$$\doteq \alpha_{23} = \frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2} = 2e_{23},$$

or

$$\alpha_{23} = 2e_{23}.$$

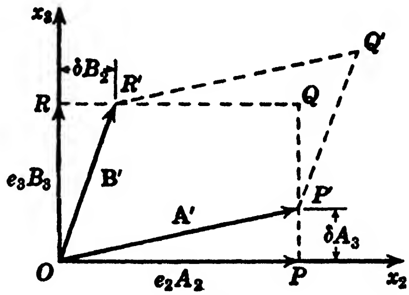

FIG. 3

Hence a positive value of $2e_{23}$ represents a decrease in the right angle between the vectors $\mathbf{A}$ and $\mathbf{B}$, which were initially directed along the positive x_2- and x_3-axes.

Again, from (4.4) and Fig. 3 we see that

$$\angle POP' \doteq \tan POP' = \frac{\delta A_3}{A_2} = e_{23},$$
$$\angle ROR' \doteq \tan ROR' = \frac{\delta B_2}{B_3} = e_{23}.$$

Since the angles POP' and ROR' are equal, it follows that, by rotating the parallelogram $R'OP'Q'$ through an angle e_{23} about the origin, one can obtain the configuration shown in Fig. 4. Obviously it represents a slide or a shear of the elements parallel to the x_1x_2-plane, where the amount of slide is proportional to the distance x_3 of the element from the x_1x_2-plane.

A similar interpretation can obviously be made in regard to the components e_{12} and e_{31}.

It is clear that the areas of the rectangle and the parallelogram in Fig. 4 are equal. Likewise an element of volume originally cubical is deformed into a parallelepiped, and the volumes of the cube and parallelepiped are equal if one disregards the products of the changes in the linear elements. Such deformation is called *pure shear*.

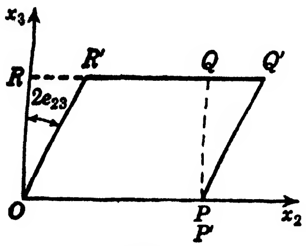

FIG. 4

The characterization of strain presented in Secs. 3 and 4 is essentially due to Cauchy. It should be noted that the strain components e_{ij} refer to the chosen set of coordinate axes; if the axes are changed, the e_{ij} will, in general, assume different values.

5. Strain Quadric of Cauchy. With each point $P^0(x^0)$ of a continuous medium, we shall associate a quadric surface, the *quadric of deformation*, which enables one to determine the elongation of any vector

$$\mathbf{A} = \mathbf{e}_i(x_i - x_i^0)$$

that runs from the point $P^0(x^0)$ to some point $P(x)$.

Now if a local system of axes x_i is introduced, with origin at the initial point $P^0(x^0)$ of the vector $\mathbf{A}$ and with axes parallel to the space-fixed axes, then formula (4.1) characterizing the extension $e = \delta A/A$ of $\mathbf{A}$ can be written as

(5.1) $$eA^2 = e_{ij}x_ix_j.$$

We consider the quadratic function

(5.2) $$2G(x_1, x_2, x_3) \equiv e_{ij}x_ix_j$$

and constrain the end point $P(x)$ of the vector $\mathbf{A}$, as yet unspecified, to lie on the quadric surface

(5.3) $$2G(x_1, x_2, x_3) = \pm k^2,$$

where k is any real constant and the sign is chosen so as to make the surface real. Comparison of (5.3) with (5.1) leads to the relation

(5.4) $$e = \pm \frac{k^2}{A^2},$$

and the strain quadric takes the form

(5.5) $$e_{ij}x_ix_j = \pm k^2.$$

From (5.4) we see that *the extension of any line through $P^0(x^0)$ is inversely proportional to the square of the radius vector that runs along the line from the point (x^0), at which the strain is being studied, to a point (x) on the quadric*

surface. Accordingly, the maximum and minimum elongations will be in the directions of the axes of the quadric (5.5)

We refer the *quadric surface of deformation* (5.5) to a new coordinate system x'_1, x'_2, x'_3, obtained from the old by a rotation of axes. Let the directions of the new coordinate axes x'_i be specified relative to the old system x_i by the table of direction cosines

	x_1	x_2	x_3
x'_1	l_{11}	l_{12}	l_{13}
x'_2	l_{21}	l_{22}	l_{23}
x'_3	l_{31}	l_{32}	l_{33}

in which l_{ij} is the cosine of the angle between the x'_i- and the x_j-axes. The old and the new coordinates are related by the equations

$$\begin{aligned} x_1 &= l_{11}x'_1 + l_{21}x'_2 + l_{31}x'_3, \\ x_2 &= l_{12}x'_1 + l_{22}x'_2 + l_{32}x'_3, \\ x_3 &= l_{13}x'_1 + l_{23}x'_2 + l_{33}x'_3, \end{aligned}$$

or, more compactly,

(5.6) $$x_i = l_{\alpha i}x'_\alpha.$$

It is readily shown that the inverse transformation is of the form[1]

$$x'_i = l_{i\alpha}x_\alpha.$$

The well-known orthogonality relations between the direction cosines can be written in the form

(5.7) $$l_{i\alpha}l_{j\alpha} = \delta_{ij}, \qquad l_{\alpha i}l_{\alpha j} = \delta_{ij}.$$

When the quadric surface (5.5) is referred to the x'_i coordinate system, a new set of strains e'_{ij} is determined and (5.5) is replaced by the new equation of the surface, namely,

$$e'_{ij}x'_ix'_j = \pm k^2.$$

The right-hand member of (5.5), however, has a geometrical meaning that is independent of the choice of coordinate system ($\pm k^2 = eA^2$); consequently

(5.8) $$e_{ij}x_ix_j = e'_{ij}x'_ix'_j.$$

In other words, the quadratic form $e_{ij}x_ix_j$ is invariant with respect to an orthogonal transformation of coordinates.

Equations (5.6) and (5.8) together yield

$$e_{ij}l_{\alpha i}l_{\beta j}x'_\alpha x'_\beta = e'_{\alpha\beta}x'_\alpha x'_\beta,$$

[1] See Prob. 5 at the end of this chapter

and since the x'_α are arbitrary,

$$e'_{\alpha\beta} = l_{\alpha i} l_{\beta j} e_{ij}. \tag{5.9}$$

Similarly it can be shown that

$$e_{\alpha\beta} = l_{i\alpha} l_{j\beta} e'_{ij}. \tag{5.10}$$

A set of quantities e_{ij} transforming according to the law (5.9) is said to represent a *cartesian tensor of rank* 2. We shall meet several such tensors in the subsequent discussion.

Differentiating $2G(x_1, x_2, x_3) = e_{ij}x_ix_j$ and noting from (3.7) that for a pure deformation $\delta A_i = e_{ij}A_j = e_{ij}x_j$, we find that

$$\frac{\partial G}{\partial x_i} = e_{ij}x_j = \delta A_i. \tag{5.11}$$

But $\dfrac{\partial G}{\partial x_i}$ are the direction ratios of the normal $\boldsymbol{\nu}$ to the quadric surface (5.5) at the point (x_i), and it follows that the vector $\delta\mathbf{A}$ is directed along the normal to the plane tangent to the surface $e_{ij}x_ix_j = \pm k^2$ (see Fig. 5). This property of the strain quadric will prove useful in the next section, where we discuss the principal axes of the quadric surface and their significance for the deformation.

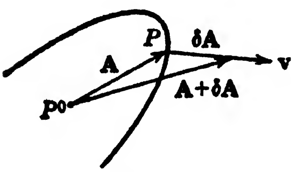

Fig. 5

6. Principal Strains. Invariants. We seek now the direction ratios of the lines through (x^0) whose orientation is left unchanged by the deformation $\delta A_i = e_{ij}A_j$. If the direction of the vector $\mathbf{A}$ is not altered by the strain, then $\delta\mathbf{A}$ and $\mathbf{A}$ are parallel and their components are proportional.[1] Therefore

$$\delta A_i = eA_i.$$

It should be noted that $e = \dfrac{\delta A_i}{A_i}$ is the extension of each component of $\mathbf{A}$ and is thus the extension of $\mathbf{A}$ itself, or $e = \delta A/A$. Equation (5.1) then shows that the extension e is given by the expression $e = e_{ij}x_ix_j/A^2$. We return now to $\delta A_i = e_{ij}A_j$, from which it is seen that

$$e_{ij}A_j = eA_i = e\delta_{ij}A_j \tag{6.1}$$

or

$$(e_{ij} - e\delta_{ij})A_j = 0. \tag{6.2}$$

This set (6.2) of three homogeneous equations in the unknowns A_j possesses a nonvanishing solution if, and only if, the determinant of the

[1] In other words, the directions we seek are those of the axes of the quadric (5.5); that is, we seek the directions yielding the extreme values of the elongations e.

coefficients of the A_j is equal to zero; that is,

(6.3) $$|e_{ij} - e\delta_{ij}| = 0,$$

or

$$\begin{vmatrix} e_{11} - e & e_{12} & e_{13} \\ e_{21} & e_{22} - e & e_{23} \\ e_{31} & e_{32} & e_{33} - e \end{vmatrix} = 0.$$

We prove next that the three roots e_1, e_2, e_3 of this cubic equation in the elongation e are all real.

Let the three directions determined by the three numbers e_j be given by the vectors[1] $\overset{j}{\mathbf{A}}$. In this notation, formula (6.1) becomes, for any root $e = e_1$,

$$e_1\overset{1}{A}_j = e_{jk}\overset{1}{A}_k.$$

We multiply both sides by $\overset{2}{A}_j$ and sum over j, getting

(6.4) $$e_1\overset{1}{A}_j\overset{2}{A}_j = e_{jk}\overset{1}{A}_k\overset{2}{A}_j.$$

Similarly, from $e_2\overset{2}{A}_j = e_{jk}\overset{2}{A}_k$ we have

(6.5) $$e_2\overset{1}{A}_j\overset{2}{A}_j = e_{jk}\overset{1}{A}_j\overset{2}{A}_k = e_{kj}\overset{1}{A}_k\overset{2}{A}_j = e_{jk}\overset{1}{A}_k\overset{2}{A}_j,$$

where j and k have been interchanged and the symmetry of e_{jk} exploited. Comparsion of (6.4) and (6.5) shows that

(6.6) $$(e_1 - e_2)\overset{1}{A}_j\overset{2}{A}_j = 0.$$

Now if we assume tentatively that (6.3) has complex roots, then these can be written

$$e_1 = E_1 + iE_2, \qquad e_2 = E_1 - iE_2, \qquad e_3,$$

where E_1, E_2, e_3 are real. If $e_2 = E_1 - iE_2$ is substituted for e in (6.2), it will be found that the resulting solutions $\overset{2}{A}_j \equiv a_j - ib_j$ are the complex conjugates of $\overset{1}{A}_j \equiv a_j + ib_j$, where the latter are obtained by putting $e = e_1 = E_1 + iE_2$. Therefore

$$\begin{aligned}\overset{1}{A}_j\overset{2}{A}_j &= (a_j + ib_j)(a_j - ib_j) \\ &= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 \neq 0.\end{aligned}$$

Hence it follows from (6.6) that $e_1 - e_2 \equiv 2iE_2 = 0$, or $E_2 = 0$, and the roots e_i are all real.

[1] The index j over $\mathbf{A}$ indicates not the jth component but rather the jth vector and its dependence upon the root e_j of the determinantal equation (6.3).

From (6.6) it follows that, if the roots e_1 and e_2 are distinct, then

$$\overset{1}{A}_i\overset{2}{A}_i = \overset{1}{\mathbf{A}} \cdot \overset{2}{\mathbf{A}} = 0,$$

so that the corresponding directions are orthogonal. These directions $\overset{i}{\mathbf{A}}$ are called the *principal directions of strain*, and the strains e_i, which are the extensions of the vectors $\overset{i}{\mathbf{A}}$ in the principal directions, are termed the *principal strains*.

We have seen that at any point (x^0) there are three mutually perpendicular directions $\overset{i}{\mathbf{A}}$ (assuming, for the moment, that the e_i are distinct) that are left unaltered by the deformation; consequently the vectors $\overset{i}{\mathbf{A}}$, the deformed vectors $\overset{i}{\mathbf{A}} + \delta\overset{i}{\mathbf{A}}$, and $\delta\overset{i}{\mathbf{A}}$ are collinear. But (5.11) shows that $\delta\mathbf{A}$ is always normal to the quadric surface (5.5), and therefore the principal directions of strain are also normal to the surface and must be the three principal axes of the quadric $e_{ij}x_ix_j = eA^2$. If some of the principal strains e_i are equal, then the associated directions become indeterminate but one can always select three directions that are mutually orthogonal. If the quadric surface is a surface of revolution, then one direction $\overset{1}{\mathbf{A}}$, say, will be directed along the axis of revolution and any two mutually perpendicular vectors lying in the plane normal to $\overset{1}{\mathbf{A}}$ may be taken as the other two principal axes. If $e_1 = e_2 = e_3$, the quadric is a sphere and any three orthogonal lines may be chosen as the principal axes.

We recall that e_1, e_2, e_3 are the extensions of vectors along the principal axes, while e_{11}, e_{22}, e_{33} are the extensions of vectors along the coordinate axes. If the coordinate axes x_i are taken along the principal axes of the quadric, then the shear strains e_{12}, e_{23}, e_{31} disappear from the equation of the quadric surface and the latter takes the form

$$e_1x_1^2 + e_2x_2^2 + e_3x_3^2 = \pm k^2.$$

The cubic equation (6.3) can be written in the form

$$|e_{ij} - e\delta_{ij}| = -e^3 + \vartheta_1 e^2 - \vartheta_2 e + \vartheta_3 = 0, \tag{6.7}$$

where ϑ_1, ϑ_2, ϑ_3 are the sums of the products of the roots taken one, two, and three at a time:

$$\begin{cases} \vartheta_1 = e_1 + e_2 + e_3 \equiv \vartheta, \\ \vartheta_2 = e_2e_3 + e_3e_1 + e_1e_2, \\ \vartheta_3 = e_1e_2e_3. \end{cases} \tag{6.8}$$

By expanding the determinant (6.7), we see that these expressions can also be written as

$$
(6.9)\quad \left\{
\begin{aligned}
\vartheta &= e_{11} + e_{22} + e_{33},\\
\vartheta_2 &= e_{22}e_{33} + e_{33}e_{11} + e_{11}e_{22} - e_{31}^2 - e_{12}^2 - e_{23}^2\\
&= \begin{vmatrix} e_{22} & e_{23} \\ e_{23} & e_{33} \end{vmatrix} + \begin{vmatrix} e_{11} & e_{31} \\ e_{31} & e_{33} \end{vmatrix} + \begin{vmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{vmatrix},\\
\vartheta_3 &= e_{11}e_{22}e_{33} + 2e_{12}e_{23}e_{31} - e_{11}e_{23}^2 - e_{22}e_{31}^2 - e_{33}e_{12}^2,\\
&= \begin{vmatrix} e_{11} & e_{12} & e_{31} \\ e_{12} & e_{22} & e_{23} \\ e_{31} & e_{23} & e_{33} \end{vmatrix}.
\end{aligned}
\right.
$$

The expressions for ϑ_2 and ϑ_3 can be written compactly by introducing the *generalized Kronecker delta*, $\delta^{ijk\cdots}_{pqr\cdots}$, which we now define. If the subscripts $p, q, r, \ldots$ are distinct and if the superscripts $i, j, k, \ldots$ are the same set of numbers as the subscripts, then the value of $\delta^{ijk\cdots}_{pqr\cdots}$ is defined to be $+1$ or -1 according as the subscripts and superscripts differ by an even or an odd permutation; the value is zero in all other cases. We can now rewrite the formulas (6.9) in the form

$$
(6.10)\quad \left\{
\begin{aligned}
\vartheta &= e_{ii}, && (i = 1, 2, 3),\\
\vartheta_2 &= \frac{1}{2!}\,\delta^{ij}_{pq}e_{pi}e_{qj}, && (i, j, p, q = 1, 2, 3),\\
\vartheta_3 &= \frac{1}{3!}\,\delta^{ijk}_{pqr}e_{pi}e_{qj}e_{rk}, && (i, j, k, p, q, r = 1, 2, 3).
\end{aligned}
\right.
$$

Since the principal strains, that is, the roots e_1, e_2, e_3 of (6.7), have a geometrical meaning that is independent of the choice of coordinate system, it is clear that ϑ, ϑ_2, and ϑ_3 are invariant with respect to an orthogonal transformation of coordinates. [Note that this invariance could have been used to derive expressions (6.8) from (6.9).]

The quantity ϑ has a simple geometrical meaning. Consider as a volume element a rectangular parallelepiped whose edges are parallel to the principal directions of strain, and let the lengths of these edges be l_1, l_2, l_3. Upon deformation, this element becomes again a rectangular parallelepiped but with edges of lengths $l_1(1 + e_1)$, $l_2(1 + e_2)$, $l_3(1 + e_3)$. Hence the change δV in the volume V of the element is

$$
\begin{aligned}
\delta V &= l_1l_2l_3(1 + e_1)(1 + e_2)(1 + e_3) - l_1l_2l_3\\
&= l_1l_2l_3(e_1 + e_2 + e_3)
\end{aligned}
$$

plus terms of higher order in e_i. Thus

$$
e_1 + e_2 + e_3 = \vartheta = \frac{\delta V}{V},
$$

and the first strain invariant ϑ represents the expansion of a unit volume

due to strain produced in the medium. For this reason ϑ is called the *cubical dilatation* or simply the *dilatation*.

PROBLEMS

1. Determine the principal directions by finding the extremal values of

$$e = \frac{e_{ij}x_ix_j}{A^2}.$$

Note that the $x_i/A = \nu_i$ are the direction cosines so that $e = e_{ij}\nu_i\nu_j$. Maximize this subject to the constraining condition $\nu_i\nu_i = 1$.

2. Refer the quadric of deformation to a set of principal axes, and discuss the nature of deformation when the quadric is an ellipsoid and when it is a hyperboloid. Draw appropriate figures and note that if $e_1 > 0$, $e_2 > 0$, $e_3 < 0$, then, depending on the direction of the vector **A** from the origin of the quadric, one must consider the surfaces $e_1x_1^2 + e_2x_2^2 - |e_3|x_3^2 = \pm k^2$.

7. General Infinitesimal Deformation. In the preceding sections, we have discussed the infinitesimal affine transformation (3.7), which carries the vector A_i into the vector $A_i' \equiv A_i + \delta A_i$, where

$$\delta A_i = \alpha_{ij}A_j = \left(\frac{\alpha_{ij} + \alpha_{ji}}{2} + \frac{\alpha_{ij} - \alpha_{ji}}{2}\right) A_j = (e_{ij} + \omega_{ij})A_j; \tag{7.1}$$

the e_{ij} and ω_{ij} were constants and so small that their products could be neglected in comparison with their first powers. Now we consider the general functional transformation and its relation to the affine deformation.

Consider an arbitrary material point $P^0(x_1^0, x_2^0, x_3^0)$ in a continuous medium, and let the same material point assume after deformation the position $P^{0\prime}(x_1^{0\prime}, x_2^{0\prime}, x_3^{0\prime})$ (see Fig. 1). We denote the small displacement of the point P^0 by

$$u_i(x_1^0, x_2^0, x_3^0) = x_i^{0\prime} - x_i^0.$$

The quantities u_1, u_2, u_3 are called the *components of displacement*. It is clear from physical considerations that it is desirable to demand that the functions u_i be single-valued and continuous throughout the region occupied by the body. For reasons that will become apparent in Sec. 10, it will be assumed that the functions $u_i(x_1, x_2, x_3)$ are of class C^3 (that is, the u_i together with their first, second, and third derivatives are continuous).

The character of the deformation in the neighborhood of the point P^0 can be determined by analyzing the change in the vector **A** joining the point $P^0(x_1^0, x_2^0, x_3^0)$ with an arbitrary neighboring point $P(x_1, x_2, x_3)$ of the undeformed medium. If $P'(x_1', x_2', x_3')$ is the deformed position of P, then the displacement u_i at the point P is

$$u_i(x_1, x_2, x_3) = u_i(x_1^0 + A_1, x_2^0 + A_2, x_3^0 + A_3) = x_i' - x_i$$

The deformed vector $\mathbf{A}'$ has components $A'_i = x'_i - x_i^{0\prime}$, and for the components of $\delta\mathbf{A} = \mathbf{A}' - \mathbf{A}$ we have

$$\begin{aligned}\delta A_i &= (x'_i - x_i^{0\prime}) - (x_i - x_i^0)\\ &= (x'_i - x_i) - (x_i^{0\prime} - x_i^0)\\ &= u_i(x_1^0 + A_1, x_2^0 + A_2, x_3^0 + A_3) - u_i(x_1^0, x_2^0, x_3^0)\\ &= \left(\frac{\partial u_i}{\partial x_j}\right)_0 A_j\end{aligned}$$

plus the remainder in the Taylor's expansion of the function $u_i(x_1^0 + A_1, x_2^0 + A_2, x_3^0 + A_3)$. The subscript zero indicates that the derivative is to be evaluated at the point P^0. The derivative $\dfrac{\partial u_i}{\partial x_j}$ will be written by introducing the symbol $u_{i,j}$ so that

$$\frac{\partial u_i}{\partial x_j} \equiv u_{i,j}, \tag{7.2}$$

and the subscript can be dropped without confusion, since we shall deal only with vectors at P^0. If the region in the vicinity of P^0 is chosen sufficiently small, that is, if A is sufficiently small, then one has the formulas analogous to (7.1),

$$\delta A_i = u_{i,j}A_j. \tag{7.3}$$

Comparison of formulas (7.3) and (7.1) shows that the transformation of the neighborhood of the point P^0 is affine and that

$$a_{ij} = u_{i,j}.$$

Now if we assume further that the displacements u_i, as well as their partial derivatives, are so small that their products can be neglected then (7.3) defines an infinitesimal affine transformation of the neighborhood of the point in question. Hence the considerations of the earlier sections are immediately applicable; the transformation (7.3) can be decomposed into pure deformation and rigid body motion,

$$\begin{aligned}\delta A_i = u_{i,j}A_j &= \left(\frac{u_{i,j} + u_{j,i}}{2} + \frac{u_{i,j} - u_{j,i}}{2}\right) A_j\\ &= (e_{ij} + \omega_{ij})A_j,\end{aligned} \tag{7.4}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{7.5}$$

It follows from (3.5) that if u_i is a small displacement at $P(x)$, then

$$\begin{aligned}u_1 &= a_1 + \omega_2 x_3 - \omega_3 x_2,\\ u_2 &= a_2 + \omega_3 x_1 - \omega_1 x_3,\\ u_3 &= a_3 + \omega_1 x_2 - \omega_2 x_1,\end{aligned}$$

where ω_i is the infinitesimal rotation vector about (0, 0, 0) and the a_i are constants representing a translation. It is clear that the transformation defined by (7.5) is in general no longer homogeneous, inasmuch as both the strain components e_{ij} and the components of rotation ω_{ij} are functions of the coördinates of the medium. The dilatation

$$\begin{aligned}\vartheta &= e_{11} + e_{22} + e_{33}\\ &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = u_{i,i}\end{aligned}$$

or the *divergence* of the displacement vector u_i will likewise differ, in general, from point to point of the body.[1]

In order to indicate the advantages of notation adopted here over the customary one in use by writers of technical treatises on elasticity and hydrodynamics, we rewrite (7.5) by setting

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad e_{11} = e_{xx}, \qquad e_{12} = e_{xy}, \text{ etc.},$$

and denote the components of the displacement vector (u_1, u_2, u_3) by (u, v, w). The components of the strain tensor become:

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z},$$

$$e_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad e_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

[1] Some of the important relations of vector analysis will now be written in tensor notation. In cartesian coordinates the divergence, gradient, and Laplacian operators can be written as follows:

$$\begin{aligned}\text{div } \mathbf{A} &\equiv \nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \frac{\partial A_i}{\partial x_i} = A_{i,i},\\ \text{grad } \varphi &\equiv \nabla\varphi = \frac{\partial \varphi}{\partial x_i} \equiv \varphi_{,i},\\ \nabla^2\varphi &= \text{div }(\nabla\varphi) = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} = \frac{\partial^2\varphi}{\partial x_i\,\partial x_i} = \varphi_{,ii}.\end{aligned}$$

The Green-Gauss Theorem, namely,

$$\int_\tau \text{div } \mathbf{A}\, d\tau = \int_\sigma \mathbf{A} \cdot \boldsymbol{\nu}\, d\sigma,$$

takes the form

$$\int_\tau A_{i,i}\, d\tau = \int_\sigma A_i \nu_i\, d\sigma,$$

where $d\sigma$ is an element of area, $d\tau$ is an element of volume, and $\boldsymbol{\nu}$ is the exterior normal to the surface σ. If we set $A_i = \dfrac{\partial \varphi}{\partial x_i} \equiv \varphi_{,i}$, then there results the identity

$$\int_\tau \varphi_{,ii}\, d\tau = \int_\sigma \varphi_{,i}\nu_i\, d\sigma,$$

or

$$\int_\tau \nabla^2\varphi\, d\tau = \int_\sigma \frac{d\varphi}{d\nu}\, d\sigma.$$

so that the dilatation ϑ is

$$\vartheta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

The components of rotation ω_{ij} read:

$$\omega_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \omega_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right),$$

$$\omega_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

While the unabridged notation, just explained, has some advantages in the discussion of specific problems, the compactness of the tensor notation and the economy of thought to which it leads in general developments are unquestionable.

8. Examples of Strain. Several important examples of strain will be considered next, and since there are no great advantages in using tensor notation in specific problems, we make use of the unabridged notation explained in the closing paragraphs of the preceding section.

a. Uniform Dilatation. If the strain quadric is a sphere, then any three orthogonal lines through the point may be used as the principal axes. In this case, the strain quadric has the equation

$$e_{xx}x^2 + e_{yy}y^2 + e_{zz}z^2 = \pm k^2, \tag{8.1}$$

where $e_{xx} = e_{yy} = e_{zz} \equiv e$ and $e_{xy} = e_{yz} = e_{xz} = 0$. The linear extension (or contraction) in any direction is the same and is equal to one-third of the dilatation, since

$$\vartheta = e_{xx} + e_{yy} + e_{zz} = 3e.$$

b. Simple Extension. Consider a simple extension of magnitude e in the direction of the x'-axis, whose direction cosines relative to the system of axes x, y, z are (l_{11}, l_{12}, l_{13}). Referred to the x', y', z' coordinate system, the strain quadric has the equation $ex'^2 = k^2$. By use of the transformation equations (5.10), we obtain in the x, y, z-system

$$\begin{cases} e_{xx} = el_{11}^2, & e_{yy} = el_{12}^2, & e_{zz} = el_{13}^2, \\ e_{xy} = el_{11}l_{12}, & e_{yz} = el_{12}l_{13}, & e_{zx} = el_{13}l_{11}. \end{cases} \tag{8.2}$$

Thus, a simple extension in the direction (l_{11}, l_{12}, l_{13}) may be specified in any x, y, z coordinate system by means of the six strain components given in (8.2).

c. Shearing Strain. Let the equation of the strain quadric when referred to the x', y', z'-system of coordinates be given by

$$2sx'y' = \pm k^2, \tag{8.3}$$

so that the only strain component is a shearing strain of magnitude s along

the directions of the x'- and y'-axes. This is the equation of a hyperbolic cylinder asymptotic to the $x'z'$- and $y'z'$-planes. The equation of the quadric (8.3) assumes the form

$$sx^2 - sy^2 = \pm k^2$$

when the axes are rotated through an angle of 45° about the z'-axis. A comparison of this equation with the general equation of the strain quadric when the latter is referred to the principal axes of strain,

$$e_{xx}x^2 + e_{yy}y^2 + e_{zz}z^2 = \pm k^2,$$

shows that we must have $e_{zz} = 0$, $e_{xx} = -e_{yy} = s$. Thus equal extension and contraction of two orthogonal linear elements is equivalent to a shearing strain of equal magnitude, which is associated with directions bisecting the angles between the elements.

d. Plane Strain. Suppose that the principal extension in the direction of the z'-axis is zero. Then for the x, y, z-system (assuming the directions of the z'- and z-axes to be the same), the strain quadric has the form

$$e_{xx}x^2 + e_{yy}y^2 + 2e_{xy}xy = \pm k^2,$$

corresponding to

$$e'_{x'x'}x'^2 + e'_{y'y'}y'^2 = \pm k^2 \qquad \text{in the } x',\ y',\ z'\text{-system},$$

$e'_{x'x'}$ and $e'_{y'y'}$ being principal extensions. In the case of simple extension (see part *b*), the quadric consists of two parallel planes; in the case of shearing strain (see part *c*), it consists of a rectangular hyperbolic cylinder. If the quadric is a circular cylinder, the state of strain is such that there is equal extension (or contraction) in all directions perpendicular to that of the z'-axis.

In the case of plane strain, the relative displacements u and v are functions of x and y alone, and w is a constant.

PROBLEMS

1. Verify the invariance of the functions ϑ, ϑ_2, and ϑ_3 [see Eqs. (6.10)] of the strains in the case of simple extension.

2. Find the dilatation and the principal strains, and describe the strain quadric for the case of simple extension.

3. Show that the examples of strain given in this section can be described by the following displacement components:

a. Uniform dilatation, $u = ex$, $v = ey$, $w = ez$.
b. Simple extension, $u' = ex'$, $v' = w' = 0$.
c. Shearing strain, $u' = 2sy'$, $v' = w' = 0$.
d. Plane strain, $u = u(x, y)$, $v = v(x, y)$, $w = 0$.

4. Show that in the examples of strain given in this section the rotation components are given by:

a. Uniform dilatation, $\omega_{xy} = \omega_{yz} = \omega_{zx} = 0$.
b. Simple extension, $\omega'_{xy} = \omega'_{yz} = \omega'_{zx} = 0$.

c. Shearing strain, $\omega'_{xy} = S$, $\omega'_{yz} = \omega'_{zx} = 0$.

d. Plane strain, $\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$, $\omega_{yz} = \omega_{zx} = 0$.

9. Notation. The values of the shear components e_{xy}, e_{zx}, e_{yz} of the strain tensor e_{ij}, defined in (7.5), differ from the quantities e_{xy}, e_{zx}, e_{yz} used by Love,[1] who writes

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

The factor ½ was inserted in the formulas (7.5) in order that the set of quantities may transform according to the tensor laws.

Trefftz[2] writes for the components of his strain tensor

$$\gamma_{xx} = 2\frac{\partial u}{\partial x}, \qquad \gamma_{yy} = 2\frac{\partial v}{\partial y}, \qquad \gamma_{zz} = 2\frac{\partial w}{\partial z},$$

$$\gamma_{yz} \quad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

while Timoshenko[3] uses

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

for the components of normal strain and agrees in notation with Trefftz for the components of shearing strain.

REFERENCES FOR COLLATERAL READING

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R. V. Southwell: Theory of Elasticity for Engineers and Physicists, Oxford University Press, New York, Secs. 292–307, pp. 285–297.

A. G. Webster· Dynamics of Particles and Rigid Bodies, Verlag von Julius Springer, Berlin, Sec. 169.

10. Equations of Compatibility. The defining formulas for the strain components e_{ij}, namely

$$u_{i,j} + u_{j,i} = 2e_{ij}, \tag{10.1}$$

will be looked upon in this section as a system of partial differential equations for the determination of the displacements u_i when the strain components e_{ij} are prescribed functions of the coordinates. We shall discuss first a necessary condition for the uniqueness of the solutions u_i of Eqs. (10.1). Thereupon we shall raise the question:

[1] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity.

[2] E. Trefftz, Handbuch der Physik, vol. 6, Mechanik der elastischen Körper.

[3] S. Timoshenko and J. N. Goodier, Theory of Elasticity.

What restrictions must be placed on the given functions $e_{ij}(x_1, x_2, x_3)$ to ensure the existence of single-valued continuous solutions $u_i(x_1, x_2, x_3)$ of Eqs. (10.1)?

It is clear first of all that specification of the e_{ij} does not determine the displacements u_i uniquely, for the strain components characterize the pure deformation of the medium in the neighborhood of the point (x), while the functions u_i may involve rigid body motions which do not affect the e_{ij}. In fact, if one obtains some solution

$$(10.2) \qquad u_i = u_i(x_1, x_2, x_3)$$

of the system (10.1), and if $P^0(x_1^0, x_2^0, x_3^0)$ is an arbitrary point of the body, then the addition to the right-hand member of (10.2) of the terms[1]

$$(10.3) \qquad u_j = u_j^0 + \omega_{jk}^0(x_k - x_k^0),$$

representing the motion of the body as a whole, will not affect the values of the prescribed components of strain entering into (10.1). It thus becomes clear that the solution of the system (10.1) cannot be unique unless one specifies the components of displacement u_i^0 and the components of rotation ω_{ij}^0 of some point P^0 of the medium, and we shall suppose in the following discussion that this has been done.

Inasmuch as there are six conditions imposed on the three functions u_i by Eqs. (10.1), one cannot expect in general that the system (10.1) will possess a solution for an arbitrary choice of the functions e_{ij}. We seek the further conditions that must be imposed on the functions e_{ij} if the system of Eqs. (10.1) is to possess a solution for the triplet of functions u_i.

The fact that the strain components e_{ij} cannot be prescribed arbitrarily can be seen from the following rough geometrical considerations: Imagine that a body τ is subdivided into small volume elements, which in the interior of τ may be assumed to have the form of cubes. The strain components e_{ij} are given on the faces of each cube, and the displacements u_i of those faces are to be calculated. If each individual cube is subjected to a deformation so that it becomes a parallelepiped, then it may happen that it is impossible to arrange the parallelepipeds to form a continuous distorted body τ'. The points that were coincident on the interfaces of the cubes may no longer coincide on the interfaces of the parallelepipeds. In fact, there may even be gaps between the pairs of initially coincident points. The requirements of continuity and single-valuedness imposed on the components of displacement place some restrictions on the choice of the strain components e_{ij} if the differential equations (10.1) are to possess solutions.

Let $P^0(x_1^0, x_2^0, x_3^0)$ be some point of a simply connected region[2] τ, at

[1] Cf. formulas (3.5).

[2] A region of space is said to be simply connected if every closed curve drawn in the region can be shrunk to a point, by continuous deformation, without passing out of the

which the displacements $u_j^0(x_1^0, x_2^0, x_3^0)$ and the components of rotation $\omega_{ij}^0(x_1^0, x_2^0, x_3^0)$ are known. We determine the displacements u_j at any other point $P'(x_1', x_2', x_3')$ in terms of the known functions[1] e_{ij} by means of a line integral over a simple[2] continuous curve C joining the points P^0 and P':

$$\begin{aligned} u_j(x_1', x_2', x_3') &= u_j^0 + \int_{P^0}^{P'} du_j = u_j^0 + \int_{P^0}^{P'} u_{j,k}\, dx_k \\ &= u_j^0 + \int_{P^0}^{P'} e_{jk}\, dx_k + \int_{P^0}^{P'} \omega_{jk}\, dx_k, \end{aligned}$$

where the last step comes from the definition (7.5). An integration by parts yields

$$\begin{aligned} \int_{P^0}^{P'} \omega_{jk}\, dx_k &\equiv \int_{P^0}^{P'} \omega_{jk}\, d(x_k - x_k') \\ &= (x_k' - x_k^0)\omega_{jk}^0 + \int_{P^0}^{P'} (x_k' - x_k)\omega_{jk,l}\, dx_l, \end{aligned}$$

and hence

$$u_j(x_1', x_2', x_3') = u_j^0 + (x_k' - x_k^0)\omega_{jk}^0 + \int_{P^0}^{P'} [e_{jl} + (x_k' - x_k)\omega_{jk,l}]\, dx_l. \tag{10.4}$$

We express the derivatives of the components of rotation $\omega_{jk,l}$ in terms of the known functions e_{ij} by using the definitions (7.5) and writing

$$\begin{aligned} \omega_{jk,l} &= \frac{\partial}{\partial x_l} \frac{1}{2} (u_{j,k} - u_{k,j}) \\ &= \frac{1}{2} (u_{j,kl} - u_{k,jl}) + \frac{1}{2} (u_{l,jk} - u_{l,jk}) \\ &= \frac{\partial}{\partial x_k} \frac{1}{2} (u_{l,j} + u_{j,l}) - \frac{\partial}{\partial x_j} \frac{1}{2} (u_{k,l} + u_{l,k}), \end{aligned}$$

where the continuity of the mixed derivatives has been used. It follows from the preceding equation that

$$\omega_{jk,l} = e_{lj,k} - e_{kl,j}. \tag{10.5}$$

When (10.5) is inserted in (10.4), it is seen that the determination of the displacements u_i at any point (x) has now been reduced to a quadrature,

$$u_j(x_1', x_2', x_3') = u_j^0 + (x_k' - x_k^0)\omega_{jk}^0 + \int_{P^0}^{P'} U_{jl}\, dx_l, \tag{10.6}$$

where the integrand

$$U_{jl} = e_{jl} + (x_k' - x_k)(e_{lj,k} - e_{kl,j}) \tag{10.7}$$

is a known function.

Inasmuch as the displacements u_i must be independent of the path of integration, the integrands $U_{jl}\, dx_l$ must be exact differentials. Hence, applying a necessary and sufficient condition that the integrands in

boundaries of the region. Thus the region between two concentric spheres is simply connected, but the interior of an anchor ring (torus) is not.

[1] The functions e_{ij} are assumed to be of class C^3 (see Sec. 7).

[2] We use the term *simple* curve in the sense of *rectifiable* curve.

(10.6) be exact differentials, namely

$$U_{ji,l} - U_{jl,i} = 0,$$

we have

$$(10.8)\quad e_{ji,l} - \delta_{kl}(e_{ij,k} - e_{ki,j}) - e_{jl,i} + \delta_{ki}(e_{lj,k} - e_{kl,j}) + (x_k' - x_k)(e_{ij,kl} - e_{ki,jl} - e_{lj,ki} + e_{kl,ji}) = 0.$$

The first line of (10.8) vanishes identically, and since this equation must be true for an arbitrary choice of $x_k' - x_k$, it follows that

$$(10.9)\qquad e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$$

The system (10.9) consists of $3^4 = 81$ equations, but some of these are identically satisfied, and some are repetitions because of the symmetry in indices ij and kl. A little reflection will show that only 6 of the 81 equations (10.9) are essential, and when these are written out in unabridged notation, one has

$$(10.10)\qquad \begin{cases} \dfrac{\partial^2 e_{xx}}{\partial y\,\partial z} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial e_{yz}}{\partial x} + \dfrac{\partial e_{zx}}{\partial y} + \dfrac{\partial e_{xy}}{\partial z}\right), \\ \dfrac{\partial^2 e_{yy}}{\partial z\,\partial x} = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial e_{zx}}{\partial y} + \dfrac{\partial e_{xy}}{\partial z} + \dfrac{\partial e_{yz}}{\partial x}\right), \\ \dfrac{\partial^2 e_{zz}}{\partial x\,\partial y} = \dfrac{\partial}{\partial z}\left(-\dfrac{\partial e_{xy}}{\partial z} + \dfrac{\partial e_{yz}}{\partial x} + \dfrac{\partial e_{zx}}{\partial y}\right), \\ 2\dfrac{\partial^2 e_{xy}}{\partial x\,\partial y} = \dfrac{\partial^2 e_{xx}}{\partial y^2} + \dfrac{\partial^2 e_{yy}}{\partial x^2}, \\ 2\dfrac{\partial^2 e_{yz}}{\partial y\,\partial z} = \dfrac{\partial^2 e_{yy}}{\partial z^2} + \dfrac{\partial^2 e_{zz}}{\partial y^2}, \\ 2\dfrac{\partial^2 e_{zx}}{\partial z\,\partial x} = \dfrac{\partial^2 e_{zz}}{\partial x^2} + \dfrac{\partial^2 e_{xx}}{\partial z^2}. \end{cases}$$

The six equations (10.10) ensuring the continuity of displacements are known as the *equations of compatibility* and were obtained by Saint-Venant in 1860, in a way different from that outlined above.[1]

One can verify by direct substitution that the displacements given by (10.6) actually satisfy the differential equations (10.1). We have already seen that the displacements specified by (10.3) contribute nothing to the strain components e_{ij}. Equation (10.6) shows that, conversely, if the e_{ij} vanish identically, then the resulting solutions are given by Eqs. (10.3), and these obviously represent a rigid body motion.

[1] The essential features of the method of derivation of the compatibility equations given above are due to E. Cesaro, *Rendiconto dell' accademia delle scienze fisiche e matematiche* (*Società reale di Napoli*) (1906). See also a memoir by V. Volterra, "L'Equilibre des corps élastiques," *Annales de l'école normale supérieure*, vol. 24 (1907). The necessity of conditions (10.10) can be proved easily. See Prob. 6 at the end of this chapter.

If the region of integration τ is multiply connected, then the functions u_i may turn out to be multiple-valued. As is well known, a multiply connected region may be reduced to a simply connected one, provided suitable barriers or crosscuts are introduced. In this case, the displacements u_i will be single-valued functions of the coordinates when evaluated by means of a line integral taken along any curve C that does not pass through one of these crosscuts. If the curve C does intersect the crosscut, then, to ensure that the u_i be single-valued, we must demand in addition to the satisfaction of the compatibility relations that the limiting values of $u_i(x_1, x_2, x_3)$ be the same when the cut is approached from either side.

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11. Finite Deformations. The preceding sections of this chapter contain all the principal results of the classical theory of infinitesimal strain. It is clear from the general discussion of the affine transformation in Sec. 2 that the linearization of the equations appearing there led to a consideration of infinitesimal transformations that permits the application of the principle of superposition of effects. Many technically important problems in elasticity, including those of buckling and stability, call for a consideration of finite deformations, that is, deformations in which the displacements u together with their derivatives are no longer small. This section contains only a brief introduction to the theory of finite strains and provides an admirable illustration of the complications that appear in the development of a theory when the fundamental equations become nonlinear.

There are two modes of description of the deformation of a continuous medium, the *Lagrangian* and the *Eulerian*. The Lagrangian description employs the coordinates a_i of a typical particle in the initial state as the independent variables, while in the case of Eulerian coordinates the independent variables are the coordinates x_i of a material point in the deformed state.

In the preceding sections, we have used the Lagrangian viewpoint as the natural means of describing the deformation of the neighborhood of the point (a_1, a_2, a_3). When we come, in the next chapter, to the discussion of the stresses acting throughout the medium, we shall find that these stresses must satisfy equilibrium conditions in the deformed body and hence Eulerian coordinates are indicated. In this section, we shall describe the deformation from both points of view, and we shall see that when the deformation is infinitesimal (that is, when products of the

derivatives of the displacements can be neglected), these two viewpoints, Lagrangian and Eulerian, coalesce, and we need make no distinction between them.

Consider an aggregate of particles in a continuous medium that lie along a curve C in the undeformed state. Just as in the preceding sections, it will be convenient to use the same reference frame for the location of particles in the deformed and undeformed states. Let the coordinates of a particle lying on a curve C_0 (before deformation) be denoted by (a_1, a_2, a_3), and let the coordinates of the same particle after deformation (now lying on some curve C) be (x_1, x_2, x_3). Then the elements of arc of the curves C_0 and C are given, respectively, by

$$ds_0^2 = da_1^2 + da_2^2 + da_3^2 = da_i\, da_i, \tag{11.1}$$

and

$$ds^2 = dx_i\, dx_i. \tag{11.2}$$

We consider first the Eulerian description of the strain and write $a_i = a_i(x_1, x_2, x_3)$. Substituting[1] $da_i = a_{i,j}\, dx_j = a_{i,k}\, dx_k$ in (11.1) yields

$$ds_0^2 = a_{i,j}a_{i,k}\, dx_j\, dx_k,$$

while $ds^2 = dx_i\, dx_i = \delta_{jk}\, dx_j\, dx_k$. It is evident that the equality of ds^2 and ds_0^2 for all curves C_0 is the necessary and sufficient condition that the transformation $a_i = a_i(x_1, x_2, x_3)$ be one of rigid body motion; hence we shall take the difference $ds^2 - ds_0^2$ as the measure of the strain and write

$$ds^2 - ds_0^2 = 2\eta_{jk}\, dx_j\, dx_k. \tag{11.3}$$

From the expressions given above for ds^2 and ds_0^2, we get

$$2\eta_{jk} = \delta_{jk} - a_{i,j}a_{i,k}$$

We now write the strains η_{jk} in terms of the displacement components $u_i = x_i - a_i$. Since $a_i = x_i - u_i$, we have

$$\begin{aligned} a_{i,j}a_{i,k} &= (\delta_{ij} - u_{i,j})(\delta_{ik} - u_{i,k}) \\ &= \delta_{jk} - u_{j,k} - u_{k,j} + u_{i,j}u_{i,k} \end{aligned}$$

and hence

$$2\eta_{jk} = u_{j,k} + u_{k,j} - u_{i,j}u_{i,k}. \tag{11.4}$$

The functions η_{jk} are called the *Eulerian strain components.*

If, on the other hand, Lagrangian coordinates are used, so that the a_i are regarded as the independent variables and the equations of transformation are of the form $x_i = x_i(a_1, a_2, a_3)$, then we can write $dx_i = x_{i,j}\, da_j$ and

$$ds^2 = dx_i\, dx_i = x_{i,j}x_{i,k}\, da_j\, da_k,$$

[1] The notation $a_{i,j} = \partial a_i/\partial x_j$ denotes differentiation with respect to the jth independent variable, which in this case is x_j.

while $ds_0^2 = da_j\, da_j = \delta_{jk}\, da_j\, da_k$. The Lagrangian components of strain ϵ_{jk} are defined by

$$ds^2 - ds_0^2 = 2\epsilon_{jk}\, da_j\, da_k, \tag{11.5}$$

and since $x_i = a_i + u_i$, we have

$$\begin{aligned} x_{i,j}x_{i,k} &= (\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) \\ &= \delta_{jk} + u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}, \end{aligned}$$

and

$$\begin{aligned} ds^2 - ds_0^2 &= 2\epsilon_{jk}\, da_j\, da_k \\ &= (u_{j,k} + u_{k,j} + u_{i,j}u_{i,k})\, da_j\, da_k, \end{aligned}$$

with

$$2\epsilon_{jk} = u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}. \tag{11.6}$$

In order to exhibit the fact that the differentiation in (11.4) is carried out with respect to the variables x_i, while in (11.6) the a_i are regarded as the independent variables, we write out the typical expressions for η_{ij} and ϵ_{ij} in unabridged notation,

$$\begin{aligned} \eta_{xx} &= \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \\ \epsilon_{aa} &= \frac{\partial u}{\partial a} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial a}\right)^2 + \left(\frac{\partial w}{\partial a}\right)^2\right], \\ 2\eta_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \\ 2\epsilon_{ab} &= \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} + \left(\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right). \end{aligned}$$

It was shown in Sec. 4 that e_{11}, e_{22}, and e_{33} can be interpreted as extensions of vectors originally parallel to the coordinate axes, while e_{12}, e_{23}, and e_{31} represent shears or changes of angle between vectors originally at right angles. When the strain components are large, however, it is no longer possible to give simple geometrical interpretations of the functions ϵ_{ij} and η_{ij}.

Consider a line element with $ds_0 = da_1$, $da_2 = da_3 = 0$, and define the extension E_1 of this element by $E_1 = (ds - ds_0)/ds_0$, or

$$ds = (1 + E_1)\, ds_0. \tag{11.7}$$

From (11.5) we have

$$ds^2 - ds_0^2 = 2\epsilon_{jk}\, da_j\, da_k = 2\epsilon_{11}\, da_1^2,$$

and the insertion of (11.7) in this expression yields

$$(1 + E_1)^2 = 1 + 2\epsilon_{11},$$

or

$$E_1 = \sqrt{1 + 2\epsilon_{11}} - 1. \tag{11.8}$$

When ϵ_{11} is small, this reduces to

$$E_1 \doteq \epsilon_{11},$$

as was shown in the discussion of infinitesimal strain in Sec. 4.

Consider next two line elements, $ds_0 = da_2$, $da_1 = da_3 = 0$, and $d\bar{s}_0 = d\bar{a}_3$, $d\bar{a}_1 = d\bar{a}_2 = 0$, that lie initially along the a_2- and a_3-axes. Let θ denote the angle between the corresponding deformed elements dx_i and $d\bar{x}_i$, of lengths ds and $d\bar{s}$, respectively. Then

$$\begin{aligned} ds\, d\bar{s} \cos\theta &= dx_i\, d\bar{x}_i = x_{i,\alpha} x_{i,\beta}\, da_\alpha\, d\bar{a}_\beta \\ &= x_{i,2} x_{i,3}\, da_2\, d\bar{a}_3 = 2\epsilon_{23}\, da_2\, d\bar{a}_3. \end{aligned}$$

If $\alpha_{23} = \pi/2 - \theta$ denotes the change in the right angle between the line elements in the initial state, then we have

$$\sin\alpha_{23} = 2\epsilon_{23} \frac{da_2\, d\bar{a}_3}{ds\, d\bar{s}},$$

and by (11.7) and (11.8)

$$\sin\alpha_{23} = \frac{2\epsilon_{23}}{\sqrt{1 + 2\epsilon_{22}}\,\sqrt{1 + 2\epsilon_{33}}} \tag{11.9}$$

Again, if the strains ϵ_{ij} are so small that their products can be neglected, then

$$\alpha_{23} \doteq 2\epsilon_{23},$$

as was seen in Sec. 4.

If the displacements and their derivatives are small, then it is immaterial whether the derivatives of the displacements are calculated at the position of a point before or after deformation. In this case, we may neglect the nonlinear terms in the partial derivatives in (11.4) and (11.6) and reduce both sets of formulas to Eqs. (7.5), which were obtained for an infinitesimal transformation. Unless a statement to the contrary is made, we shall deal with infinitesimal strain and shall write

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

The ratio of the volume element in the strained state to the corresponding element of volume in the unstrained state is equal to the functional determinant

$$\frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)} = \left|\frac{\partial x_i}{\partial a_j}\right| = \left|\frac{\partial(a_i + u_i)}{\partial a_j}\right| = |\delta_{ij} + u_{i,j}|. \tag{11.10}$$

If this is denoted by $1 + \Delta$, then Δ is the change of volume per unit volume at a point and is called the *cubical dilatation*. It is obvious that for small strains

$$\Delta \doteq u_{i,i} = e_{11} + e_{22} + e_{33} = \vartheta.$$

As was done in the infinitesimal case, Eqs. (11.4) [or (11.6)] can be looked upon as the differential equations for the determination of the func-

tions u_i, where the components of strain η_{ij} (or ϵ_{ij}) are prescribed functions. Since these equations are nonlinear, the problem of integration is much more involved. While it is not difficult to formulate the conditions on the function η_{ij} (or ϵ_{ij}) if the set of Eqs. (11.4) [or (11.6)] is to possess a solution with suitable properties, this will not be pursued here.[1]

In concluding this brief treatment of finite strains, it should be emphasized that the transformations of finite homogeneous strain are not in general commutative and that the simple superposition of effects is no longer applicable to finite deformations. These facts are responsible, in the main, for the absence of satisfactory solutions for all but the simplest problems, such as homogeneous strain, simple tension, and torsion of an elliptical cylinder, which become trivial when the equations are linearized.

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PROBLEMS

1. Show that a tensor α_{ij} can be decomposed into a symmetric tensor $e_{ij} = e_{ji}$ and a skew-symmetric tensor $\omega_{ij} = -\omega_{ji}$ in one, and only one, way. *Hint:* Assume that the decomposition can be made in two ways:

$$\alpha_{ij} = e_{ij} + \omega_{ij} = \bar{e}_{ij} + \bar{\omega}_{ij}$$

2. From $\delta A_i = e_{ij}A_j$, find $\delta\mathbf{A}$ and δA for a vector lying initially along the x-axis, that is, $\mathbf{A} = \mathbf{i}A$, and justify the statement of Sec. 4 that in this case $\dfrac{\delta A}{A} = e_{xx}$. Does $\delta\mathbf{A}$ lie along the x-axis?

3. Derive Eq. (5.10) from (5.8) and (5.9).

4. Derive Eq. (8.2) by using the invariance of the strain quadric and the equations of transformation $x'_i = l_{ij}x_j$.

5. Show that the inverse of the transformation (5.6) is $x'_i = l_{i\alpha}x_\alpha$.

6. Show by differentiation of the strain components

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

[1] A detailed discussion of the basic equations of nonlinear theory is contained in I. S. Sokolnikoff, Tensor Analysis (1951), pp. 290–319. See also F. D. Murnaghan, Finite Deformations of an Elastic Solid (1951); V. V. Novozhilov, Foundations of Nonlinear Theory of Elasticity (1953); A. E. Green and W. Zerna, Theoretical Elasticity (1954). A critical appraisal of the literature on nonlinear mechanics of continua was made by C. A. Truesdell, *Journal of Rational Mechanics and Analysis*, vol. 1 (1952), pp. 125–300; vol. 2 (1953), pp. 593–616.

that the equations of compatibility are necessary conditions for the existence of continuous single-valued displacements. *Hint:*

$$e_{ij,kl} = \tfrac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

and, by interchange of i, k and likewise of j, l, $e_{kl,ij} = \tfrac{1}{2}(u_{k,lij} + u_{l,kij})$. Add these, interchange j and k, and show that the compatibility equations (10.9) result.

7. Show that the shear strain e_{23}, for example, can be interpreted as the extension of the diagonal OQ of the rectangle $OPQR$ (Fig. 4), provided the rectangle is a square.

CHAPTER 2

ANALYSIS OF STRESS

12. Body and Surface Forces. Consider a continuous medium, the points of which are referred to a rectangular cartesian system of axes, and let τ represent the region occupied by the medium and $\Delta\tau$ an element of volume of τ. In analyzing the forces acting on the volume element $\Delta\tau$, it is necessary to take into account two types of forces:

1. Body (or volume) forces; that is, the forces which are proportional to the mass contained in the volume element $\Delta\tau$;

2. Surface forces, which act on the surface $\Delta\sigma$ of the volume element $\Delta\tau$.

It will be assumed throughout this discussion that the volume forces are continuous functions of class C^1 and the surface forces are piecewise continuous functions of the coordinates (x_1, x_2, x_3) of the points of the medium.

As a typical example of body force, one can take the force of gravity, $\rho g\,\Delta\tau$, acting on the mass contained in the volume element $\Delta\tau$ of the medium whose density is ρ, and where g is the gravitational acceleration. An example of surface force is the tension acting on any horizontal section of a steel rod suspended vertically. Thus, if one imagines that the rod is cut by a horizontal plane into two parts, the upper and the lower, then the action of the weight of the lower part of the rod is transmitted to the upper part across the surface of the cut. A hydrostatic pressure on the surface of a submerged solid body provides another example of surface force.

Let the vector $\mathbf{F} = \mathbf{e}_i F_i$ represent the body force per unit volume of the medium. The resultant $\mathbf{R} = \mathbf{e}_i R_i$ of the body forces $\mathbf{F}$ can be represented as the volume integral $\mathbf{R} = \int_\tau \mathbf{F}\, d\tau$, or

$$R_i = \int_\tau F_i\, d\tau. \tag{12.1}$$

The resultant moment $\mathbf{M} = \mathbf{e}_i M_i$ due to the body force $\mathbf{F}$ can be written as the integral over τ of the vector product of the position vector $\mathbf{r} = \mathbf{e}_i x_i$ and the force vector $\mathbf{F}$; that is, $\mathbf{M} = \int_\tau (\mathbf{r} \times \mathbf{F})\, d\tau$, or[1]

$$M_i = \int_\tau e_{ijk} x_j F_k\, d\tau. \tag{12.2}$$

[1] The alternating tensor e_{ijk} is defined to be zero if any two subscripts are equal,

Consider next an element $\Delta\sigma$ of a surface situated either in the interior or on the boundary of the medium, and let the force acting on the element $\Delta\sigma$ be $\bar{\mathbf{T}}\,\Delta\sigma$. Because of the assumed continuity of forces, we have $\lim_{\Delta\sigma\to 0} \frac{\bar{\mathbf{T}}\,\Delta\sigma}{\Delta\sigma} = \mathbf{T}(x_1, x_2, x_3)$, where the vector $\mathbf{T}$ represents the surface force per unit area of the surface acting at the point (x_i) and is called the *stress vector*.

If $\Delta\sigma$ is a surface element in the interior of the medium, we agree to call one side of the element $\Delta\sigma$ positive and the other side negative; the force $\mathbf{T}\,\Delta\sigma$ will be thought to represent the action of the part of the body lying on the positive side upon the part on the negative side. Hence, if a unit normal $\boldsymbol{\nu}$ is drawn (Fig. 6) to the surface element $\Delta\sigma$ so that it points in the direction of the positive side, then the action of the matter lying on the negative side of the normal upon that on the positive is $-\mathbf{T}\,\Delta\sigma$. This latter statement follows directly from Newton's third law of motion.

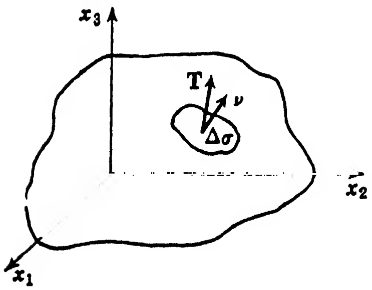

FIG 6

It is obvious that the surface forces developed in a solid body are of much more complicated character than those in an ideal fluid at rest, since they need not be normal to the elements of surface. Clearly, the surface forces depend not only on the position of the surface element but on its orientation as well. In order to bring into explicit evidence the dependence of the stress $\mathbf{T}$ on the orientation of the element of surface, the stress vector will be written as $\overset{\nu}{\mathbf{T}}$. It must be noted that in general $\overset{\nu}{\mathbf{T}}$ is not in the direction $\boldsymbol{\nu}$.

13. Stress Tensor. It will be shown in this section that the state of stress at any point of the medium is completely characterized by the specification of nine quantities, called the *components of stress tensor*. The introduction of these quantities in elasticity is due to Cauchy.

Let $P(x)$ be any point in the medium and $\overset{\nu}{\mathbf{T}}$ the stress vector acting on an element of surface at P, with the normal $\boldsymbol{\nu}$. Draw through P three planar elements parallel to the coordinate planes, and pass the fourth plane ABC (Fig. 7) normal to $\boldsymbol{\nu}$ and at a small distance h from P.

Denote by $\overset{1}{\mathbf{T}}$ the stress acting on the face PBC of the small tetrahedron

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$+1$ if i, j, k is a cyclic permutation of 1, 2, 3, and -1 if i, j, k is a cyclic permutation of 1, 3, 2. We have, for example,

$$M_1 = \int_\tau (x_2F_3 - x_3F_2)\,d\tau.$$

$PABC$ and by $\overset{2}{\mathbf{T}}$ and $\overset{3}{\mathbf{T}}$ the stresses acting on the faces PAC and PAB, respectively. Thus $\overset{i}{\mathbf{T}}$ is the stress vector acting on a planar surface element normal to the x_i-axis. The resolution of the vector $\overset{i}{\mathbf{T}}$ into components along the coordinate axes gives $\overset{i}{\mathbf{T}} = \mathbf{e}_j \overset{i}{T}_j$. It will be convenient to write

$$\overset{i}{T}_j \equiv \tau_{ij}, \tag{13.1}$$

so that

$$\overset{i}{\mathbf{T}} = \mathbf{e}_j \tau_{ij}.$$

We shall show that the nine scalar quantities τ_{ij} are the components of a tensor, the *stress tensor*, and that the τ_{ij} serve to determine completely the

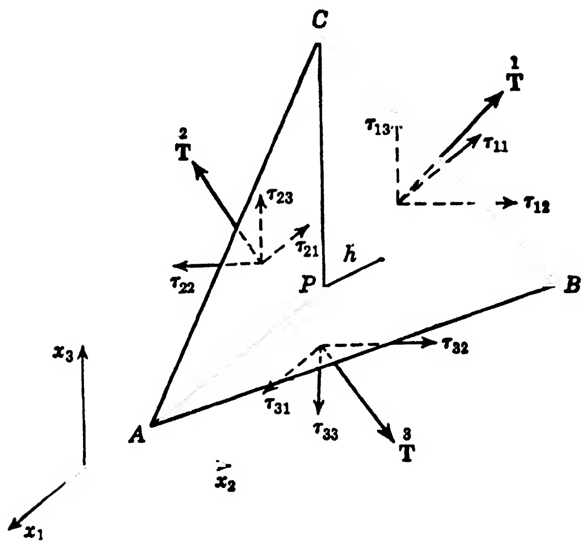

FIG. 7

state of stress at the point P. The stress vector $\overset{\nu}{\mathbf{T}}$ can then be calculated from the τ_{ij} for any orientation $\boldsymbol{\nu}$ of the surface element at P. The meaning of the subscripts in the components τ_{ij} should be carefully noted. Observe that in τ_{23}, for example, the first subscript, 2, indicates the *coordinate axis* normal to the element of area on which the stress $\overset{2}{\mathbf{T}}$ acts, while the second subscript, 3, indicates the direction of the component of this stress vector.

If the volume element is taken in the shape of a rectangular parallelepiped, with faces parallel to the coordinate planes, and $\overset{i}{\mathbf{T}}$ is the stress

vector acting on a face of the parallelepiped perpendicular to the x_i-axis, the components τ_{ij} are shown in Fig. 8. The convention in regard to the signs of the scalar quantities is the following: If one draws an exterior normal to a given face of the parallelepiped, then the components τ_{ij} are reckoned positive if the corresponding components of force act in the directions of increasing x_1, x_2, x_3 when the normal has the same sense as the positive direction of the axis to which the face is perpendicular; if, on the other hand, the exterior normal to a given face points in the direction opposite to that of the positive coordinate axis, then the positive values of the components τ_{ij} are associated with forces directed oppositely to the positive directions of coordinate axes. The arrows in Fig. 8 indicate vectors representing forces which, for positive values of the τ_{ij} are

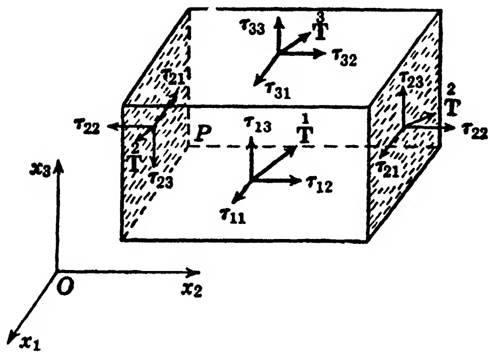

FIG. 8

exerted by the material exterior to the parallelepiped on the matter within it.

The components τ_{11}, τ_{22}, and τ_{33} are called the *normal components* of stress; the other components are called the *tangential*, or *shearing*, components. It follows from our convention concerning signs that the positive values of τ_{11}, τ_{22}, and τ_{33} are associated with tension and the negative ones with compression of the medium. In Fig. 8, the τ_{22}, for example, produces tension along the x_2-axis, while forces $\mathbf{e}_1\tau_{21}$ and $\mathbf{e}_3\tau_{23}$, lying in the shaded faces, give rise to shear.

We return now to our tetrahedral element in Fig. 7 and proceed to establish a connection between the τ_{ij} and the stress vector $\overset{\nu}{\mathbf{T}}$. Let the area of the face ABC be σ; then the face normal to the x_i-axis will have an area $\sigma_i = \sigma \cos (x_i, \nu) = \sigma\nu_i$. The equilibrium of the tetrahedral element $PABC$ requires the vanishing of the resultant force acting on the matter within $PABC$, and we proceed to calculate the x_i-component of this force.

Let $\overset{\nu}{T}_i$, τ_{ij}, and F_i be the values of the stress vector, stress tensor, and

body force at the point P; then, on account of the assumed continuity of the stress vector $\overset{\nu}{T}_i$, the x_i-component of the force acting on the face ABC of the tetrahedron is $(\overset{\nu}{T}_i + \epsilon_i)\sigma$, where $\lim_{h\to 0} \epsilon_i = 0$. The corresponding component of force due to stresses acting on the faces of areas σ_j is $(-\tau_{ji} + \epsilon_{ji})\sigma_j$, where $\lim_{h\to 0} \epsilon_{ij} = 0$ and the τ_{ij} are taken with the negative sign because the exterior normals to the faces of areas σ_j are directed oppositely to the direction of increasing x_j-coordinate. Finally, the contribution of the body force F_i to the x_i-component of the resultant force is $(F_i + \epsilon_i')\frac{1}{3}h\sigma$, where $\frac{1}{3}h\sigma = \Delta\tau$ is the volume of the element $PABC$ and $\lim_{h\to 0} \epsilon_i' = 0$. Thus, for equilibrium of the tetrahedron we must have

$$(\overset{\nu}{T}_i + \epsilon_i)\sigma + (-\tau_{ji} + \epsilon_{ji})\sigma_j + (F_i + \epsilon_i')\tfrac{1}{3}h\sigma = 0. \tag{13.2}$$

If in (13.2) we set $\sigma_j = \sigma\nu_j = \sigma \cos(x_j, \nu)$, divide through by σ, and pass to the limit as $h \to 0$, we get

$$\overset{\nu}{T}_i = \tau_{ji}\nu_j. \tag{13.3}$$

It is clear from (13.3) that, having specified the components of the stress tensor τ_{ij} at any point $P(x)$ of the medium, one can calculate the stress $\overset{\nu}{\mathbf{T}}$ on any element of surface whose orientation is determined by $\boldsymbol{\nu}$ and which passes through the point in question.

14. Note on Notation and Units. There is a deplorable lack of uniformity of notation and terminology in use by various writers on the theory of elasticity. Many British writers have adopted the notation for the components of the stress tensor introduced by Kirchhoff and write

$$\tau_{11} \equiv X_x, \qquad \tau_{12} \equiv Y_x, \qquad \tau_{21} \equiv X_y, \qquad \ldots, \qquad \tau_{33} \equiv Z_z.$$

Most American writers (as well as many Russian and German authors) write

$$\sigma_x \equiv \tau_{11}, \qquad \sigma_y = \tau_{22}, \qquad \sigma_z = \tau_{33}$$

for the *normal stresses* and denote the remaining six *tangential*, or *shear*, stresses τ_{12}, τ_{31}, etc., by τ_{xy}, τ_{zx}, etc.

The notation

$$\tau_{11} = \widehat{xx}, \qquad \tau_{12} = \widehat{xy}, \qquad \ldots, \qquad \tau_{33} = \widehat{zz}$$

has been suggested by K. Pearson and is quite convenient when one contemplates using orthogonal curvilinear coordinates. When it appears desirable to exhibit the dependence of the components of the stress tensor on the x, y, z-system of coordinates, we shall write $\tau_{11} = \tau_{xx}$, $\tau_{22} = \tau_{yy}$, $\tau_{23} = \tau_{yz}$, $\tau_{21} = \tau_{yx}$, etc. In this notation, formulas (13.3) read:

$$\overset{\nu}{T}_x = \tau_{xx} \cos(x, \nu) + \tau_{yx} \cos(y, \nu) + \tau_{zx} \cos(z, \nu),$$
$$\overset{\nu}{T}_y = \tau_{xy} \cos(x, \nu) + \tau_{yy} \cos(y, \nu) + \tau_{zy} \cos(z, \nu),$$
$$\overset{\nu}{T}_z = \tau_{xz} \cos(x, \nu) + \tau_{yz} \cos(y, \nu) + \tau_{zz} \cos(z, \nu).$$

From the definition of the stress vector, it follows that the stress vector **T** has the dimensions of

$$\frac{\text{force}}{\text{area}} \quad \text{or} \quad \frac{M}{LT^2}.$$

In the cgs system, the stress is measured in dynes per square centimeter, while in English practical units it is measured in pounds per square inch or in tons per square inch.

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A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Secs. 41–48, pp. 74–80.
E. Trefftz: Handbuch der Physik, Verlag von Julius Springer, Berlin, vol. 6, Secs. 1–3.
R. V. Southwell: Theory of Elasticity for Engineers and Physicists, Oxford University Press, New York, Secs. 258–268, pp. 259–264.

15. Equations of Equilibrium. Consider a continuous medium every portion of which, contained within the volume τ and bounded by the closed surface σ, is in equilibrium. For equilibrium, the resultant force acting on the matter within τ must vanish, and we calculate now the x_i-component of this force.

Both body forces **F** and surface forces $\overset{\nu}{\mathbf{T}}$ must be considered; the condition of equilibrium of forces requires that

$$\int_\tau F_i \, d\tau + \int_\sigma \overset{\nu}{T}_i \, d\sigma = 0,$$

or, making use of (13.3),

$$\int_\tau F_i \, d\tau + \int_\sigma \tau_{ji}\nu_j \, d\sigma = 0. \tag{15.1}$$

Now if it is assumed that the functions τ_{ji} and their first partial derivatives $\tau_{ji,k} \equiv \dfrac{\partial \tau_{ji}}{\partial x_k}$ are continuous and single-valued in τ, then the Divergence Theorem[1] can be applied to the surface integral in (15.1) to yield

$$\int_\sigma \tau_{ji}\nu_j \, d\sigma = \int_\tau \tau_{ji,j} \, d\tau,$$

and (15.1) takes the form

$$\int_\tau (F_i + \tau_{ji,j}) \, d\tau = 0. \tag{15.2}$$

[1] See p. 22.

Since the region of integration τ is arbitrary (every part of the medium is in equilibrium!) and since the integrand of (15.2) is continuous, it follows that the latter must vanish identically. Thus, at every interior point in τ, we have,

$$\tau_{ji,j} = -F_i, \tag{15.3}$$

or, when written out in full in the notation explained in Sec. 14,

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= -F_x,\\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= -F_y,\\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= -F_z.
\end{aligned}$$

Consider next the consequence of the vanishing of the resultant moment, which is produced by body and surface forces. Recalling the formula (12.2), the condition that the resultant moment due to body and surface forces vanishes can be written as

$$M_i = \int_\tau e_{ijk}x_jF_k\,d\tau + \int_\sigma e_{ijk}x_j\overset{\nu}{T}_k\,d\sigma = 0. \tag{15.4}$$

With the aid of (13.3) and the Divergence Theorem, the surface integral in (15.4) can be transformed as follows:

$$\int_\sigma e_{ijk}x_j\overset{\nu}{T}_k\,d\sigma = \int_\sigma e_{ijk}x_j\tau_{lk}\nu_l\,d\sigma = \int_\tau (e_{ijk}x_j\tau_{lk})_{,l}\,d\tau = \int_\tau e_{ijk}(x_j\tau_{lk,l} + \delta_{jl}\tau_{lk})\,d\tau.$$

But $\delta_{jl}\tau_{lk} = \tau_{jk}$, and from equilibrium equations (15.3),

$$\tau_{lk,l} = -F_k,$$

so that the foregoing expression gives

$$\int_\sigma e_{ijk}x_j\overset{\nu}{T}_k\,d\sigma = \int_\tau e_{ijk}(-x_jF_k + \tau_{jk})\,d\tau.$$

Accordingly, Eq. (15.4) becomes

$$\int_\tau e_{ijk}\tau_{jk}\,d\tau = 0,$$

and since the integrand is continuous and the volume τ is arbitrary, we must have

$$e_{ijk}\tau_{jk} = 0. \tag{15.5}$$

Equation (15.5) can be expanded to give, for example,

$$e_{123}\tau_{23} + e_{132}\tau_{32} = 0,$$

or since $e_{123} = -e_{132} = +1$, $\tau_{23} = \tau_{32}$; one obtains similarly

$$\tau_{12} = \tau_{21} \qquad \text{and} \qquad \tau_{13} = \tau_{31}.$$

In short,

$$\tau_{ij} = \tau_{ji}; \tag{15.6}$$

that is, the stress tensor is symmetric. The symmetry of the components of the stress tensor allows us to write (15.3) as $\tau_{ij,j} = -F_i$ or, recalling the definition (13.1), $\overset{i}{T}_{j,j} = -F_i$; that is,

$$\operatorname{div} \overset{i}{\mathbf{T}} = -F_i. \tag{15.7}$$

Since the nine stress components τ_{ij} are bound by the three relations (15.6), we see that the state of stress at any point is completely characterized by the six quantities τ_{11}, τ_{22}, τ_{33}, $\tau_{12} = \tau_{21}$, $\tau_{23} = \tau_{32}$, $\tau_{31} = \tau_{13}$.

It follows from the foregoing that the six components of stress must satisfy the three partial differential equations (15.3),

$$\tau_{ij,j} = -F_i,$$

in the interior of the medium and that on the surface bounding the medium they must satisfy the three boundary conditions (13.3),

$$\tau_{ij}\nu_j = \overset{\nu}{T}_i,$$

stemming from the equilibrium conditions on the surface. In these equations the functions F_i and $\overset{\nu}{T}_i$ are prescribed. It is clear that these equations are not sufficient for the complete determination of the state of stress, and one must have further information concerning the constitution of the body in order that the solution of Eqs. (15.3) be unique.

PROBLEM

Consider an elastic solid acted upon by body forces that exert moments M_i per unit volume (as in the case of a polarized dielectric solid under the action of an electric field). Show that in this case, Eq. (15.5) must be replaced by

$$e_{ijk}\tau_{jk} + M_i = 0.$$

What can be said in this case about the symmetry of the stress components? See in this connection Eric Reissner, "Note on the Theorem of the Symmetry of the Stress Tensor," *Journal of Mathematics and Physics*, vol. 23 (1944), pp. 192–194.

16. Transformation of Coordinates. The symmetry of the shear components of the stress tensor ($\tau_{ij} = \tau_{ji}$) established in Sec. 15 is but a special case of a general theorem that will prove useful in establishing the laws of transformation of the components of the stress tensor under an orthogonal transformation of coordinate axes. We prove the following theorem:

THEOREM: *Let the surface elements $\Delta\sigma$ and $\Delta\sigma'$, with unit normals $\mathbf{v}$ and $\mathbf{v}'$, pass through the point P; then the component of the stress vector $\overset{\nu}{\mathbf{T}}$ (acting on $\Delta\sigma$) in the direction of $\mathbf{v}'$ is equal to the component of the stress vector $\overset{\nu'}{\mathbf{T}}$ (acting on $\Delta\sigma'$) in the direction of the normal $\mathbf{v}$.*

In vector notation, the theorem reads:

$$\overset{\nu'}{\mathbf{T}} \cdot \mathbf{v} = \overset{\nu}{\mathbf{T}} \cdot \mathbf{v}'. \tag{16.1}$$

The proof of the theorem employs only Eq. (13.3) and the symmetry of the stress components. For

$$\begin{aligned} \overset{\nu'}{\mathbf{T}} \cdot \mathbf{v} &= \overset{\nu'}{T}_i \nu_i = \tau_{ji}\nu_j'\nu_i \\ &= (\tau_{ij}\nu_i)\nu_j' = \overset{\nu}{T}_j \nu_j' = \overset{\nu}{\mathbf{T}} \cdot \mathbf{v}', \end{aligned}$$

and the theorem is proved.

The formula

$$\overset{\nu'}{\mathbf{T}} \cdot \mathbf{v} = \tau_{ij}\nu_i'\nu_j, \tag{16.2}$$

obtained above, enables one to compute the component in any direction $\mathbf{v}$ of the stress vector acting on any given element with normal $\mathbf{v}'$. It will be used now to derive the formulas of transformation of the components of the stress tensor τ_{ij} when the latter is referred to a new coordinate system x_i' obtained from the old by a rotation of axes.

Since the stress component $\tau_{\alpha\beta}'$ (referred to the x_i'-system of coordinates) is the projection on the x_β'-axis of the stress vector acting on a surface element normal to the x_α'-axis, we can write

$$\tau_{\alpha\beta}' = \overset{\alpha}{T}_\beta = \overset{\nu'}{\mathbf{T}} \cdot \mathbf{v}, \tag{16.3}$$

where $\mathbf{v}'$ is parallel to the x_α'-axis and $\mathbf{v}$ is parallel to the x_β'-axis. Thus, (16.2) and (16.3) give

$$\tau_{\alpha\beta}' = \tau_{ij}\nu_i'\nu_j.$$

Then

$$\begin{aligned} \nu_i' &= \cos(x_\alpha', x_i) \equiv l_{\alpha i}, \\ \nu_j &= \cos(x_\beta', x_j) \equiv l_{\beta j}, \end{aligned}$$

and we get

$$\tau_{\alpha\beta}' = l_{\alpha i} l_{\beta j} \tau_{ij}. \tag{16.4}$$

The equations of transformation from the τ_{ij}' to $\tau_{\alpha\beta}$ have the form

$$\tau_{\alpha\beta} = l_{i\alpha} l_{j\beta} \tau_{ij}'. \tag{16.5}$$

The law of transformation (16.4) is identical with that deduced in Sec. 5 for the transformation of the strain tensor and exhibits the tensor character of the quantities τ_{ij}. Indeed, these equations represent the trans-

formation under rotation of axes of any tensor of rank 2 that is referred to a cartesian coordinate system.

If we set $\beta = \alpha$ in (16.4) and use the orthogonality relations

$$l_{\alpha i} l_{\alpha j} = l_{i\alpha} l_{j\alpha} = \delta_{ij},$$

we see that

$$\tau'_{\alpha\alpha} = l_{\alpha i} l_{\alpha j} \tau_{ij} = \delta_{ij} \tau_{ij} = \tau_{ii}$$

or

$$\tau'_{11} + \tau'_{22} + \tau'_{33} = \tau_{11} + \tau_{22} + \tau_{33}.$$

This result can be stated as a theorem.

THEOREM: *The expression*

$$\Theta = \tau_{11} + \tau_{22} + \tau_{33}$$

is invariant relative to an orthogonal transformation of coordinates.

This theorem states, in effect, that, whatever be the orientation of three mutually orthogonal planes passing through a given point, the sum of the normal stresses is independent of the orientation of the planes.

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A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Sec. 48, p. 80; Secs. 54–55, 56a, 56b, pp. 84–87.

R. V. Southwell: Theory of Elasticity for Engineers and Physicists, Oxford University Press, New York, Secs. 270–275, pp. 265–268.

E. Trefftz: Handbuch der Physik, Verlag von Julius Springer, Berlin, vol. 6, Secs. 4, 5, 7.

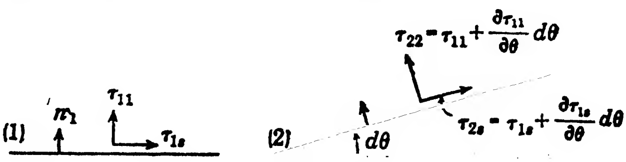
PROBLEMS

1. Show from (16.1), $\overset{\nu'}{\mathbf{T}} \cdot \boldsymbol{\nu} = \overset{\nu}{\mathbf{T}} \cdot \boldsymbol{\nu}'$, that, if $\overset{P}{\mathbf{T}}$ is the stress vector across a plane P, then the stress vector on any plane Q that contains $\overset{P}{T}$ lies in the plane P.

2. Show that the symmetry of the stress components $\tau_{ij} = \tau_{ji}$ follows from (16.1), $\overset{\nu'}{\mathbf{T}} \cdot \boldsymbol{\nu} = \overset{\nu}{\mathbf{T}} \cdot \boldsymbol{\nu}'$.

3. If $\overset{P}{\mathbf{T}}$ and $\overset{Q}{\mathbf{T}}$ are the stress vectors at a point and acting across planes P and Q, find the direction of the stress vector $\overset{R}{\mathbf{T}}$ on a plane R containing both $\overset{P}{\mathbf{T}}$ and $\overset{Q}{\mathbf{T}}$.

4. Show with the help of (16.1) that the normal stress has a stationary value (maximum or minimum) when the shear stress is zero. *Hint:* Let τ_{11} be the normal and τ_{1s} the shear stress across plane (1). Then by (16.1),



$$\left(\tau_{11} + \frac{\partial \tau_{11}}{\partial \theta} d\theta\right) \cos d\theta + \left(\tau_{1s} + \frac{\partial \tau_{1s}}{\partial \theta} d\theta\right) \sin d\theta = \tau_{11} \cos d\theta - \tau_{1s} \sin d\theta$$

or

$$\frac{\partial \tau_{11}}{\partial \theta} = -2\tau_{13}.$$

That is, the normal stress across a surface element varies as the element is rotated and at a rate which is twice the shear component (with sign changed) perpendicular to the axis of rotation.

17. Stress Quadric of Cauchy. For the purpose of studying the nature of the distribution of stresses throughout a continuous medium, we define at each point $P(x)$ a quadric surface, the stress quadric of Cauchy. The discussion of this quadric will parallel closely that of the strain quadric in Secs. 5 and 6.

Consider an element of area with normal $\boldsymbol{\nu}$ and containing a point $P^0(x^0)$, and let $\overset{\nu}{\mathbf{T}}$ be the stress vector acting on this surface element (Fig. 9). We introduce a local system of axes x_i with origin at P^0, and we denote by $\mathbf{A}$ the vector, in the direction of the normal $\boldsymbol{\nu}$, from P^0 to some point $P(x)$. The vector $\overset{\nu}{\mathbf{T}}$ may be resolved into *normal* component N along $\boldsymbol{\nu}$ and *tangential* (or *shearing*) component S orthogonal to $\boldsymbol{\nu}$. The normal component N of $\overset{\nu}{\mathbf{T}}$ can be written with the help of (16.2) as

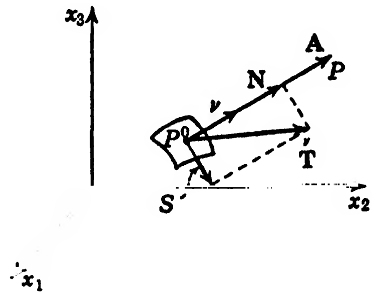


FIG. 9

$$N = \overset{\nu}{\mathbf{T}} \cdot \boldsymbol{\nu} = \overset{\nu}{T}_i \nu_i = \tau_{ij}\nu_i\nu_j,$$

or since $x_i = A\nu_i$,

$$NA^2 = \tau_{ij}x_ix_j. \tag{17.1}$$

This suggests that we consider the quadratic function

$$2F(x_1, x_2, x_3) = \tau_{ij}x_ix_j. \tag{17.2}$$

The length of the vector $\mathbf{A}$ is as yet unspecified; we restrict the coordinates x_i by requiring the end point $P(x)$ of $\mathbf{A}$ to lie on the quadric surface

$$2F(x_1, x_2, x_3) = \tau_{ij}x_ix_j = \pm k^2, \tag{17.3}$$

where k is an arbitrary real constant and where the sign is chosen so as to make the surface real. From (17.3) and (17.1) it is seen that

$$N = \pm \frac{k^2}{A^2}. \tag{17.4}$$

Since A^2 is a positive quantity, k^2 will be taken with the positive sign whenever the normal component N of $\overset{\nu}{\mathbf{T}}$ represents tension and with the negative sign when it represents compression. (Note the convention adopted in Sec. 13.)

If the coordinate axes are rotated to give a new coordinate system x'_i, then new stress components τ'_{ij} are determined and the equation of the stress quadric becomes

$$\tau'_{ij}x'_ix'_j = NA^2 = \pm k^2.$$

But both N and A have values that do not depend on the particular coordinate system used, and hence

$$\tau_{ij}x_ix_j = \tau'_{ij}x'_ix'_j. \tag{17.5}$$

Thus, the quadratic form $\tau_{ij}x_ix_j$ has a value that is independent of the choice of coordinate system. In other words, it is *invariant* with respect to an orthogonal transformation of coordinates.

The invariance of the form $\tau_{ij}x_ix_j$, shown by Eq. (17.5), affords an easy means of calculating the equations of transformation (16.4). For (cf. Sec. 5) if one substitutes in the right-hand member of (17.5) the expressions for x'_i in terms of the x_i, namely,

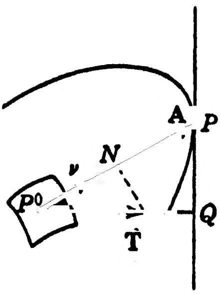


Fig. 10

$$x'_i = l_{i\alpha}x_\alpha,$$

then the resulting expression

$$\tau_{\alpha\beta}x_\alpha x_\beta = l_{i\alpha}l_{j\beta}\tau'_{ij}x_\alpha x_\beta$$

is an identity in the variables x_i. From this we get the equations of transformation (16.5),

$$\tau_{\alpha\beta} = l_{i\alpha}l_{j\beta}\tau'_{ij}.$$

From $2F(x_1, x_2, x_3) = \tau_{ij}x_ix_j$ and Eq. (13.3), it is seen that

$$\frac{\partial F}{\partial x_i} = \tau_{ij}x_j = \tau_{ij}\nu_jA = A\overset{\nu}{T}_i. \tag{17.6}$$

Thus, the quadratic form $F(x_1, x_2, x_3)$ has some attributes of a potential function, since its derivatives with respect to the variables x_i are proportional to the corresponding components of force.

Since the $\dfrac{\partial F}{\partial x_i}$ are the direction ratios of the normal **n** to the plane tangent to the quadric surface (17.3) at the point $P(x)$, we see from (17.6) that the stress vector $\overset{\nu}{\mathbf{T}}$ is also normal to this tangent plane. This gives an easy means of constructing the stress vector $\overset{\nu}{\mathbf{T}}$ from the knowledge of its normal component N. All that is necessary is to draw the quadric surface (17.3) and construct the tangent plane to the quadric through the terminus P(x) of the vector **A** (Fig. 10). Then the vector $\overset{\nu}{\mathbf{T}}$ is directed along the perpendicular P^0Q to the tangent plane. If the magnitude of N is known, one can readily determine the length of the vector $\overset{\nu}{\mathbf{T}}$.

If the direction $\mathbf{v}$ is taken along one of the axes of the quadric, then $\mathbf{v}$ (and $\mathbf{A}$) will be normal to the plane tangent to the surface at (x_i). But $\overset{\nu}{\mathbf{T}}$ is perpendicular to the tangent plane so that, in this case, $\overset{\nu}{\mathbf{T}}$ and $\mathbf{v}$ coincide in direction; hence their components must be proportional. Thus,[1]

$$\overset{\nu}{T}_i = \tau \nu_i = \tau\, \delta_{ij}\nu_j \tag{17.7}$$

when $\mathbf{v}$ lies along an axis of the stress quadric. Since $\mathbf{v}$ is a unit vector and $\overset{\nu}{\mathbf{T}} = \tau\mathbf{v}$, the constant τ denotes the magnitude of the stress vector $\overset{\nu}{\mathbf{T}}$ that acts on an element normal to the axis of the surface. For any direction $\mathbf{v}$ we have $\overset{\nu}{T}_i = \tau_{ij}\nu_j$, and therefore $\tau_{ij}\nu_j = \tau\, \delta_{ij}\nu_j$, or

$$(\tau_{ij} - \tau\delta_{ij})\nu_j = 0. \tag{17.8}$$

This set of three homogeneous equations in the unknown directions $\mathbf{v}$ has a nonvanishing solution if, and only if, the determinant of the coefficients of the ν_j is equal to zero; that is,

$$|\tau_{ij} - \tau\delta_{ij}| = 0. \tag{17.9}$$

This cubic equation in the stress τ is entirely analogous to Eq. (6.3) for the principal strains. Like the latter equation, it has three real roots τ_1, τ_2, τ_3, which are called the *principal stresses*. If τ in (17.8) is replaced by any one of these roots τ_i, then the resulting set of equations may be solved for the corresponding direction $\overset{i}{\mathbf{v}}$. The three directions $\overset{i}{\mathbf{v}}$ are termed the *principal directions of stress*, and the argument of Sec. 6 shows that these directions are orthogonal. The planes normal to the principal directions are called the *principal planes of stress*. If the vector $\mathbf{v}$ is a principal direction $\overset{i}{\mathbf{v}}$, then the associated stress vector $\overset{\nu}{\mathbf{T}} = \tau\overset{i}{\mathbf{v}}$ lies along the normal $\overset{i}{\mathbf{v}}$ and the stress is normal. In other words, there is no shearing stress on a surface element tangent to a principal plane.

In general, there are only three mutually orthogonal principal axes of the quadric, so that at each point $P^0(x^0)$ of the medium one can find three mutually orthogonal directions $\overset{i}{\mathbf{v}}$ such that the surface elements normal to these directions will experience no tangential stress. If the quadric surface is a surface of revolution, there will be infinitely many such directions $\overset{i}{\mathbf{v}}$; one of them will be directed along the axis of revolution, and any two mutually perpendicular directions lying in the plane normal to the axis of revolution may be taken as the remaining principal axes. If $\tau_1 = \tau_2 = \tau_3$, the quadric is a sphere and any three orthogonal lines may be chosen as the principal axes. In this case, whatever be the orientation

[1] It is clear from (17.4) that the normal components N of the stress vector assume extreme values when the radius vector $\mathbf{A}$ is taken along the axes of the quadric.

of the surface element at the center of the sphere, the stress experienced by it will be purely normal.

We recall that τ_1, τ_2, τ_3 are the only stresses acting on the surface elements perpendicular to the principal directions $\overset{1}{\mathbf{v}}, \overset{2}{\mathbf{v}}, \overset{3}{\mathbf{v}}$, while $\tau_{11}, \tau_{22}, \tau_{33}$ are the normal stresses on elements perpendicular to the coordinate axes. If the coordinate axes are taken along the axes of the quadric, then the shear stresses $\tau_{12}, \tau_{23}, \tau_{31}$ disappear from the equation of the surface $\tau_{ij}x_ix_j = \pm k^2$, which now takes the form

$$\tau_1x_1^2 + \tau_2x_2^2 + \tau_3x_3^2 = \pm k^2 = NA^2. \tag{17.10}$$

The cubic equation (17.9) can be written as

$$|\tau_{ij} - \tau\,\delta_{ij}| = -\tau^3 + \Theta_1\tau^2 - \Theta_2\tau + \Theta_3 = 0,$$

where $\Theta_1, \Theta_2, \Theta_3$ are the invariants of the stress tensor:[1]

$$\begin{cases} \Theta_1 = \tau_1 + \tau_2 + \tau_3 = \tau_{11} + \tau_{22} + \tau_{33} \equiv \Theta, \\ \Theta_2 = \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 \\ \quad = \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{23} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{vmatrix}, \\ \Theta_3 = \tau_1\tau_2\tau_3 \\ \quad = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}. \end{cases} \tag{17.11}$$

A reference to formulas (16.5) shows that one can write down at once the expressions for the components of the stress tensor τ_{ij} in terms of the principal stresses. Thus, if the direction cosines of the principal axes of stress X_i are given by the table

	X_1	X_2	X_3
x_1	l_{11}	l_{12}	l_{13}
x_2	l_{21}	l_{22}	l_{23}
x_3	l_{31}	l_{32}	l_{33}

then one has the simple formula

$$\tau_{ij} = \sum_{\alpha=1}^{3} l_{i\alpha}l_{j\alpha}\tau_\alpha. \tag{17.12}$$

The character of the distribution of stress at the point $P^0(x^0)$ depends on the signs of the principal stresses. (Note the agreement above con-

[1] Cf. Eq. (6.9).

cerning the choice of the sign of k^2.) If the principal stresses are all positive, then the equation of the stress quadric has the form

$$\tau_1 x_1^2 + \tau_2 x_2^2 + \tau_3 x_3^2 = k^2,$$

and the surface is an ellipsoid. Equation (17.4) now reads $N = k^2/A^2$, from which it follows that the force acting on every surface element passing through the point P^0 is tensile. If, on the other hand, all τ_i are negative, then (17.10) takes the form

$$\tau_1 x_1^2 + \tau_2 x_2^2 + \tau_3 x_3^2 = -k^2.$$

This surface is again an ellipsoid, but the normal component N of the stress vector $\overset{\nu}{\mathbf{T}}$ this time is $N = -k^2/A^2$, and the stress is compressive.

Consider next the case when $\tau_1 > 0, \tau_2 > 0, \tau_3 < 0$; Eq. (17.10) has one of the forms

$$\tau_1 x_1^2 + \tau_2 x_2^2 - |\tau_3| x_3^2 = k^2$$

or

$$\tau_1 x_1^2 + \tau_2 x_2^2 - |\tau_3| x_3^2 = -k^2,$$

depending on the orientation of the surface element at $P^0(x^0)$. The first of these equations represents an unparted hyperboloid and the second a biparted one (Fig. 11). If the normal to the surface element at P^0 cuts the biparted hyperboloid, then $N = -k^2/A^2$, so that the stress is compressive, while if the normal cuts the unparted hyperboloid, then $N = k^2/A^2$, and the stress is tensile. Vectors $\mathbf{A}$ that lie on the surface of the asymptotic cone

$$\tau_1 x_1^2 + \tau_2 x_2^2 - |\tau_3| x_3^2 = 0$$

do not cut either of the hyperboloids. In this case, $NA^2 = 0$, and hence $N = 0$. Accordingly, the elements of surface whose normals are directed along the generators of the cone experience only tangential stress.

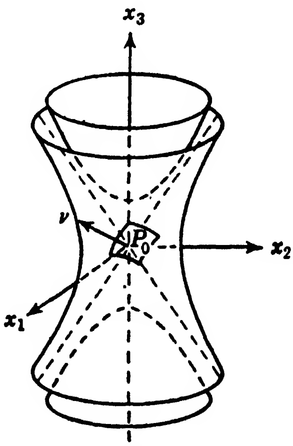


FIG. 11

It is easily shown that the case of $\tau_1 < 0, \tau_2 < 0, \tau_3 > 0$ does not differ essentially from that just considered. The only difference is in the regions in which the medium experiences compression and tension.

18. Maximum Normal and Shear Stresses. Mohr's Diagram. We have shown in the preceding section [Eq. (17.4)] that the component N of the stress vector $\overset{\nu}{\mathbf{T}}$ in the direction $\boldsymbol{\nu}$, normal to the surface element, is inversely proportional to the square of the radius vector $A\boldsymbol{\nu}$ to the stress quadric. The extreme values of the radius vector lie along the axes of

the quadric. Hence the extreme values of N, which we have denoted by τ_1, τ_2, τ_3, are the extreme values of the normal components of the stress vector acting at P^0 as the surface element assumes different orientations. These extreme values are obviously of moment in the study of failure of materials. In some theories of failure it is also important to know the extreme values of the shearing component S of $\overset{\nu}{\mathbf{T}}$ and the directions $\boldsymbol{\nu}$ associated with them. These are easily determined. If we direct the coordinate axes at P^0 along the principal directions of stress, the components τ_{12}, τ_{23}, and τ_{13} vanish and $\tau_{11} = \tau_1$, $\tau_{22} = \tau_2$, $\tau_{33} = \tau_3$. From the basic relation

$$\overset{\nu}{T}_i = \tau_{ij}\nu_j,$$

we then have

$$\overset{\nu}{T}_1 = \tau_1\nu_1, \qquad \overset{\nu}{T}_2 = \tau_2\nu_2, \qquad \overset{\nu}{T}_3 = \tau_3\nu_3, \tag{18.1}$$

and since

$$N = \overset{\nu}{T}_i\nu_i = \tau_{ij}\nu_i\nu_j,$$

we get

$$N = \tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2. \tag{18.2}$$

But from Fig. 9

$$S^2 = |\overset{\nu}{\mathbf{T}}|^2 - N^2,$$

and on substituting in this formula from (18.1) and (18.2) we obtain

$$S^2 = \tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2 - (\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2)^2. \tag{18.3}$$

It is clear from (18.3) that if the directions $\boldsymbol{\nu}$ are taken along the axes of the stress quadric so that

$$\begin{aligned} \nu_1 &= \pm 1, & \nu_2 &= \nu_3 = 0, \\ \nu_2 &= \pm 1, & \nu_3 &= \nu_1 = 0, \\ \nu_3 &= \pm 1, & \nu_1 &= \nu_2 = 0, \end{aligned}$$

then $S = 0$. This merely verifies the known fact that the planar elements normal to the principal directions of stress are free from shear. Thus the minimum (zero) values of $|S|$ are associated with the principal directions. To determine the directions associated with the maximum values of $|S|$, we maximize the function in the right-hand member of (18.3), subject to the constraining relation $\nu_i\nu_i = 1$. The simplest way of doing this is to use the method of Lagrange multipliers and seek the free extremum of the function

$$F = S^2 - \lambda\nu_i\nu_i.$$

This leads to the three equations,

$$\frac{\partial F}{\partial \nu_i} = 0.$$

in λ and ν_i which, together with the relation $\nu_i\nu_i = 1$, serve to determine the desired directions.

We dispense with the elementary computations and record the final results in the accompanying table, the last column of which gives the values of $|N|$ associated with the extreme values of $|S|$.

TABLE OF EXTREMAL VALUES OF S

ν_1	ν_2	ν_3	$\|S\|_{extr}$	$\|N\|$
0	0	± 1	0	$\|\tau_3\|$
0	± 1	0	0	$\|\tau_2\|$
± 1	0	0	0	$\|\tau_1\|$
0	$\pm \frac{\sqrt{2}}{2}$	$\pm \frac{\sqrt{2}}{2}$	$\frac{1}{2}\|\tau_2 - \tau_3\|$	$\frac{1}{2}\|\tau_2 + \tau_3\|$
$\pm \frac{\sqrt{2}}{2}$	0	$\pm \frac{\sqrt{2}}{2}$	$\frac{1}{2}\|\tau_3 - \tau_1\|$	$\frac{1}{2}\|\tau_3 + \tau_1\|$
$\pm \frac{\sqrt{2}}{2}$	$\pm \frac{\sqrt{2}}{2}$	0	$\frac{1}{2}\|\tau_1 - \tau_2\|$	$\frac{1}{2}\|\tau_1 + \tau_2\|$

If $\tau_3 < \tau_2 < \tau_1$, so that τ_1 is the maximum value of N and τ_3 is its minimum value, then the maximum value of $|S|$ is,

$$|S| = \tfrac{1}{2}(\tau_3 - \tau_1).$$

We see from the table that the maximum shearing stress acts on the surface element containing the x_2 principal axis and bisecting the angle between the x_1- and x_3-axes. If $\tau_2 = \tau_3$, there will be infinitely many directions associated with the surface elements that are subjected to a maximum shearing stress. We summarize the main results of this section in the following theorem:

THEOREM: *The maximum shearing stress is equal to one-half the difference between the greatest and least normal stresses and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses.*

The results of this section can be further illuminated by constructing a diagram proposed[1] by O. Mohr.

If we rewrite Eqs. (18.2) and (18.3) in the form

$$\begin{aligned} N &= \tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2, \\ S^2 + N^2 &= \tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2, \end{aligned}$$

[1] Otto Mohr, *Zivilingenieur* (1882), p. 113. See also his book Technische Mechanik, 2d ed. (1914).

recall that $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$, and solve for the ν_i^2, we obtain

$$
\left\{
\begin{aligned}
\nu_1^2 &= \frac{S^2 + (N - \tau_2)(N - \tau_3)}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)}, \\
\nu_2^2 &= \frac{S^2 + (N - \tau_3)(N - \tau_1)}{(\tau_2 - \tau_3)(\tau_2 - \tau_1)}, \\
\nu_3^2 &= \frac{S^2 + (N - \tau_1)(N - \tau_2)}{(\tau_3 - \tau_1)(\tau_3 - \tau_2)}.
\end{aligned}
\right. \tag{18.4}
$$

We are assuming that

$$\tau_3 < \tau_2 < \tau_1,$$

so that $\tau_1 - \tau_2 > 0$ and $\tau_1 - \tau_3 > 0$, and since ν_i^2 is nonnegative, we conclude from the first of Eqs. (18.4) that

$$S^2 + (N - \tau_2)(N - \tau_3) \geq 0. \tag{18.5}$$

We consider now the space of the variables (S, N) and plot in the cartesian SN-plane (Fig. 12) the values of S as ordinates and those of N as abscissas.

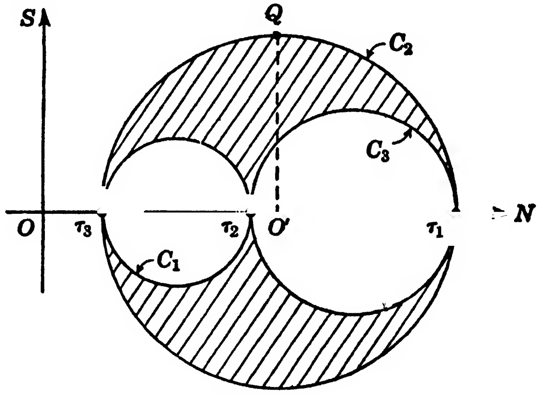


FIG. 12

The equation

$$S^2 + (N - \tau_2)(N - \tau_3) = 0 \tag{18.6}$$

represents a circle C_1 with center on the N-axis and passing through the points $(\tau_2, 0)$, $(\tau_3, 0)$. Hence the region defined by (18.5) is exterior to the circle (18.6) and includes its boundary. Further, $\tau_2 - \tau_3 > 0$, $\tau_2 - \tau_1 < 0$, and we conclude from the second of Eqs. (18.4) that

$$S^2 + (N - \tau_3)(N - \tau_1) \leq 0. \tag{18.7}$$

Thus the region defined by (18.7) is a closed region, interior to the circle C_2 (Fig. 12), whose equation is

$$S^2 + (N - \tau_3)(N - \tau_1) = 0.$$

Finally the third of Eqs. (18.4) yields the result that

$$S^2 + (N - \tau_1)(N - \tau_2) \geq 0, \tag{18.8}$$

since $\tau_3 - \tau_1 < 0$ and $\tau_3 - \tau_2 < 0$.

The region defined by (18.8) is exterior to the circle C_3 (Fig. 12), with center on the N-axis and passing through the points $(\tau_1, 0)$, $(\tau_2, 0)$.

It follows from inequalities (18.5), (18.7), and (18.8) that the admissible values of S and N lie in the crescent-shaped regions (shaded in Fig. 12) bounded by the circles C_1, C_2, and C_3.

The maximum shearing stress S, as is clear from Fig. 12, is represented by the greatest ordinate $O'Q$ of the circle C_2, and hence

$$S_{\max} = \frac{\tau_1 - \tau_3}{2}.$$

To determine the orientation of the surface elements that support this stress, we make use of formulas (18.4). The value of N, corresponding to $S_{\max}$ (shown as OO' in Fig. 12), is

$$N = \frac{\tau_1 + \tau_3}{2},$$

and the substitution of this value and $S_{\max} = \frac{1}{2}(\tau_1 - \tau_3)$ in (18.4) yields $\nu_1^2 = \nu_3^2 = \frac{1}{2}$, $\nu_2^2 = 0$. These coincide with the values appearing in the table on page 51.

PROBLEMS

Discuss the Mohr circle diagram for the case where $\tau_2 = \tau_3$, and determine the orientation of surface elements experiencing extreme shearing stresses. Consider also the case where $\tau_1 = \tau_2 = \tau_3$.

19. Examples of Stress. This section contains several examples closely paralleling those in Sec. 8. As in that section, we prefer to use the unabridged notation.

a. Purely Normal Stress. If for every plane passing through a point $P^0(x^0)$ the stress vector $\overset{\nu}{\mathbf{T}}$ is normal to the plane, that is, if it is directed along the normal $\boldsymbol{\nu}$ or opposite to it, then for any choice of rectangular coordinates

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0, \qquad \text{and} \qquad \tau_{xx} = \tau_{yy} = \tau_{zz}.$$

The stress quadric in this case is a sphere whose equation is

$$x^2 + y^2 + z^2 = \frac{\pm k^2}{\tau_{xx}}.$$

Any set of orthogonal axes that pass through the point P^0 may be taken as principal axes of the quadric. This case corresponds to hydrostatic pressure if τ_{xx} is negative.

b. Simple Tension or Compression. A state of simple tension or compression is characterized by the fact that the stress vector for one plane through the point is normal to that plane and the stress vector for any plane perpendicular to this one vanishes. Hence if the x'-, y'-, and z'-axes coincide with the principal axes of stress, then the stress quadric (17.3) has the equation

$$\tau_1 x'^2 = \pm k^2.$$

Transforming to any other orthogonal coordinate system x, y, z with the aid of (17.12), we obtain the following stress components:

$$\begin{aligned} \tau_{xx} &= \tau_1 l_{11}^2, & \tau_{yy} &= \tau_1 l_{21}^2, & \tau_{zz} &= \tau_1 l_{31}^2, \\ \tau_{xy} &= \tau_1 l_{11} l_{21}, & \tau_{yz} &= \tau_1 l_{21} l_{31}, & \tau_{zx} &= \tau_1 l_{31} l_{11}, \end{aligned}$$

where l_{11}, l_{21}, l_{31} are the direction cosines of the x'-axis relative to the axes x, y, z. A positive value of τ_1 represents tension, and a negative represents compression.

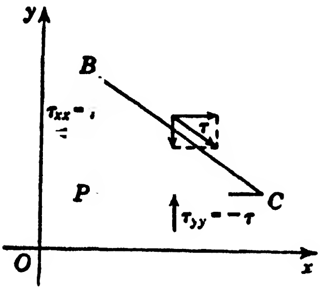


FIG. 13

c. Shearing Stress. Consider a stress quadric

$$2\tau x'y' = \pm k^2, \tag{19.1}$$

which is a hyperbolic cylinder whose elements are parallel to the z'-axis and which represents a shearing stress of magnitude τ. Equation (19.1) takes the form

$$\tau x^2 - \tau y^2 = \pm k^2,$$

when the axes are rotated through an angle of 45° about the z'-axis. A comparison of this equation with the general equation of the stress quadric

$$\tau_{xx} x^2 + \tau_{yy} y^2 + \tau_{zz} z^2 = \pm k^2 \tag{19.2}$$

when the latter is referred to the principal axes of stress shows that we must have

$$\tau_{zz} = 0, \qquad \tau_{xx} = -\tau_{yy} = \tau.$$

Thus, the shearing stress is equivalent to tension across one plane and compression of equal magnitude across a perpendicular plane. This can also be shown geometrically by considering the equilibrium of the element *PBC* (Fig. 13). Hence the stress on the face *BC* is a pure shear of magnitude $\tau = -\tau_{yy} = +\tau_{xx}$. This type of shearing stress would tend to slide planes of the material originally perpendicular to the y'-axis in a direction parallel to the x'-axis and planes of the material originally perpendicular to the x'-axis in a direction parallel to the y'-axis.

d. Plane Stress. If one of the principal stresses vanishes, then the stress quadric becomes a cylinder whose base is a conic, the stress conic.

A state of stress, in this case, is said to be *plane*. The base of the cylinder lies in a plane containing the directions of the nonvanishing principal stresses. For example, if this plane is perpendicular to the z-axis, the equation of the quadric is

$$\tau_{xx}x^2 + \tau_{yy}y^2 + 2\tau_{xy}xy = \pm k^2.$$

For simple tension in the x-direction, the stress conic reduces to the pair of lines

$$x = \pm \sqrt{\frac{\pm k^2}{\tau_{xx}}}.$$

For the case of shear, the stress conic is a rectangular hyperbola

$$xy = \pm \frac{k^2}{2\tau_{xy}}.$$

If the stress conic is a circle, there is equal tension or compression in all directions in the plane of the circle.

CHAPTER 3

EQUATIONS OF ELASTICITY

20. Hooke's Law. It has already been noted that the treatment contained in Chaps. 1 and 2 is applicable to all material media that can be represented with sufficient accuracy as continuous bodies; this chapter will be concerned with the characterization of elastic solids.

The first attempt at a scientific description of the strength of solids was made by Galileo. He treated bodies as inextensible, however, since at that time there existed neither experimental data nor physical hypotheses that would yield a relation between the deformation of a solid body and the forces responsible for the deformation. It was Robert Hooke who, some forty years after the appearance of Galileo's Discourses (1638), gave the first rough law of proportionality between the forces and displacements. Hooke published his law first in the form of an anagram "ceiiinosssttuu" in 1676, and two years later gave the solution of the anagram: "*ut tensio sic vis*," which can be translated freely as "the extension is proportional to the force." To study this statement further, we discuss the deformation of a thin rod subjected to a tensile stress.

Consider a thin rod (of a low-carbon steel, for example), of initial cross-sectional area a_0, which is subjected to a variable tensile force F. If the stress is assumed to be distributed uniformly over the area of the cross section, then the *nominal stress* $T = F/a_0$ can be calculated for any applied load F. The *actual stress* is obtained, under the assumption of a uniform stress distribution, by dividing the load at any stage of the test by the actual area of the cross section of the rod at that stage. The difference between the nominal and the actual stress is negligible, however, throughout the elastic range of the material.

If the nominal stress T is plotted as a function of the extension e (change in length per unit length of the specimen), then for some ductile metals a graph like that in Fig. 14 is secured. The graph is very nearly a straight line with the equation

$$T = Ee \tag{20.1}$$

until the stress reaches the *proportional limit* (point P in Fig. 14). The position of this point, however, depends to a considerable extent upon the sensitivity of the testing apparatus. The constant of proportionality E is known as *Young's modulus.*

In most metals, especially in soft and ductile materials, careful observation will reveal very small permanent elongations which are the results of very small tensile forces. In many metals, however (steel and wrought iron, for example), if these very small permanent elongations are neglected (less than 1/100,000 of the length of a bar under tension), then the graph of stress against extension is a straight line, as noted above, and practically all the deformation disappears after the force has been removed. The greatest stress that can be applied without producing a permanent deformation is called the *elastic limit* of the material. When the applied force is increased beyond this fairly sharply defined limit, the material exhibits both elastic and plastic properties. The determination of this limit requires successive loading and unloading by ever larger forces until a permanent set is recorded. For many materials the proportional limit is very nearly equal to the elastic limit, and the distinction between the two is sometimes dropped, particularly since the former is more easily obtained.

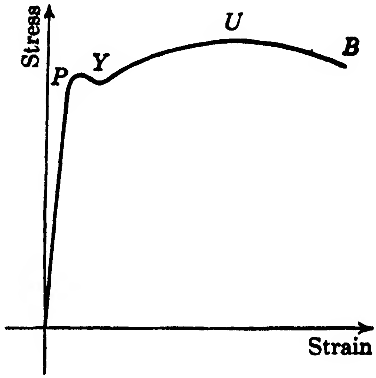


FIG. 14

When the stress increases beyond the elastic limit, a point is reached (Y on the graph) at which the rod suddenly stretches with little or no increase in the load. The stress at point Y is called the *yield-point stress.*

The nominal stress T may be increased beyond the yield point until the *ultimate stress* (point U) is reached. The corresponding force $F = Ta_0$ is the greatest load that the rod will bear. When the ultimate stress is reached, a brittle material (such as a high-carbon steel) breaks suddenly, while a rod of some ductile metal begins to "neck"; that is, its cross-sectional area is greatly reduced over a small portion of the length of the rod. Further elongation is accompanied by an increase in actual stress but by a decrease in total load, in cross-sectional area, and in nominal stress until the rod breaks (point B).

The elastic limit of low-carbon steels is about 35,000 lb per sq in.; the ultimate stress is about 60,000 lb per sq in. Hard steels may be prepared with an ultimate strength greater than 200,000 lb per sq in.

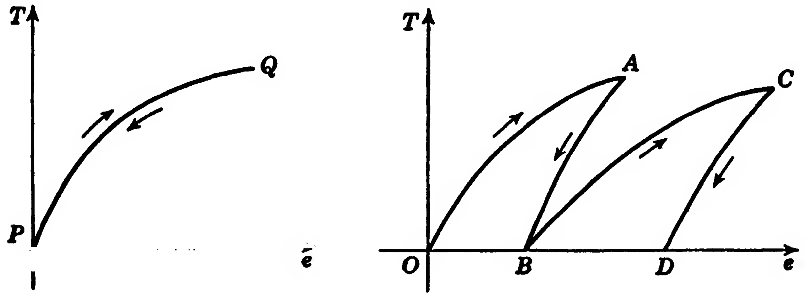
We shall consider only the behavior of elastic materials subjected to stresses below the proportional limit; that is, we shall be concerned only with those materials and situations in which Hooke's law, expressed by Eq. (20.1), or a generalization of it, is valid.[1]

[1] In order to give the reader some feeling regarding the magnitude of deformations with which the theory of elasticity deals, note that a 1-in.-long rod of iron with proportional limit of 25,000 lb per sq in., a yield point of 30,000 lb per sq in., and Young's mod-

Some materials subjected to tensile tests have an extremely small range of values of extensions e for which the law (20.1) is valid. In this case, the stress-strain curve above the proportional limit may have the appearance indicated in Fig. 15a. In the process of loading and unloading specimens made of such materials, the same curve PQ may be traced out, and if there is no residual deformation, the material is elastic with the stress-strain law of the form

$$T = f(e),$$

where f is a single-valued nonlinear function. More frequently, however, the loading-unloading diagram has the appearance shown in Fig. 15b. In this diagram the curve OA is associated with the loading of the specimen and AB with the unloading. In this instance there is a residual



(a) (b)

FIG. 15

deformation, represented by OB, which characterizes the plastic behavior. For plastic materials the relationship between T and e is no longer one-to-one, and after repeated loadings and unloadings a saw-tooth pattern indicated in Fig. 15b may be obtained.

A natural generalization of Hooke's law immediately suggests itself, namely, one can invoke the principle of superposition of effects and assume that at each point of the medium the strain components e_{ij} are linear functions of the stress components τ_{ij}. Such a generalization was made by Cauchy, and the resulting law is known as the *generalized Hooke's law*. We discuss it in the following section.

21. Generalized Hooke's Law. We saw in the preceding chapters that the state of stress in continuous media is completely determined by the stress tensor τ_{ij}, and the state of deformation by the strain tensor e_{ij}. We shall now assume that when an elastic medium is maintained at a

---

ulus of 30,000,000 lb per sq in. will elongate under a load of 13,000 lb per sq in. about 0.0004 in. Even if the rod is loaded to the yield point, the determination of the extension will require very refined measurements.

fixed temperature there is a one-to-one analytic relation

$$\tau_{ij} = F_{ij}(e_{11}, e_{22}, \ldots, e_{12}), \qquad (i, j = 1, 2, 3)$$

between the τ_{ij} and e_{ij} and that the τ_{ij} vanish when the strains e_{ij} are all zero. This last assumption implies that in the initial unstrained state the body is unstressed. Now, if the functions F_{ij} are expanded in the power series in e_{ij} and only the linear terms retained in the expansions, we get

$$\tau_{ij} = c_{ijkl}e_{kl} \qquad (i, j, k, l = 1, 2, 3). \tag{21.1}$$

The coefficients c_{ijkl}, in the linear forms (21.1), in general will vary from point to point of the medium. If, however, the c_{ijkl} are independent of the position of the point, the medium is called *elastically homogeneous*. Henceforth we confine our attention to those media in which the c_{ijkl} do not vary throughout the region under consideration. The law (21.1) is a natural generalization of Hooke's law, and it is used in all developments of the linear theory of elasticity.[1]

Inasmuch as the components τ_{ij} are symmetric, an interchange of the indices i and j in (21.1) does not alter these formulas, so that

$$c_{ijkl} = c_{jikl}.$$

Moreover, we can assume, without loss of generality, that the c_{ijkl} are also symmetric with respect to the last two indices. For if the constants c'_{ijkl} and c''_{ijkl} are defined by the formulas

$$c'_{ijkl} = \tfrac{1}{2}(c_{ijkl} + c_{ijlk}),$$
$$c''_{ijkl} = \tfrac{1}{2}(c_{ijkl} - c_{ijlk}),$$

then, clearly, $c'_{ijkl} = c'_{ijlk}$ and $c''_{ijkl} = -c''_{ijlk}$. Thus c_{ijkl} can be written as the sum

$$c_{ijkl} = c'_{ijkl} + c''_{ijkl},$$

in which the c'_{ijkl} are symmetric and the c''_{ijkl} are skew-symmetric with respect to k and l. Accordingly, the law (21.1) can always be written in the form

$$\tau_{ij} = c'_{ijkl}e_{kl} + c''_{ijkl}e_{kl}.$$

However, the double sum in the second term of this expression vanishes inasmuch as $e_{kl} = e_{lk}$ and $c''_{ijkl} = -c''_{ijlk}$. Thus,

$$\tau_{ij} = c'_{ijkl}e_{kl},$$

where the c'_{ijkl} are symmetric with respect to the first two and the last two indices.

[1] It is important to note that the generalized Hooke's law (21.1) is also used in some investigations where the strains are finite, in the sense of Sec. 11. For many materials a linear relationship (21.1) holds for an appreciable range of values of the e_{ij}. The linear theory of elasticity, however, is based on the use of the infinitesimal strains, defined in Sec. 7, and on the linear law (21.1).

We shall consider henceforth that the c_{ijkl} in (21.1) have been symmetrized, so that there are at most 36 independent constants in the general stress-strain law (21.1).

To avoid dealing with double sums, we can introduce the notation

$$\begin{array}{llllll} \tau_{11} = \tau_1, & \tau_{22} = \tau_2, & \tau_{33} = \tau_3, & \tau_{23} = \tau_4, & \tau_{31} = \tau_5, & \tau_{12} = \tau_6, \\ e_{11} = e_1, & e_{22} = e_2, & e_{33} = e_3, & 2e_{23} = e_4, & 2e_{31} = e_5, & 2e_{12} = e_6, \end{array}$$

and write the six equations (21.1) in the form

$$\begin{gathered} \tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{14}e_4 + c_{15}e_5 + c_{16}e_6, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_6 = c_{61}e_1 + c_{62}e_2 + c_{63}e_3 + c_{64}e_4 + c_{65}e_5 + c_{66}e_6, \end{gathered}$$

or, more compactly,

$$\tau_i = c_{ij}e_j \qquad (i, j = 1, 2, \ldots, 6). \tag{21.2}$$

The relations between the τ_i and e_i must be reversible, hence $|c_{ij}| \neq 0$, and we can write

$$e_i = C_{ij}\tau_j.$$

The constants c_{ij} in (21.2) are called the *elastic constants*, or *moduli*, of the material. Inasmuch as the strains e_i are dimensionless, the c_{ij} have the same dimensions as the stress components.[1]

We have just remarked that the maximum number of the independent elastic constants is 36, but this number reduces to 21 whenever there exists a function

$$W = \tfrac{1}{2}c_{ij}e_ie_j, \tag{21.3}$$

with the property,

$$\frac{\partial W}{\partial e_i} = \tau_i. \tag{21.4}$$

For one can always suppose that the quadratic form (21.3) is symmetric, and it then follows from (21.4) that

$$\tau_i = c_{ij}e_j,$$

where $c_{ij} = c_{ji}$.

The potential function W was introduced by George Green,[2] and it is called the *strain-energy density function.* Its existence for the isothermal and adiabatic processes[3] has been argued on the basis of the first and

[1] See Sec. 14.

[2] G. Green, *Transactions of the Cambridge Philosophical Society*, vol. 7 (1839), p. 121.

[3] See references in this book to Lord Kelvin's papers in the *Historical Sketch*. The isothermal process corresponds to the case of slow loading and unloading involving no temperature changes of the medium. It is of interest in elastostatic problems. The adiabatic process is approximated in those dynamical problems where bodies execute small and rapid vibrations. The elastic constants c_{ij} in the two cases cannot be expected to be identical.

second laws of thermodynamics, and it is now generally accepted that, for the most general case of an anisotropic elastic body, the number of independent elastic constants in the generalized Hooke's law is 21. The matter of the number of elastic constants required to describe the stress-strain law of the form (21.1) was the subject of a lengthy controversy. Cauchy and Poisson argued,[1] on the basis of special mathematical models of molecular interaction, that the number of independent constants cannot exceed 15. Their arguments proved wanting and are in contradiction to experimental evidence.

If an elastic medium exhibits a geometrical symmetry of internal structure (crystallographic form, regular arrangement of fibers or molecules, etc.) then its elastic properties become identical in certain directions.[2] The geometric symmetry, however, is not equivalent to elastic symmetry because there may be certain other directions for which the elastic properties are the same but the geometric ones are not.

If the medium is elastically symmetric in certain directions, then the number of independent constants c_{ij} in (21.2) is further reduced. Because of their practical importance, we discuss in this section two particular types of elastic symmetry. These are (1) symmetry with respect to a plane (in which 13 independent elastic constants are involved), and (2) symmetry with respect to three mutually perpendicular planes (involving 9 independent constants c_{ij}). In the next section we prove that when the elastic properties of a body are identical in all directions, that is, if the body is elastically *isotropic*, the number of essential elastic constants reduces to 2.

It is obvious from (21.2) that the coefficients c_{ij}, in general, depend on the chosen reference frame inasmuch as the stress components τ_i and the strain components e_i vary with the choice of coordinate systems. For certain media the coefficients c_{ij} may remain invariant under a given transformation of coordinates, and it is this invariance which determines the elastic symmetry of the medium under consideration.[3]

[1] A. L. Cauchy, *Exercises de mathématique*, vol. 3 (1828*a*), p. 213; vol. 3 (1828*b*), p. 328.

S. D. Poisson, *Mémoires de l'académie*, Paris, vol. 8 (1829); vol. 18 (1842).

See also in this connection:

M. Born, Dynamik der Kristalgitter (1915) and Atomtheorie des festen Zustandes, 2d ed. (1923), and comments on Born's work by I. Stakgold, *Quarterly of Applied Mathematics*, vol. 8 (1950), pp. 169–186.

P. S. Epstein, *Physical Review*, vol. 70 (1946), pp. 915–922.

The arguments of Green and Lord Kelvin, in support of the 21-constant theory, are presented in Chap. III of Love's Treatise on the Mathematical Theory of Elasticity.

[2] This is the principle expressed by F. Neumann in Vorlesungen über die Theorie der Elastizität (1885). See also Love's Treatise (1927), p. 155.

[3] The reader familiar with the rudiments of tensor analysis will recognize that when

Consider a substance elastically symmetric with respect to the x_1x_2-plane. This symmetry is expressed by the statement that the c_{ij} are invariant under the transformation

$$x_1 = x_1', \qquad x_2 = x_2', \qquad x_3 = -x_3'.$$

The table of direction cosines of this transformation is

	x_1	x_2	x_3
x_1'	1	0	0
x_2'	0	1	0
x_3'	0	0	-1

and from formulas (5.9) and (16.4) it is seen that

$$\tau_i' = \tau_i, \qquad e_i' = e_i, \qquad (i = 1, 2, 3, 6),$$
$$\tau_4' = -\tau_4, \qquad e_4' = -e_4, \qquad \tau_5' = -\tau_5, \qquad e_5' = -e_5.$$

The first equation of (21.2) becomes

$$\tau_1' = c_{11}e_1' + c_{12}e_2' + c_{13}e_3' + c_{14}e_4' + c_{15}e_5' + c_{16}e_6',$$

or

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 - c_{14}e_4 - c_{15}e_5 + c_{16}e_6.$$

Comparison of this equation with the expression for τ_1 given by (21.2) shows that

$$c_{14} = c_{15} = 0.$$

Similarly, by considering $\tau_2', \ldots, \tau_6'$, we find that

$$c_{24} = c_{25} = c_{34} = c_{35} = c_{64} = c_{65} = 0,$$
$$c_{41} = c_{42} = c_{43} = c_{46} = c_{51} = c_{52} = c_{53} = c_{56} = 0.$$

For a material with one plane of elastic symmetry (which is taken to be the x_1x_2-plane), the matrix of the coefficients of the linear forms in (21.2) can be written as follows,

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the law (21.1) is written in the form

$$\tau_{ij} = c_{ij}^{kl}e_{kl}, \qquad (i, j, k, l = 1, 2, 3),$$

valid in all coordinate systems, then it follows from the tensor character of the τ_{ij} and e_{kl} that the c_{ij}^{kl} are components of a tensor of rank 4. Consequently, under a transformation of coordinates from the system X to X', the c_{ij}^{kl} transform according to the law

$$(a) \qquad c'^{kl}_{ij} = \frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_k'}{\partial x_\gamma}\frac{\partial x_l'}{\partial x_\delta}c_{\alpha\beta}^{\gamma\delta}.$$

If the c_{ij}^{kl} are invariant (so that $c'^{kl}_{ij} = c_{ij}^{kl}$) under a given coordinate transformation, then the transformation characterizes the nature of elastic symmetry. The $\frac{\partial x_i}{\partial x_j'}$ figuring in the law (a) are the direction cosines appearing in the tables of this section, because the systems X and X' are cartesian.

$$(21.5)\qquad \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{pmatrix}$$

Such materials as wood, for example, have three mutually orthogonal planes of elastic symmetry and are said to be *orthotropic*. In considering such materials, we shall choose the axes of coordinates so that the coordinate planes coincide with the planes of elastic symmetry. In this case, some of the coefficients c_{ij} exhibited in the array (21.5) vanish. Besides the symmetry with respect to the x_1x_2-plane, expressed by (21.5), the elastic constants c_{ij} must also be invariant under the transformation of coordinates defined by the following table of direction cosines.

	x_1	x_2	x_3
x_1'	-1	0	0
x_2'	0	1	0
x_3'	0	0	1

This change of coordinates is a reflection in the x_2x_3-plane and leaves the τ_i and e_i unchanged with the following exceptions:

$$\tau_5' = -\tau_5, \qquad e_5' = -e_5, \qquad \tau_6' = -\tau_6, \qquad e_6' = -e_6.$$

From (21.5) we have

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{16}e_6.$$

This becomes

$$\tau_1' = c_{11}e_1' + c_{12}e_2' + c_{13}e_3' + c_{16}e_6',$$

or

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 - c_{16}e_6,$$

from which it follows that $c_{16} = 0$. By considering in a similar way the transformed expressions for $\tau_2, \ldots, \tau_6$, we find that[1]

$$c_{26} = c_{36} = c_{45} = c_{54} = c_{61} = c_{62} = c_{63} = 0.$$

Thus, for orthotropic media the matrix of the c_{ij} takes the following form.

$$(21.6)\qquad \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}$$

[1] Note that elastic symmetry in the x_1x_2-plane and in the x_2x_3-plane implies elastic symmetry in the x_1x_3-plane.

If the coefficients c_{ij} are symmetric, that is,

(21.7) $$c_{ij} = c_{ji}, \qquad (i, j = 1, 2, \ldots, 6),$$

we see that there are 13 essential constants in the array (21.5) and 9 in (21.6). This symmetry has not been assumed, however, in establishing the forms of the arrays of coefficients (21.5) and (21.6), nor will it be used in the next section, where the law (21.2) is specialized to that for an isotropic medium.

It is worth noting that the statement of the law (21.2) is not devoid of inconsistency. In the process of formulating the notion of the components of strain e_{ij}, it was assumed that the components of displacement u_i are functions of the coordinates (x_1, x_2, x_3) of the body in its undeformed state; that is, Lagrangian coordinates were used. On the other hand, Eulerian coordinates were employed in defining the components of the stress tensor τ_{ij}; that is, it was assumed that the τ_{ij} are functions of the coordinates (x'_1, x'_2, x'_3) of the stressed (and hence deformed) medium. Of course, if the displacements u_i and their derivatives are small, then the values of $\tau_{ij}(x)$ and $\tau_{ij}(x')$ cannot differ by a great deal. As an indication of the order of approximation involved here, note that, if $x'_k = x_k + u_k$, then

$$\frac{\partial \tau_{ij}}{\partial x_k} = \frac{\partial \tau_{ij}}{\partial x'_l}\frac{\partial x'_l}{\partial x_k} = \frac{\partial \tau_{ij}}{\partial x'_l}\left(\delta_{kl} + \frac{\partial u_l}{\partial x_k}\right) = \frac{\partial \tau_{ij}}{\partial x'_k} + \frac{\partial \tau_{ij}}{\partial x'_l}\frac{\partial u_l}{\partial x_k}.$$

Hence, in writing $\dfrac{\partial \tau_{ij}}{\partial x_k} = \dfrac{\partial \tau_{ij}}{\partial x'_k}$, we assume that the displacement derivatives are small compared with unity. In what follows, it will be assumed that both the components of strain e_{ij} and the components of stress τ_{ij} are functions of the initial coordinates (x_1, x_2, x_3).

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A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Secs. 60–65, pp. 92–100.

Chap. VI of Love's treatise is given to a discussion of the equilibrium of nonisotropic elastic solids and contains further references on the subject. Voigt's Lehrbuch der Kristallphysik is a standard treatise on the subject.

L. Lecornu: Théorie mathématique de l'élasticité, Mémorial des sciences mathématiques, Gauthiers-Villars & Cie, Paris, pp. 12–18.

Contains a discussion of the theory of Poincaré regarding the number of elastic constants in the generalized Hooke's law.

PROBLEMS

1. Are the principal axes of strain coincident with those of stress for an anisotropic medium with Hooke's law expressed by Eq. (21.2)? For a medium with one plane of elastic symmetry? For an orthotropic medium? *Hint:* Take the coordinate axes along the principal axes of strain so that $e_4 = e_5 = e_6 = 0$.

2. Show directly from the generalized Hooke's law [Eq. (21.2)] that in an isotropic body the principal axes of strain coincide with those of stress. *Hint:* Take the coordinate axes along the principal axes of strain ($e_4 = e_5 = e_6 = 0$), and consider the effect on τ_{23} and τ_{31} of a rotation of axes by 180° about the x_3-axis.

22. Homogeneous Isotropic Media. Most structural materials are formed of crystalline substances, and hence very small portions of such materials cannot be regarded as being isotropic. Nevertheless, the assumption of isotropy and homogeneity, when applied to an entire body, often does not lead to serious discrepancies between the experimental and theoretical results.[1] The reason for this agreement lies in the fact that the dimensions of most crystals are so small in comparison with the dimensions of body and they are so chaotically distributed that, in the large, the substance behaves as though it were isotropic.

From the definition of the isotropic medium, it follows that its elastic properties are independent of the orientation of coordinate axes. In particular, the coefficients c_{ij} must remain invariant when we introduce new coordinate axes x_1', x_2', x_3', obtained by rotating the x_1, x_2, x_3-system through a right angle about the x_1-axis. By considering the transformed stress components τ_i', in exactly the same way as was done in the preceding section, it is found that

$$c_{12} = c_{13}, \qquad c_{31} = c_{21}, \qquad c_{32} = c_{23}, \qquad c_{33} = c_{22}, \qquad c_{66} = c_{55}.$$

Similarly, a rotation of axes through a right angle about the x_3-axis leads to the relations

$$c_{21} = c_{12}, \qquad c_{22} = c_{11}, \qquad c_{23} = c_{13}, \qquad c_{31} = c_{32}, \qquad c_{55} = c_{44}.$$

We introduce, finally, the coordinate system x_1', x_2', x_3', got from the x_1, x_2, x_3-system by rotating the latter through an angle of 45° about the x_3-axis. In this case, we have

$$\tau_{12}' = -\tfrac{1}{2}\tau_{11} + \tfrac{1}{2}\tau_{22}, \qquad e_{12}' = -\tfrac{1}{2}e_{11} + \tfrac{1}{2}e_{22},$$

or, noting the definitions on page 60,

$$\tau_6' = -\tfrac{1}{2}\tau_1 + \tfrac{1}{2}\tau_2, \qquad e_6' = -e_1 + e_2.$$

From (21.6) and the relation $c_{66} = c_{44}$, we have

$$\tau_6 = c_{44}e_6.$$

When referred to the x_1', x_2', x_3'-axes, this becomes $\tau_6' = c_{44}e_6'$ or

$$-\tfrac{1}{2}\tau_1 + \tfrac{1}{2}\tau_2 = c_{44}(-e_1 + e_2). \tag{22.1}$$

Now from (21.6)

$$\begin{aligned} \tau_1 &= c_{11}e_1 + c_{12}e_2 + c_{13}e_3, \\ \tau_2 &= c_{21}e_1 + c_{22}e_2 + c_{23}e_3, \end{aligned}$$

[1] Many cast metals are notable exceptions. The processes of rolling and drawing frequently produce a definite orientation of crystals, so that many rolled and drawn metals are anisotropic.

and from the relations given above, namely,

$$c_{22} = c_{11}, \qquad c_{23} = c_{13} = c_{21} = c_{12},$$

we get

$$-\tfrac{1}{2}\tau_1 + \tfrac{1}{2}\tau_2 = \tfrac{1}{2}(c_{11} - c_{12})(-e_1 + e_2).$$

Comparison of this equation with (22.1) yields the result

$$c_{44} = \tfrac{1}{2}(c_{11} - c_{12}) \equiv \mu, \tag{22.2}$$

so that

$$\tau_6 = \mu e_6.$$

We shall find it convenient to write the generalized Hooke's law for an isotropic body in terms of the two constants λ and μ, where μ is defined by (22.2) and where we put

$$c_{12} = \lambda.$$

From (21.6) we can now write

$$\begin{aligned}\tau_{11} &= c_{11}e_{11} + c_{12}e_{22} + c_{12}e_{33}\\ &= c_{12}(e_{11} + e_{22} + e_{33}) + (c_{11} - c_{12})e_{11}\\ &= \lambda\vartheta + 2\mu e_{11}.\end{aligned}$$

Thus, the generalized Hooke's law for a homogeneous isotropic body can be written in the following form:

$$\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}, \qquad (i, j = 1, 2, 3). \tag{22.3}$$

Equation (22.3) yields a simple relation connecting the invariants $\vartheta = e_{ii}$ and $\Theta = \tau_{ii}$.

Putting $j = i$ in (22.3) and noting that $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$, one finds that

$$\Theta \equiv \tau_{ii} = 3\lambda\vartheta + 2\mu e_{ii},$$

or

$$\Theta = (3\lambda + 2\mu)\vartheta. \tag{22.4}$$

Equations (22.3) can now be solved easily for the strains e_{ij} in terms of the stresses τ_{ij}. We have

$$e_{ij} = \frac{-\lambda}{2\mu}\,\delta_{ij}\vartheta + \frac{1}{2\mu}\,\tau_{ij},$$

or

$$e_{ij} = \frac{-\lambda\delta_{ij}}{2\mu(3\lambda + 2\mu)}\,\Theta + \frac{1}{2\mu}\,\tau_{ij}. \tag{22.5}$$

It is clear from (22.5) that we must require that $\mu \neq 0$ and $3\lambda + 2\mu \neq 0$.

The constants λ and μ were introduced by G. Lamé and are called the *Lamé constants.*

We have shown that the stress-strain law for isotropic media involves no more than two elastic constants. The fact that no further reduction is possible is physically obvious from the simple tensile tests, but an

analytic proof of this, utilizing the properties of isotropic tensors, can be constructed.[1]

If the axes x_i are directed along the principal axes of strain, then $e_{23} = e_{31} = e_{12} = 0$. But from (22.3) we see that in this case τ_{23}, τ_{31}, and τ_{12} also vanish. Hence the axes x_i must lie along the principal axes of stress, and we have the result that *the principal axes of stress are coincident with the principal axes of strain if the medium is isotropic.* This property was used by Cauchy to define the isotropic elastic medium.

Henceforth no distinction will be made between the principal axes of strain and those of stress, and such axes will be referred to simply as the *principal axes.*

23. Elastic Moduli for Isotropic Media. Simple Tension. Pure Shear. Hydrostatic Pressure. In order to gain some insight into the physical significance of elastic constants entering in formulas (22.3), we consider the behavior of elastic bodies subjected to simple tension, pure shear, and hydrostatic pressure.

Assume that a right cylinder with the axis parallel to the x_1-axis is subjected to the action of longitudinal forces applied to the ends of the cylinder. If the applied forces give rise to a uniform tension T in every cross section of the cylinder, then

$$\tau_{11} = T = \text{const}, \qquad \tau_{22} = \tau_{33} = \tau_{12} = \tau_{23} = \tau_{31} = 0. \tag{23.1}$$

Since the body forces are not present, the state of stress determined by (23.1) satisfies the equilibrium equations (15.3) in the interior of the cylinder, and equations (13.3) show that the lateral surface of the cylinder is free of tractions.

The substitution from (23.1) in (22.5) yields the appropriate values of strains, namely,[2]

$$\begin{cases} e_{11} = \dfrac{(\lambda + \mu)T}{\mu(3\lambda + 2\mu)}, & e_{22} = e_{33} = \dfrac{-\lambda T}{2\mu(3\lambda + 2\mu)}, \\ e_{12} = e_{23} = e_{31} = 0, \end{cases} \tag{23.2}$$

which clearly satisfy the compatibility equations (10.9). Accordingly, the state of stress (23.1) actually corresponds to the one that can exist in a deformed elastic body.

Noting that

$$\frac{e_{22}}{e_{11}} = \frac{-\lambda}{2(\lambda + \mu)},$$

we introduce the abbreviations

$$\sigma \equiv \frac{\lambda}{2(\lambda + \mu)}, \qquad E \equiv \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{23.3}$$

[1] H. Jeffreys, Cartesian Tensors (1931).

[2] The integration of Eqs. (23.2), yielding the displacements u_i, is carried out in Sec. 30.

Then Eqs. (23.2) can be written in the form

$$(23.4)\qquad \begin{cases} e_{11} = \dfrac{1}{E}\,T, & e_{22} = e_{33} = \dfrac{-\sigma}{E}\,T = -\sigma e_{11}, \\ e_{12} = e_{23} = e_{31} = 0. \end{cases}$$

If the stress T represents tension, so that $T > 0$, then a tensile stress will produce an extension in the direction of the axis of the cylinder and a contraction in its cross section. Accordingly, for $T > 0$, we have $e_{11} > 0$, $e_{22} < 0$, $e_{33} < 0$. It follows that E and σ are both positive.

Physical interpretations of the elastic moduli E and σ are easily obtained. It follows from the first of the formulas (23.4) that the quantity

$$E = \frac{T}{e_{11}}$$

represents the ratio of the tensile stress T to the extension e_{11} produced by the stress T. Again, from (23.4), it is seen that

$$\sigma = \left|\frac{e_{22}}{e_{11}}\right| = \left|\frac{e_{33}}{e_{11}}\right|;$$

thus σ denotes the ratio of the contraction of the linear elements perpendicular to the axis of the cylinder to the longitudinal extension of the rod. The quantity E is known as *Young's modulus*, and the number σ is called the *Poisson ratio.*

It is easy to verify that one can express the constants λ and μ in terms of Young's modulus and Poisson's ratio as

$$(23.5)\qquad \lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)}.$$

Consider next the state of pure shear characterized by the stress components

$$\tau_{23} = T = \text{const}, \qquad \tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = \tau_{31} = 0.$$

Substituting these values in (22.5) yields

$$(23.6)\qquad e_{23} = \frac{1}{2\mu}\,T, \qquad e_{11} = e_{22} = e_{33} = e_{12} = e_{31} = 0.$$

These formulas show that a rectangular parallelepiped $OPQR$, whose faces are parallel to the coordinate planes, is sheared in the x_2x_3-plane (see Fig. 4) so that the right angle between the edges of the parallelepiped parallel to the x_2- and x_3-axes is diminished, for $T > 0$, by the angle $\alpha_{23} = 2e_{23}$. From (23.6) we have

$$\mu = \frac{T}{\alpha_{23}}.$$

Thus the number μ represents the ratio of the shearing stress T to the change in angle α_{23} produced by the shearing stress. For this reason the

quantity μ is called the *modulus of rigidity*, or the *shear modulus*. Since E and σ are both positive, it follows from the second of Eqs. (23.5) that μ is also positive.

Finally consider a body τ of arbitrary shape subjected to a hydrostatic pressure of uniform intensity p distributed over its surface. The components $\overset{\nu}{T}_i$ of the stress vector acting on the surface are then

$$\overset{\nu}{T}_i = -p\nu_i,$$

where ν_i are the direction cosines of the normal $\boldsymbol{\nu}$ to the surface.

The system of stresses

$$\text{(23.7)} \qquad \begin{cases} \tau_{11} = \tau_{22} = \tau_{33} = -p, & \tau_{12} = \tau_{23} = \tau_{31} = 0, \\ \Theta = \tau_{11} + \tau_{22} + \tau_{33} = -3p, & \end{cases}$$

satisfies the equilibrium equation in the interior of τ and on its surface. From (22.5) we deduce the expressions[1]

$$\text{(23.8)} \qquad e_{11} = e_{22} = e_{33} = -\frac{p}{3\lambda + 2\mu}, \qquad e_{12} = e_{23} = e_{31} = 0,$$

which, clearly, satisfy the compatibility equations (10.9). The cubical compression $\vartheta = e_{ii}$ can be obtained either from (23.8) or from the general relations (22.4) and (23.7). We get

$$\vartheta = e_{11} + e_{22} + e_{33} = -\frac{p}{\lambda + \frac{2}{3}\mu},$$

which can be written as

$$\vartheta = -\frac{p}{k}, \qquad \text{or} \qquad k = -\frac{p}{\vartheta},$$

by introducing the abbreviation

$$\text{(23.9)} \qquad k = \lambda + \tfrac{2}{3}\mu.$$

[1] If the substitution from (23.5) in (23.8) is made, we find that

$$e_{11} = e_{22} = e_{33} = \frac{-p(1-2\sigma)}{E}, \qquad e_{ij} = 0 \text{ for } i \neq j.$$

Since $u_{i,j} + u_{j,i} = 2e_{ij}$, we have for the determination of displacements the system of equations,

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} = -p\,\frac{1-2\sigma}{E}, \qquad \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, \qquad i \neq j.$$

The integration of these equations yields [cf. Sec. 30]

$$u_i = -\frac{p}{3k}x_i + \alpha_{0i} + \alpha_{ij}x_j, \qquad k = \frac{E}{3(1-2\sigma)},$$

where $\alpha_{ij} = -\alpha_{ji}$ and the α_{0i} are the integration constants. These integration constants are associated with the rigid body motion. If we fix the point $x_i = 0$ (assumed to be in the body) and impose the condition that the rotation vector ω_i (Sec. 7) vanishes, we get

$$u_i = -\frac{p}{3k}x_i.$$

Thus, the quantity k represents the ratio of the compressive stress to the cubical compression, and for this reason it is called the *modulus of compression.* Since for all physical substances a hydrostatic pressure tends to diminish the bulk, it is clear that k is positive. Substituting in (23.9) the expressions for λ and μ from (23.5) gives

$$k = \frac{E}{3(1 - 2\sigma)}.$$

Since k is positive for all physical substances, *it follows that σ is less than one-half, and hence* [see (23.5)] λ *is positive.* For most structural materials, the value of σ does not deviate much from one-third. If the material is highly incompressible (rubber, for example), σ is nearly one-half and $\mu \doteq E/3$.

The stress-strain relations (22.5), when written by making the substitutions from (23.5), assume the simple form

$$e_{ij} = \frac{1 + \sigma}{E} \tau_{ij} - \frac{\sigma}{E} \delta_{ij}\Theta, \tag{23.10}$$

where $\Theta = \tau_{ii}$. If we recall the notation of Sec. 14, these relations can also be given in the following form:

$$\tag{23.11} \left\{ \begin{aligned} e_{xx} &= \frac{1}{E}[\tau_{xx} - \sigma(\tau_{yy} + \tau_{zz})], \\ e_{yy} &= \frac{1}{E}[\tau_{yy} - \sigma(\tau_{zz} + \tau_{xx})], \\ e_{zz} &= \frac{1}{E}[\tau_{zz} - \sigma(\tau_{xx} + \tau_{yy})], \\ e_{yz} &= \frac{1+\sigma}{E}\tau_{yz}, \qquad e_{zx} = \frac{1+\sigma}{E}\tau_{zx}, \qquad e_{xy} = \frac{1+\sigma}{E}\tau_{xy}. \end{aligned} \right.$$

The following table gives average values of E, μ, and σ for several elastic materials; the moduli E and μ are given in millions of pounds per square inch.[1]

	E	μ	σ (experimental)	$\sigma = \frac{E}{2\mu} - 1$
Carbon steels	29.5	11.5	0.29	0.283
Wrought iron	28.0	11.0	0.28	0.273
Cast iron	16.5	6.5	0.25	0.269
Copper (hot-rolled)	15.0	5.6	0.33	0.339
Brass, 2:1 (cold-drawn)	13.0	4.9	0.33	0.327
Glass	8.0	3.2	0.25	0.250
Spruce (along the grain)	1.5	0.08		

[1] In the engineering literature, the modulus of shear is often denoted by G, and the reciprocal of Poisson's ratio σ is denoted by m; that is, $m = 1/\sigma$.

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A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Secs. 69–71.

E. Trefftz: Handbuch der Physik, Verlag von Julius Springer, Berlin, vol. 6, Secs. 11–12.

PROBLEMS

1. Show that Hooke's law in the form (23.11) can be obtained by the following argument: An elementary rectangular parallelepiped subjected to tensile stresses τ_{xx} on opposite faces will experience a longitudinal extension $e_{xx} = \tau_{xx}/E$ and lateral contractions $e_{yy} = e_{zz} = -\sigma e_{xx}$. Now consider the effect of stresses τ_{xx}, τ_{yy}, τ_{zz}, and superpose the resulting strains to get Eq. (23.11).

2. Use Hooke's law to show that the stress invariant $\Theta = \tau_{ii}$ and the strain invariant $\vartheta = e_{ii}$ are connected by the relation $\Theta = 3k\vartheta$, where k is the modulus of compression.

3. Show that a stress vector cannot cross a free surface (one on which there is no external load). *Hint:* Let ν be the normal to the free surface. Then $\overset{\nu}{\mathbf{T}} = 0$ and, from (16.1), $\overset{\nu'}{\mathbf{T}} \cdot \nu = \overset{\nu}{\mathbf{T}} \cdot \nu' = 0$.

4. Derive the following relations between the Lamé coefficients λ and μ, Poisson's ratio σ, Young's modulus E, and the bulk modulus k:

$$\begin{aligned}
\lambda &= \frac{2\mu\sigma}{1-2\sigma} = \frac{\mu(E-2\mu)}{3\mu-E} = k - \frac{2}{3}\mu = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \\
&= \frac{3k\sigma}{1+\sigma} = \frac{3k(3k-E)}{9k-E}, \\
\mu &= \frac{\lambda(1-2\sigma)}{2\sigma} = \frac{3}{2}(k-\lambda) = \frac{E}{2(1+\sigma)} = \frac{3k(1-2\sigma)}{2(1+\sigma)} \\
&= \frac{3kE}{9k-E}, \\
\sigma &= \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{3k-\lambda} = \frac{E}{2\mu} - 1 = \frac{3k-2\mu}{2(3k+\mu)} \\
&= \frac{3k-E}{6k}, \\
E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma} = \frac{9k(k-\lambda)}{3k-\lambda} \\
&= 2\mu(1+\sigma) = \frac{9k\mu}{3k+\mu} = 3k(1-2\sigma), \\
k &= \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\sigma)}{3\sigma} = \frac{2\mu(1+\sigma)}{3(1-2\sigma)} \\
&= \frac{\mu E}{3(3\mu-E)} = \frac{E}{3(1-2\sigma)}.
\end{aligned}$$

24. Equilibrium Equations for an Isotropic Elastic Solid. The complete system of equations of equilibrium of a homogeneous isotropic elastic solid is made up of the following equations:

a. Equations of Equilibrium. From (15.3)

$$\tau_{ij,j} + F_i = 0, \qquad (i, j = 1, 2, 3); \tag{24.1}$$

b. Stress-Strain Relations. From (22.3)

$$\tau_{ij} = \lambda\, \delta_{ij}\vartheta + 2\mu e_{ij}, \tag{24.2}$$

where

$$\vartheta = e_{ii},$$

and [from (7.5)]

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{24.3}$$

The systems of Eqs. (24.1) and (24.2) must be satisfied at every interior point of the body τ, and on the surface Σ of the body τ the stresses must fulfill the equilibrium conditions (13.3)

$$\tau_{ij}\nu_j = \overset{\nu}{T}_i, \tag{24.4}$$

where the ν_i are the direction cosines of the exterior normal $\mathbf{v}$ to the surface Σ, and $\overset{\nu}{\mathbf{T}}$ is the stress vector acting on the surface element with normal $\mathbf{v}$. To these equations one must adjoin the equations of compatibility [from (10.9)]

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0. \tag{24.5}$$

It will be shown in Sec. 27 that the system of Eqs. (24.1) and (24.2), subject to the conditions of equilibrium on the surface (24.4), is complete in the sense that, if there exists a solution of the system, then that solution is unique. There are nine equations in the system on the set of nine unknown functions τ_{ij}, u_i $(i, j = 1, 2, 3)$. Once the displacements u_i are determined, the strain components e_{ij} entering into (24.2) are readily calculated with the aid of the formulas (24.3). We have assumed that the displacements u_i are continuous functions of class C^3 throughout the region τ, and a reference to (24.2) shows that the components of stress τ_{ij} are continuous of class C^2 in the same region. The equations of equilibrium (24.1) contain the components F_i of the body force $\mathbf{F}$, and they are assumed to be prescribed functions of the coordinates x_i of the undeformed body. Typical examples of the body forces $\mathbf{F}$, occurring in practical applications, are centrifugal forces and forces of gravitation. Furthermore, the components $\overset{\nu}{T}_i$ of the external surface force $\overset{\nu}{\mathbf{T}}$ are assumed to be prescribed functions of the coordinates x_i of the undeformed surface Σ of the body.

In order that the solution of the problem may exist, it is clear that one cannot prescribe the body force $\mathbf{F}$ and the surface force $\overset{\nu}{\mathbf{T}}$ in a perfectly arbitrary manner, inasmuch as Eqs. (24.1) were established on the hypothesis that the body is in equilibrium. Hence one must demand that the distribution of the forces $\mathbf{F}$ and $\overset{\nu}{\mathbf{T}}$, acting on the body τ, be such that the resultant force and the resultant moment vanish.[1]

[1] That is, $\mathbf{F}$ and $\overset{\nu}{\mathbf{T}}$ must be sufficiently regular and satisfy, for the body as a whole, the equations immediately preceding (15.1), and (15.4).

It is clear from physical considerations that, instead of prescribing the distribution of the surface force $\overset{\nu}{\mathbf{T}}$ acting on Σ, one could prescribe the displacements u_i on the surface Σ and that the state of stress established in the interior of the body by deforming its surface Σ must also be characterized in a unique way. Thus, we are led to consider the following fundamental boundary-value problems of elasticity:

Problem 1. *Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known.*

Problem 2. *Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the displacements of the points on the surface of the body are prescribed functions.*

In many applications, it is important to consider a problem resulting from the combination of the problems stated above. Thus, one may have the displacements of the points on part of the surface prescribed and the distribution of forces specified over the remaining portion. Such a problem will be referred to as a mixed boundary-value problem.

It should be noted that in Prob. 1 the external forces are assigned over the initial, or undeformed, surface of the body, while the equilibrium under these forces is reached when the body is in the final deformed state. Since the displacements are small, the error introduced in this approximation has the order of magnitude implicit in the formulation of the stress-strain relations, as stated in the concluding paragraph of Sec. 21.

The formulation of the fundamental boundary-value problems of elasticity given above suggests the desirability of expressing the differential equations for Prob. 1 entirely in terms of stresses and those for Prob. 2 entirely in terms of displacements. This is not difficult to do.

Let us first obtain the equations in terms of displacements u_i by substituting in (24.1) the expressions for stresses in terms of displacements. Making use of the formulas (24.3), we can write the system (24.2) in the form

$$\tau_{ij} = \lambda\, \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i}). \tag{24.6}$$

Substituting the values of the stress components (24.6) in the equilibrium equations (24.1) gives

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + F_i = 0,$$

or

$$\mu \nabla^2 u_i + (\lambda + \mu) \frac{\partial \vartheta}{\partial x_i} + F_i = 0 \tag{24.7}$$

where

$$\vartheta = e_{ii} = u_{i,i} = \operatorname{div} \mathbf{u}.$$

Equations (24.7) are associated with the name of Navier.

Note that we need not adjoin the compatibility equations (10.9), for the only purpose of the latter is to impose restrictions on the strain components that shall ensure that the e_{ij} yield single-valued continuous displacements u_i, when the region τ is simply connected.

It is clear that Prob. 2 is completely solved if one obtains the solution of the system (24.7) subject to the boundary conditions

$$u_i = f_i(x_1, x_2, x_3), \qquad (i = 1, 2, 3),$$

where the f_i are prescribed continuous functions on the boundary of the undeformed solid. From the knowledge of the functions u_i, one can determine the strains, and hence the stresses by making use of the relations (24.2).

We now turn our attention to the first boundary-value problem. It was noted earlier that not every solution of the system of three equations of equilibrium (24.1) corresponds to a possible state of strain in an elastic body, because the components of strain, defined by the system of Eqs. (23.10), must satisfy the equations of compatibility (24.5). We proceed to derive the compatibility equations in terms of the stresses. If the expressions (23.10)

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta$$

are inserted in the compatibility equations (24.5)

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0,$$

we obtain

$$(24.8) \quad \tau_{ij,kl} + \tau_{kl,ij} - \tau_{ik,jl} - \tau_{jl,ik} = \frac{\sigma}{1+\sigma}(\delta_{ij}\Theta_{,kl} + \delta_{kl}\Theta_{,ij} - \delta_{ik}\Theta_{,jl} - \delta_{jl}\Theta_{,ik}).$$

Since the indices i, j, k, l assume values 1, 2, 3, there are $3^4 = 81$ equations in the system (24.5), but not all these are independent, for an interchange of i and j or of k and l obviously does not yield new equations. Also for certain values of the indices (such as $i = j = k = l$), Eqs. (24.5) are identically satisfied, and, as already noted in Sec. 10, the set of Eqs. (24.5) contains only six independent equations obtained by setting

$$\begin{array}{lll} k = l = 1, & i = j = 2; & \\ k = l = 2, & i = j = 3; & \\ k = l = 3, & i = j = 1; & \\ k = l = 1, & i = 2, & j = 3; \\ k = l = 2, & i = 3, & j = 1; \\ k = l = 3, & i = 1, & j = 2. \end{array}$$

Inasmuch as Eqs. (23.10) establish one-to-one correspondence between the e_{ij} and the τ_{ij}, the set of 81 equations (24.8) likewise contains only 6

independent equations. If we combine Eqs. (24.8) linearly by setting $k = l$ and summing with respect to the common index, we get

$$\tau_{ij,kk} + \tau_{kk,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}(\delta_{ij}\Theta_{,kk} + \delta_{kk}\Theta_{,ij} - \delta_{ik}\Theta_{,jk} - \delta_{jk}\Theta_{,ik}).$$

This is a set of 9 equations of which only 6 are independent because of the symmetry in i and j. Consequently, in combining Eqs. (24.8) linearly, the number of independent equations is not reduced, and hence the resultant set of equations is equivalent to the original one.

Noting that

$$\tau_{ij,kk} = \nabla^2\tau_{ij}$$

and

$$\tau_{kk} = \Theta,$$

the foregoing equations can be written as

$$(24.9)\qquad \nabla^2\tau_{ij} + \Theta_{,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}(\delta_{ij}\nabla^2\Theta + 3\Theta_{,ij} - 2\Theta_{,ij}),$$

if we make use of the continuity of the second derivatives of Θ.

Equations (24.9) can be written more neatly by utilizing the equations of equilibrium (24.1)

$$\tau_{ik,k} + F_i = 0.$$

Thus, differentiating (24.1) with respect to x_j, we get

$$(24.10)\qquad \tau_{ik,kj} = -F_{i,j}$$

and since $\tau_{ik,kj} = \tau_{ik,jk}$, we can rewrite (24.9) in the form

$$(24.11)\qquad \nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = -(F_{i,j} + F_{j,i}).$$

This set of 6 independent equations can be further simplified by expressing an invariant $\nabla^2\Theta$ in terms of the derivatives of the body force **F**. This may be done as follows:

If we set $k = i$ and $l = j$ in (24.8) and sum with respect to the common indices, we get

$$2\tau_{ij,ij} - \tau_{ii,jj} - \tau_{jj,ii} = \frac{\sigma}{1+\sigma}(2\,\delta_{ij}\Theta_{,ij} - \delta_{ii}\Theta_{,jj} - \delta_{jj}\Theta_{,ii}).$$

But

$$\tau_{ii} = \tau_{jj} = \Theta, \qquad \delta_{ij}\Theta_{,ij} = \Theta_{,ii} = \nabla^2\Theta,$$

and

$$\delta_{ii}\Theta_{,jj} = \delta_{jj}\Theta_{,ii} = 3\nabla^2\Theta.$$

The foregoing equation can be written as

$$\tau_{ij,ij} - \nabla^2\Theta = \frac{\sigma}{1+\sigma}(\nabla^2\Theta - 3\nabla^2\Theta)$$

or

$$\tau_{ij,ij} = \frac{1-\sigma}{1+\sigma} \nabla^2\Theta. \tag{24.12}$$

The differentiation of the equilibrium equation

$$\tau_{ij,i} = -F_j$$

gives

$$\tau_{ij,ij} = -F_{j,j},$$

and inserting this in the left-hand member of (24.12) yields the formula

$$\nabla^2\Theta = -\frac{1+\sigma}{1-\sigma} F_{j,j} \equiv -\frac{1+\sigma}{1-\sigma} \operatorname{div} \mathbf{F}. \tag{24.13}$$

Substituting from (24.13) in (24.11) gives the final form of the compatibility equation in terms of stresses,

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{\sigma}{1-\sigma}\delta_{ij} \operatorname{div} \mathbf{F} - (F_{i,j} + F_{j,i}). \tag{24.14}$$

Equations (24.14), when written out in unabridged notation, yield the following 6 equations of compatibility:

$$\left\{\begin{aligned}
\nabla^2\tau_{xx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\mathbf{F} - 2\frac{\partial F_x}{\partial x},\\
\nabla^2\tau_{yy} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\mathbf{F} - 2\frac{\partial F_y}{\partial y},\\
\nabla^2\tau_{zz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\mathbf{F} - 2\frac{\partial F_z}{\partial z},\\
\nabla^2\tau_{yz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z} &= -\left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right),\\
\nabla^2\tau_{zx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z\,\partial x} &= -\left(\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z}\right),\\
\nabla^2\tau_{xy} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\,\partial y} &= -\left(\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x}\right).
\end{aligned}\right. \tag{24.15}$$

Equations (24.15) were obtained by Michell in 1900 and, for the case when the body forces are absent, by Beltrami in 1892. They are known as the Beltrami-Michell *compatibility equations*. Thus, in order to determine the state of stress in the interior of an elastic body, one must solve the system of equations consisting of (24.1) and (24.15) subject to the boundary conditions (24.4).

The system of Eqs. (24.1) and (24.15) is equivalent to the system consisting of Eqs. (24.1), (24.2), and (24.5).

If the field of body force $\mathbf{F}$ is conservative, so that

$$\mathbf{F} = \nabla\varphi$$

or

$$F_j = \varphi_{,j},$$

then

$$\operatorname{div} \mathbf{F} \equiv F_{j,j} = \varphi_{,jj} \equiv \nabla^2\varphi,$$

and

$$F_{i,j} = \varphi_{,ij}, \qquad F_{j,i} = \varphi_{,ji} = \varphi_{,ij},$$

so that (24.14) can be written as

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\,\Theta_{,ij} = -\frac{\sigma}{1-\sigma}\,\delta_{ij}\nabla^2\varphi - 2\varphi_{,ij}. \tag{24.16}$$

We shall consider two particular cases of body forces, namely, the case in which **F** is a constant vector and that in which the potential function φ is harmonic (that is, $\operatorname{div} \mathbf{F} = \nabla^2\varphi = 0$).

If **F** is constant, then φ is a linear function. In this case the right-hand member of (24.16) vanishes, and we obtain the equations of Beltrami,

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\,\Theta_{,ij} = 0. \tag{24.17}$$

From (24.13) it follows that in this case

$$\nabla^2\Theta = 0,$$

so that $\Theta = \tau_{ii}$ is a harmonic function. Equation (22.4) shows that the strain invariant $\vartheta = e_{ii}$ is also harmonic; that is,

$$\nabla^2\vartheta = 0$$

whenever Θ is harmonic. From (24.17) it is seen that, if the τ_{ij} are of class C^4, the components of stress satisfy the *biharmonic* equation

$$\nabla^2\nabla^2\tau_{ij} \equiv \nabla^4\tau_{ij} = 0,$$

and since the strain components e_{ij} are linear functions of the τ_{ij}, we have

$$\nabla^4 e_{ij} = 0.$$

A function V of class C^4, and satisfying the equation $\nabla^4 V = 0$, is called a *biharmonic function*.

If the body force **F** is derived from a harmonic potential function, so that

$$\operatorname{div} \mathbf{F} = \nabla^2\varphi = 0,$$

then from (24.13) and (22.4) we see that

$$\nabla^2\Theta = 0, \qquad \text{and} \qquad \nabla^2\vartheta = 0.$$

We can thus enunciate a theorem.

THEOREM: *When the components of the body force* **F** *are constant, the invariants Θ and ϑ are harmonic functions and the stress components τ_{ij} and strain components e_{ij} are biharmonic functions.*

When **F** *is derived from a harmonic potential function, the invariants* Θ *and* ϑ *are also harmonic.*

It will be shown, with the aid of some general theorems to be established in Sec. 26, that Probs. 1 and 2 have essentially unique solutions. Before proceeding to derive these theorems, however, we may note that, on account of the linear character of Eqs. (24.1), (24.2), and (24.3), the principle of superposition is applicable to the fundamental problems of elasticity.

Thus, suppose that one finds a set of nine functions

$$\tau_{ij}^{(1)},\ u_i^{(1)}, \qquad (i, j = 1, 2, 3),$$

which satisfy the systems (24.1) and (24.2) with prescribed body forces $F_i^{(1)}$. Also let a set of functions

$$\tau_{ij}^{(2)},\ u_i^{(2)}, \qquad (i, j = 1, 2, 3) \tag{24.18}$$

be the solutions of the systems corresponding to the choice of the body forces $F_i^{(2)}$. Then it is obvious that the solution

$$\tau_{ij} = \tau_{ij}^{(1)} + \tau_{ij}^{(2)}, \qquad u_i = u_i^{(1)} + u_i^{(2)}, \qquad (i = j = 1, 2, 3) \tag{24.19}$$

will correspond to the choice of the body force whose components are $F_i^{(1)} + F_i^{(2)}$. If the set of functions (24.18) represents a solution of the homogeneous system, that is, when $F_i^{(2)} = 0$, then the expressions (24.19) represent a solution of the problem corresponding to the choice of the body force with components $F_i^{(1)}$.

PROBLEMS

1. Show that the following stress components are not the solution of a problem in elasticity, even though they satisfy the equations of equilibrium with zero body forces:

$$\begin{aligned}
\tau_{xx} &= c[y^2 + \sigma(x^2 - y^2)], \qquad c \neq 0,\\
\tau_{yy} &= c[x^2 + \sigma(y^2 - x^2)],\\
\tau_{zz} &= c\sigma(x^2 + y^2),\\
\tau_{xy} &= -2c\sigma xy,\\
\tau_{yz} &= \tau_{zx} = 0.
\end{aligned}$$

2. The solutions of many problems in elasticity are either exactly or approximately independent of the value chosen for Poisson's ratio. This fact suggests that approximate solutions may be found by so choosing Poisson's ratio as to simplify the problem. Show that, if one takes $\sigma = 0$, then

$$\lambda = 0, \qquad \mu = \tfrac{1}{2}E, \qquad k = \tfrac{1}{3}E,$$

and Hooke's law is expressed by

$$\tau_{ij} = Ee_{ij} = \tfrac{1}{2}E(u_{i,j} + u_{j,i}).$$

Show by differentiation of these equations that

$$\tau_{ij,ij} = \tfrac{1}{2}(\tau_{ii,jj} + \tau_{jj,ii})$$

(no sum on repeated subscripts). That is, the six stress components are connected, in this case, by the three equilibrium equations

$$\tau_{ij,j} + F_i = 0$$

and by three compatibility equations, namely,

$$\frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} = \frac{1}{2}\left(\frac{\partial^2 \tau_{xx}}{\partial y^2} + \frac{\partial^2 \tau_{yy}}{\partial x^2}\right),$$

and two similar equations obtained by cyclic interchange of x, y, z. Derive these compatibility conditions from Eq. (24.8) by setting $\sigma = 0$, $k = i$, $l = j$.

A. and L. Föppl have discussed[1] the simplification of the equations of elasticity obtained by choosing for Poisson's ratio $\sigma = 0$ or $\sigma = \frac{1}{2}$. Westergaard[2] has treated the problem of obtaining the general solution from a solution for a particular choice of Poisson's ratio.

3. Define the stress function S by

$$\tau_{ij} = S_{,ij} \equiv \frac{\partial^2 S}{\partial x_i\, \partial x_j}$$

and consider the case of zero body force. Show that, if Poisson's ratio σ is assumed to vanish, then the equilibrium and compatibility equations given in the preceding problem reduce to

$$\nabla^2 S = \text{const.}$$

4. Show that, if Poisson's ratio σ has the value $\frac{1}{2}$, then

$$\mu = \tfrac{1}{3}E, \qquad \lambda = \infty, \qquad k = \infty, \qquad \vartheta \equiv e_{jj} = u_{i,i} = 0.$$

Interpret physically the situation described by these elastic coefficients. From Hooke's law (23.10) deduce the relations

$$\begin{aligned}\tau_{ij} &= 2\mu e_{ij} + \tfrac{1}{3}\, \delta_{ij}\Theta \\ &= \mu(u_{i,j} + u_{j,i}) + \tfrac{1}{3}\, \delta_{ij}\Theta.\end{aligned}$$

Show that in this case

$$u_{j,ij} = \frac{\partial \vartheta}{\partial x_i} = 0$$

and that the equilibrium equations (24.1) can be written in the form

$$\nabla^2 u_i + \frac{1}{\mu}\left(\frac{1}{3}\Theta_{,i} + F_i\right) = 0.$$

That is, putting $u_1 = u$, $u_2 = v$, etc., the four functions u, v, w, Θ are to be determined from the four equations

$$\begin{aligned}\nabla^2 u + \frac{1}{\mu}\left(\frac{1}{3}\frac{\partial \Theta}{\partial x} + F_x\right) &= 0,\\ \nabla^2 v + \frac{1}{\mu}\left(\frac{1}{3}\frac{\partial \Theta}{\partial y} + F_y\right) &= 0,\\ \nabla^2 w + \frac{1}{\mu}\left(\frac{1}{3}\frac{\partial \Theta}{\partial z} + F_z\right) &= 0,\\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.\end{aligned}$$

This case ($\sigma = \frac{1}{2}$) has been discussed at length by A. and L. Föppl.[1]

[1] A. and L. Föppl, Drang und Zwang, vol. 1, Sec. 3.

[2] H. M. Westergaard, "Effects of a Change of Poisson's Ratio Analysed by Twinned Gradients," *Journal of Applied Mechanics*, vol. 62 (1940), pp. A-113-A-116.

25. Dynamical Equations of an Isotropic Elastic Solid. The differential equations of motion of an elastic solid can be obtained at once from the equations of equilibrium (24.1) by invoking the Principle of D'Alembert and adding the forces of inertia to the components F_i of the body force. If $\rho(x_1, x_2, x_3)$ is the density of the medium, then the components of the force of inertia acting on the mass contained within the volume element $d\tau$ are[1] $-\rho \frac{\partial^2 u_i}{\partial t^2} d\tau$. Hence adding to the components F_i of the body force **F** in (24.1) the components of the force of inertia per unit volume gives the system of equations

$$\tau_{ij,j} + F_i = \rho \ddot{u}_i, \tag{25.1}$$

where we write $\frac{\partial^2 u_i}{\partial t^2} \equiv \ddot{u}_i$.

Inasmuch as the stress-strain relations (24.2) do not involve body forces, they remain valid in this case also. The displacements u_i are now regarded as functions of the space variables x_i and of the time t.

It follows that the dynamical equations in terms of the displacements u_i can be written at once by referring to the set of Eqs. (24.7). Thus,

$$\mu \nabla^2 u_i + (\lambda + \mu) \frac{\partial \vartheta}{\partial x_i} + F_i = \rho \ddot{u}_i. \tag{25.2}$$

To these equations it is necessary to adjoin the initial and the boundary conditions. Thus, at each point of the surface Σ of the undeformed medium, the surface forces $\overset{\nu}{T}_i$ or the displacements u_i must be prescribed. The functions u_i prescribed on the surface Σ, in general, are functions of the space coordinates x_i and of the time t. If the surface forces $\overset{\nu}{T}_i$ are prescribed as functions of x_i and t, then the components of stress must satisfy the usual equilibrium conditions (24.4) on the surface Σ and in addition one must know the initial conditions on the displacements u_i and on their time derivatives. We set forth these conditions explicitly for the fundamental boundary-value problems of dynamical elasticity that correspond to the problems of equilibrium in Sec. 24.

Problem 1. *Determine the displacements $u_i(x_1, x_2, x_3, t)$ that satisfy in τ the system of Eqs.* (25.2) *and satisfy the conditions*

$$u_i = u_i^0(x_1, x_2, x_3), \quad \frac{\partial u_i}{\partial t} = U_i^0(x_1, x_2, x_3), \qquad \textit{for } t = t_0 \textit{ throughout } \tau,$$

and that satisfy on the surface Σ of the region τ the boundary conditions

$$\overset{\nu}{T}_i = f_i(x_1, x_2, x_3, t) \qquad \textit{for } t \geq t_0.$$

[1] If ρ is a function of t, we write $-\frac{\partial}{\partial t}\left(\rho \frac{\partial u_i}{\partial t}\right) d\tau$.

Problem 2. *Determine the displacements* $u_i(x_1, x_2, x_3, t)$ *that satisfy in* τ *the system of Eqs.* (25.2) *and are such that on the surface* Σ *of* τ

$$u_i = U_i(x_1, x_2, x_3, t) \qquad \textit{for } t \geq t_0.$$

As in Sec. 24, we may consider a mixed boundary-value problem in which the surface forces $\overset{\nu}{T}_i$ are prescribed functions of x_i and t over part of the surface and the displacements u_i are given functions of x_i and t over the rest of the surface. As an example of such a problem, consider an elastic plate clamped at the edges. Let the plate, initially at rest, be subjected to a normal load varying with time. In this case, the displacements are known on the edges of the plate (for $t \geq t_0$), while the surface forces are given functions of x_1, x_2, x_3, t.

REFERENCES FOR COLLATERAL READING

A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Secs. 85, 86, 91, 92.

E. Trefftz: Handbuch der Physik, Verlag von Julius Springer, Berlin, vol. 6, Secs. 13-15.

26. The Strain-energy Function and Its Connection with Hooke's Law. We introduce the definition of the *unstrained*, or *natural*, state of a body as a standard state of uniform temperature and zero displacement, with reference to which all strains will be specified.

If the body is in the natural state at the instant of time $t = 0$, and if it is subjected to the action of external forces, then the latter may produce a deformation of the body and hence will do work. We shall be concerned with the rate at which work is done by the external body and surface forces. If (x_1, x_2, x_3) denote the coordinates of an arbitrary material point P of the body in the unstrained state, then at any time t the coordinates of the same material point P will be $x_i + u_i(x_1, x_2, x_3, t)$. Since the displacement of the point P in the interval of time $(t, t + dt)$ is given by

$$\frac{\partial u_i}{\partial t}\, dt \equiv \dot{u}_i\, dt,$$

it follows that the work done in dt sec by the body forces acting on the volume element $d\tau$ located at P is $F_i\dot{u}_i\, d\tau\, dt$. The work performed by the external surface forces in the same interval of time is $\overset{\nu}{T}_i\dot{u}_i\, d\sigma\, dt$, where $d\sigma$ is the element of surface. Denoting by $\mathcal{E}$ the total work done by the body and surface forces, we have the following expression for the rate of doing work on the matter originally occupying some region τ,

$$\frac{d\mathcal{E}}{dt} = \int_\tau F_i\dot{u}_i\, d\tau + \int_\Sigma \overset{\nu}{T}_i\dot{u}_i\, d\sigma; \tag{26.1}$$

here Σ denotes the original surface of the unstrained region τ.

Now the surface integral appearing in (26.1) can be expressed as a volume integral by substituting for the components of the surface force $\overset{\nu}{T}$ their values from Eqs. (24.4) and by making use of the Divergence Theorem. We have

$$\int_\Sigma \overset{\nu}{T}_i \dot{u}_i \, d\sigma = \int_\Sigma (\tau_{ij}\dot{u}_i)\nu_j \, d\sigma = \int_\tau (\tau_{ij}\dot{u}_i)_{,j} \, d\tau. \tag{26.2}$$

Carrying out the indicated differentiation in the integrand of the volume integral in (26.2) and recalling the formulas (7.5) give

$$\begin{aligned}\int_\Sigma \overset{\nu}{T}_i \dot{u}_i \, d\sigma &= \int_\tau \tau_{ij,j}\dot{u}_i \, d\tau + \int_\tau \tau_{ij}\dot{u}_{i,j} \, d\tau \\ &= \int_\tau \tau_{ij,j}\dot{u}_i \, d\tau + \int_\tau \tau_{ij}\left(\frac{\dot{u}_{i,j} + \dot{u}_{j,i}}{2} + \frac{\dot{u}_{i,j} - \dot{u}_{j,i}}{2}\right) d\tau \\ &= \int_\tau \tau_{ij,j}\dot{u}_i \, d\tau + \int_\tau (\tau_{ij}\dot{e}_{ij} + \tau_{ij}\dot{\omega}_{ij}) \, d\tau.\end{aligned}$$

But $\omega_{ij} = -\omega_{ji}$, so that $\tau_{ij}\dot{\omega}_{ij} = 0$, and hence

$$\int_\Sigma \overset{\nu}{T}_i \dot{u}_i \, d\sigma = \int_\tau (\tau_{ij,j}\dot{u}_i + \tau_{ij}\dot{e}_{ij}) \, d\tau. \tag{26.3}$$

A reference to the dynamical equations (25.1) shows that we can write

$$\tau_{ij,j}\dot{u}_i = (\rho\ddot{u}_i - F_i)\dot{u}_i.$$

When this is inserted in Eq. (26.3) and the resulting expression used in (26.1), one obtains

$$\frac{d\mathcal{E}}{dt} = \int_\tau \rho\ddot{u}_i\dot{u}_i \, d\tau + \int_\tau \tau_{ij}\dot{e}_{ij} \, d\tau. \tag{26.4}$$

The kinetic energy K of the body is defined as

$$K = \tfrac{1}{2} \int_\tau \rho\dot{u}_i\dot{u}_i \, d\tau,$$

and for the rate of change of kinetic energy we have[1]

$$\frac{dK}{dt} = \int_\tau \rho\ddot{u}_i\dot{u}_i \, d\tau.$$

Hence Eq. (26.4) can be written in the form

$$\frac{d\mathcal{E}}{dt} = \frac{dK}{dt} + \int_\tau \tau_{ij}\frac{\partial e_{ij}}{\partial t} \, d\tau. \tag{26.5}$$

We recall next the definitions,

$$\begin{array}{llllll}\tau_1 = \tau_{11}, & \tau_2 = \tau_{22}, & \tau_3 = \tau_{33}, & \tau_4 = \tau_{23}, & \tau_5 = \tau_{31}, & \tau_6 = \tau_{12}, \\ e_1 = e_{11}, & e_2 = e_{22}, & e_3 = e_{33}, & e_4 = 2e_{23}, & e_5 = 2e_{31}, & e_6 = 2e_{12},\end{array}$$

[1] It is assumed here that the variation of the density ρ with time is negligible.

used in Sec. 21, and suppose that there exists a function $W(e_1, e_2, \ldots, e_6)$ of the independent variables e_i such that[1]

$$\frac{\partial W}{\partial e_i} = \tau_i. \tag{26.6}$$

Then (26.5) can be written in the form

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \frac{dK}{dt} + \int_\tau \frac{\partial W}{\partial e_i}\frac{\partial e_i}{\partial t}\,d\tau \\ &= \frac{dK}{dt} + \frac{d}{dt}\int_\tau W\,d\tau.\end{aligned}$$

Integrating this equation with respect to t between the limits $t = 0$ and $t = t$, where $t = 0$ corresponds to the natural state, we obtain

$$\mathcal{E} = K + U, \tag{26.7}$$

where

$$U \equiv \int_\tau W\,d\tau, \tag{26.8}$$

since both K and $\mathcal{E}$ vanish in the natural state. The function W is called the *volume density of strain energy*, or the *elastic potential*, and U is the *strain energy* of the body.

Equation (26.7) has a simple physical interpretation. The work $\mathcal{E}$ done by the external forces in altering the configuration of the natural state to the state at the time t is equal to the sum of the kinetic energy K and the strain energy U. The strain energy U may be conceived as the energy stored in the body when it is brought from the configuration of the natural state to the state at the time t. If at the time t the body is in equilibrium, then $K = 0$ and $\mathcal{E} = U$.

We assume now that the strain-energy density function $W(e_1, e_2, \ldots, e_6)$ can be expanded in a power series

$$2W = c_0 + 2c_ie_i + c_{ij}e_ie_j + \cdots,$$

and discard all terms of order 3 and higher in the strains; the constant term c_0 can be disregarded[2] since we are interested only in the derivatives of W. Thus, we have, from (26.6),

$$\tau_i = c_i + \tfrac{1}{2}(c_{ij} + c_{ji})e_j.$$

If the τ_i are to vanish with the strains e_j, we must set $c_i = 0$ and thus obtain

$$W = \tfrac{1}{2}c_{ij}e_ie_j. \tag{26.9}$$

[1] We shall exhibit such a function for an isotropic elastic medium in formula (26.16).

[2] It is clearly associated with the initially prestressed state.

Hence

$$\tau_i = \frac{\partial W}{\partial e_i} = \frac{1}{2}(c_{ij} + c_{ji})e_j. \tag{26.10}$$

It is thus seen that the coefficients in the generalized Hooke's law are symmetric if the strain-energy density function, with the properties stated above, exists.

If the quadratic form (26.9) is symmetrized in advance, we can write (26.10) in the form

$$\tau_i = c_{ij}e_j, \tag{26.11}$$

where $c_{ij} = c_{ji}$.

Upon substituting from (26.11) in (26.9), we get the Clapeyron formula[1]

$$W = \tfrac{1}{2}\tau_i e_i \qquad (i = 1, 2, \ldots, 6),$$

which can also be written as

$$W = \tfrac{1}{2}\tau_{ij}e_{ij} \qquad (i, j = 1, 2, 3). \tag{26.12}$$

When the stress-strain law (26.11) is written in the form

$$e_i = C_{ij}\tau_j,$$

the formula of Clapeyron yields

$$W = \tfrac{1}{2}C_{ij}\tau_i\tau_j, \tag{26.13}$$

so that

$$\begin{aligned}\frac{\partial W}{\partial \tau_i} &= C_{ij}\tau_j \\ &= e_i.\end{aligned}$$

This result is due to A. Castigliano[2] (1847–1884).

We observe that the formula of Castigliano follows from the *assumed* linear stress-strain law (26.11). Green's formula (26.6), on the other hand, is the consequence of the assumed existence of the function W. The form of the stress-strain law defined by (26.6) depends on the structure of W. If W is the quadratic form

$$W = \tfrac{1}{2}c_{ij}e_ie_j, \qquad (i, j = 1, \ldots, 6), \tag{26.14}$$

which we suppose is symmetrized, then formula (26.6) yields the law (26.11).

In the linear theory of anisotropic elastic media W is taken in the form (26.14), which, in the most general case, contains 21 independent elastic

[1] Attributed to B. P. E. Clapeyron (1799–1864) by G. Lamé (1798–1870) in the 1852 edition of Lamé's *Leçons sur la théorie mathématique de l'élasticité des corps solides*.

[2] *Atti della reale accademia delle scienze di Torino* (1875).

constants c_{ij}. If the medium is isotropic, the stress-strain law has the form[1]

$$\tau_{ij} = \lambda\vartheta\delta_{ij} + 2\mu e_{ij}, \tag{26.15}$$

and it follows from (26.12) that

$$\begin{aligned} W &= \tfrac{1}{2}\lambda\vartheta e_{ii} + \mu e_{ij}e_{ij} \\ &= \tfrac{1}{2}\lambda\vartheta^2 + \mu(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{13}^2). \end{aligned} \tag{26.16}$$

A quadratic form that takes only positive values for every set of values of the independent variables, not all zero, is said to be *positive definite.* *Equation* (26.16) *shows that the strain-energy density W is a positive definite form in the strains e_{ij}, since both λ and μ are positive constants.* This important property of function W will be used in Sec. 27 to establish the uniqueness of solution of the fundamental boundary-value problems in the linear theory of elasticity.

As a consequence of the linear character of the stress-strain law (26.15), the function W is expressible as *a positive definite quadratic form in the stress components τ_{ij}.* Thus, on substituting from (23.10) in (26.12) we get,

$$W = -\frac{\sigma}{2E}\Theta^2 + \frac{1+\sigma}{2E}\tau_{ij}\tau_{ij}, \tag{26.17}$$

or

$$W = -\frac{\sigma}{2E}\Theta^2 + \frac{1+\sigma}{2E}(\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2) + \frac{1+\sigma}{E}(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2),$$

where $\Theta = \tau_{11} + \tau_{22} + \tau_{33}$.

It is easily checked that

$$\begin{aligned} \frac{\partial W}{\partial \tau_{ij}} &= \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\Theta\delta_{ij} \\ &= e_{ij}. \end{aligned}$$

It is clear that W, the energy of deformation per unit volume, has a physical meaning that is independent of the choice of coordinate axes, and hence it is invariant relative to all transformations of cartesian axes. It is also known that every invariant of a tensor e_{ij} can be expressed as a function of the principal invariants[2] ϑ, ϑ_2, ϑ_3. Inasmuch as W is a quadratic form in the e_{ij}, it cannot depend on ϑ_3, and hence it must involve only

$$\vartheta = e_{\text{I}} + e_{\text{II}} + e_{\text{III}} = e_{ii}$$

and

$$\vartheta_2 = e_{\text{II}}e_{\text{III}} + e_{\text{III}}e_{\text{I}} + e_{\text{I}}e_{\text{II}} = \tfrac{1}{2}\,\delta_{pq}^{ij}e_{pi}e_{qj}.$$

[1] It is worth recalling that this law was deduced in Sec. 22 without invoking the assumption that $c_{ij} = c_{ji}$ in the generalized Hooke's law (21.2).

[2] See (6.10).

We have, in fact,[1]

$$W = (\tfrac{1}{2}\lambda + \mu)\vartheta^2 - 2\mu\vartheta_2. \tag{26.18}$$

27. Uniqueness of Solution. Remarks on Existence of Solution. Before proceeding to the proof of the uniqueness of solution of the fundamental boundary-value problems of the linear theory of elasticity, we establish an important theorem concerning the strain-energy function.

CLAPEYRON'S THEOREM: *If a body is in equilibrium under a given system of body forces F_i and surface forces $\overset{\nu}{T}_i$, then the strain energy of deformation is equal to one-half the work that would be done by the external forces (of the equilibrium state) acting through the displacements u_i from the unstressed state to the state of equilibrium.*

The theorem asserts that

$$\int_\tau F_i u_i \, d\tau + \int_\Sigma \overset{\nu}{T}_i u_i \, d\sigma = 2 \int_\tau W \, d\tau. \tag{27.1}$$

Now the surface integral in (27.1) can be transformed in exactly the same way as was done in obtaining the formula (26.3). Making use of the equilibrium equations (15.3) and of the relation (26.12), we write

$$\int_\Sigma \overset{\nu}{T}_i u_i \, d\sigma = \int_\tau (\tau_{ij,j} u_i + \tau_{ij} e_{ij}) \, d\tau = \int_\tau (-F_i u_i + 2W) \, d\tau.$$

Then

$$\int_\tau F_i u_i \, d\tau + \int_\Sigma \overset{\nu}{T}_i u_i \, d\sigma = 2 \int_\tau W \, d\tau,$$

and the theorem is proved. This formula will be utilized in establishing the uniqueness of solution of the problems of equilibrium of an elastic solid.

It is clear that in order that the solutions of the equilibrium boundary-value problems (see Sec. 24) may exist, it is necessary to demand the vanishing of the resultant force and the resultant torque produced by the prescribed body and surface forces. This condition was implied in the derivation of the equilibrium equations (15.3).

In order to establish the uniqueness of solution of the boundary-value problems formulated in Sec. 24, assume that it is possible to obtain two solutions

$$u_i^{(1)}, \ \tau_{ij}^{(1)}, \qquad (i, j = 1, 2, 3), \tag{27.2}$$

and

$$u_i^{(2)}, \ \tau_{ij}^{(2)}, \qquad (i, j = 1, 2, 3). \tag{27.3}$$

Because of the linear character of the differential equations, it is clear

[1] A student interested in the anatomy of stress-strain relations in nonlinear elasticity may wish to read a paper by M. Reiner, entitled "Elasticity beyond the Elastic Limit," *American Journal of Mathematics*, vol. 70 (1948), pp. 433–446.

that the set of functions defined by the formulas

$$u_i \equiv u_i^{(1)} - u_i^{(2)}, \qquad \tau_{ij} \equiv \tau_{ij}^{(1)} - \tau_{ij}^{(2)},$$

will satisfy Eqs. (24.1) with $F_i = 0$. Thus, for the "difference" u_i, τ_{ij} of the two solutions, we have from the formula (27.1)

$$\int_\Sigma \overset{\nu}{T}_i u_i \, d\sigma = 2 \int_\tau W \, d\tau.$$

But since solutions (27.2) and (27.3) satisfy the boundary conditions, it follows that the components $\overset{\nu}{T}_i = \overset{\nu}{T}{}_i^{(1)} - \overset{\nu}{T}{}_i^{(2)}$ of the external surface forces vanish in the case of the first boundary-value problem, and the displacements $u_i = u_i^{(1)} - u_i^{(2)}$ vanish on the surface Σ for the case of the second boundary-value problem. It is also obvious that the integrand of the surface integral will vanish in the case of the mixed problem. We thus have in all cases

$$\int_\tau W \, d\tau = 0.$$

But W is a positive definite quadratic form in the components of strain, and hence the integral can vanish only when $W = 0$, that is, when $e_{ij} = 0$ $(i, j = 1, 2, 3)$. But $e_{ij} = e_{ij}^{(1)} - e_{ij}^{(2)}$, and it follows that the components of the strain tensor for the two solutions must be identical, and hence the components of the stress tensor are also identical. As regards the uniqueness of displacements, we recall from Sec. 10 that they are determined to within the quantities representing rigid body motions. In the case of the second and mixed boundary-value problems, the displacements are determined uniquely, since they are prescribed at least over part of the surface of the body.

It is important to note that the foregoing proof assumes that the displacements u_i are single-valued functions but imposes no restrictions on the connectivity of the region.

Consider now the dynamical case of Sec. 25, and assume that there are two solutions of the type (27.2) and (27.3) that satisfy the boundary conditions. Then, as above, the difference of two solutions

$$u_i, \tau_{ij}, \qquad (i, j = 1, 2, 3)$$

satisfies the differential equations when body forces are set equal to zero. We have in all cases the condition that

$$\overset{\nu}{T}_i \frac{\partial u_i}{\partial t} = 0 \qquad \text{on } \Sigma, \qquad t \geq t_0. \tag{27.4}$$

For in the case of the first dynamical problem, $\overset{\nu}{T}_i = 0$ on Σ, $t \geq t_0$, and in the case of the second problem, the components of velocity $\dfrac{\partial u_i}{\partial t} = 0$ on Σ, $t \geq t_0$, since $u_i = 0$ for all $t \geq t_0$.

Recalling that the displacements u_i correspond to the solution of Eqs. (25.2) when body forces are absent, and noting the expression (27.4), leads to the conclusion that both integrals in the formula (26.1) vanish, so that Eq. (26.5) becomes

$$\frac{dK}{dt} + \frac{dU}{dt} = 0,$$

or

$$K + U = \text{const.}$$

But the constant of integration in the above formula must be zero, since the displacements u_i and the velocities $\frac{\partial u_i}{\partial t}$ vanish at the instant $t = t_0$. Hence

$$K + U = 0,$$

and since both the kinetic energy K and the function U are essentially positive, one has

$$K = U = 0 \qquad \text{for all } t \geq t_0.$$

It follows from these equations that

$$\frac{\partial u_j}{\partial t} = 0, \qquad \text{and} \qquad e_{ij} = 0, \qquad (i, j = 1, 2, 3),$$

for all values of $t \geq t_0$. The first of the above-written relations states that we are dealing with a static case, and the second means that deformation of the body is not present, so that the solution (u_1, u_2, u_3) represents a rigid body motion. But the displacements (u_1, u_2, u_3) vanish at $t = t_0$, and hence rigid body motion cannot be present in our solution, or

$$u_1 = u_2 = u_3 = 0 \qquad \text{for all } t \geq t_0.$$

Thus, the two assumed solutions (27.2) and (27.3, are identical.

The proof of uniqueness given here is essentially due to Kirchhoff.[1] It should be noted that the crucial point in the argument is the positive definite character of the strain-energy density function W. In nonlinear theory, where large strains may be present, W need not be a positive definite quadratic form in the strains, and the proof breaks down. Indeed, problems concerned with elastic stability and buckling contemplate large deflections, and it is well known that solutions of such problems need not be unique. The reader may be familiar with the situation in the theory of Euler's columns where a column subjected to end thrusts may assume several distinct equilibrium configurations.

We conclude this section with a few remarks on the existence of solution of the fundamental boundary-value problems in linear elasticity. Because of the resemblance in the formulation of such problems to the basic problems of Potential Theory, it is natural that the early attempts to establish the existence of solution centered on methods similar to those

[1] G. Kirchhoff, *Journal für Mathematik* (*Crelle Journal*), vol. 56 (1859).

developed for the problems of Dirichlet and Neumann. The resulting proofs were not distinguished by simplicity since they depend on construction of certain auxiliary functions analogous to Green's functions in Potential Theory. The demonstration of existence of such auxiliary functions proved to be a problem of the same order of difficulty as the original problem.[1]

With the development of powerful methods of the theory of integral equations, it proved possible to demonstrate the existence of solution of the fundamental problems of elasticity under very general conditions both as regards the types of regions and the character of tractions and displacements specified on their surfaces. It suffices to suppose that the regions admit the application of the Divergence Theorem and that the functions assigned on the surfaces of such regions have piecewise continuous derivatives.[2]

28. Saint-Venant's Principle. It is obvious from the formulation of the fundamental boundary-value problems of the theory of elasticity that the exact solution of these problems is likely to present formidable mathematical difficulties because of the complicated form of the boundary conditions. Frequently it is possible to obtain a solution of the problem if the boundary conditions are somewhat modified, and it is worth noting that in the technological applications of the theory of elasticity one can only approximate the mathematical formulation of the boundary conditions, so that the mathematical solution of the problem represents only an approximation to the actual situation. In 1855, B. de Saint-Venant, in his famous memoir on torsion, proposed a principle that can be stated as follows:

[1] In the special case of a semi-infinite space bounded by the plane the auxiliary functions were constructed by V. Cerruti, *Atti della accademia nazionale dei Lincei, Memorie, Classe di scienze fisiche, matematiche, e naturali* (1882), who utilized the method of singularities developed by E. Betti, *Il Nuovo cimento*, vols. 6–10 (1872). Cerruti used this method to solve the problem of Boussinesq treated in Chap. 6.

[2] The existence of solution of basic three-dimensional problems was considered by:

I. Fredholm, *Arkiv för Matematik, Astronomi och Fysik*, vol. 2 (1906), pp. 3–8.

G. Lauricella, *Atti della accademia nazionale dei Lincei, Rendiconti, Classe di scienze fisiche, matematiche e naturali*, vol. 15 (1906), pp. 426–432; *Il Nuovo cimento*, vol. 13 (1907), pp. 104–118, 155–174, 237–262, 501–518.

A. Korn, *Annales de la faculté des sciences de l'université de Toulouse pour les sciences mathématiques, et les sciences physiques*, vol. 10 (1908), pp. 165–269; *Annales de l'école normale supérieure*, vol. 24 (1907), pp. 9–75; *Mathematische Annalen*, vol. 75 (1914), pp. 497–544.

H. Weyl, *Rendiconti del circolo matematico di Palermo*, vol. 39 (1915), pp. 1–49.

L. Lichtenstein, *Mathematische Zeitschrift*, vol. 20 (1924) pp. 21–28; vol. 24 (1925), p. 640.

D. I. Sherman, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 7 (1943), pp. 341–360.

For corresponding contributions to the two-dimensional problems of elasticity see Chap. 5.

If some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided that the two distributions of forces are statically equivalent.

The phrase "statically equivalent" means that the two distributions of forces have the same resultant force and the same resultant moment.

To illustrate the meaning of the principle, consider a long beam, one end of which is fixed in a rigid wall, while the other is acted upon by a distribution of forces that gives rise to a resultant force F and a couple of moment M. Now there are infinitely many distributions of forces that may act on the end of the beam and that will have the same resultant F and the same resultant moment M. The principle of Saint-Venant asserts that, while the distributions of stresses and strains near the region of application may differ greatly, the eccentricities of the local distribution will have no appreciable effect on the state of stress far enough from the points of application, so long as the systems of applied forces are statically equivalent. This principle is frequently used in practical applications.

One would suspect from the generality of the statement of the principle that it is not easy to justify in all cases on purely mathematical grounds.[1] In specific instances, one can calculate the distribution of stresses produced by various statically equivalent systems of forces, and in problems on beams, for example, it is commonly assumed that the local eccentricities are not felt at distances that are about five times the greatest linear dimension of the area over which the forces are distributed. However, in problems involving structural members with thin walls (box beams, shells, etc.) it is possible to apply to a small portion of the structure such eccentric distribution of forces that their effect is seriously felt throughout the structure.[2]

[1] J. Boussinesq, in Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques (1885), has shown that if the external forces act normally to the plane surface of a semi-infinite solid, and if they are confined to lie in a circle of radius ϵ, then the stresses at a fixed interior point at a distance greater than ϵ from the center of the circle are of the order of magnitude ϵ when the resultant of the external forces is zero and of the order ϵ^2 when the resultant moment is also zero. R. v. Mises has shown that these results need not be valid when the external forces are not normal to the surface. In a paper entitled "On Saint-Venant's Principle," *Bulletin of the American Mathematical Society*, vol. 51 (1945), pp. 555–562, v. Mises proposed a modification of the Saint-Venant principle, concerned essentially with the relative rather than absolute orders of magnitude of applied forces and the resulting internal stresses. See also E. Sternberg, *Quarterly of Applied Mathematics*, vol. 11 (1954), pp. 393–402.

The plausibility of the Saint-Venant principle, in its usual form, has been argued (not too convincingly) by many authors.

[2] See, for example, N. J. Hoff, *Journal of Aeronautical Sciences*, vol. 12 (1945), p. 445.

CHAPTER 4

EXTENSION, TORSION, AND FLEXURE OF BEAMS

29. Statement of Problem. This chapter is devoted to an analysis of the behavior of elastic beams bounded by a cylindrical surface (which is termed the *lateral surface* of the beam) and by a pair of planes normal to the lateral surface (which are called the *bases* of the cylinder). It contains a treatment of the technically important problem of torsion and flexure of cylinders and an account of the different methods of attack on the problems of the theory of elasticity concerned with a study of beams. An elegant method of solution of such problems, developed by N. I. Muskhelishvili and others, will be considered in detail.[1] Although it is not the purpose of this chapter to provide a compendium of the theory of beams, a number of problems will be worked out in detail, either because of their intrinsic importance in structural design, or for the sake of illustrating the methods of solution.

In dealing with special problems, no great saving of space is likely to result from the use of abridged notation; for this reason, we shall denote the variables x_1, x_2, and x_3 by x, y, and z, as was agreed in Sec. 7. We shall also write $\tau_{11} = \tau_{xx}$, $\tau_{23} = \tau_{yz}$, etc., for the components of the stress tensor and use the corresponding notation for the components of strain e_{ij}. The displacements u_i along the directions of the x, y, and z-axes will be labeled u, v, and w, and the components of body force $\mathbf{F}$ in the same directions will be written as F_x, F_y, and F_z.

Throughout this chapter, the z-axis of our coordinate system will be directed along the length of the beam parallel to the generators of the cylinder. The cylinder is assumed to be of length l, and one of its bases is taken to lie in the xy-plane, while the other is in the plane $z = l$. It is supposed in Secs. 29 to 62 that the lateral surface of the cylinder is free of external load and that the load on the beam is distributed over its bases, $z = 0$ and $z = l$, in a way that fulfills the equilibrium conditions of a rigid body.

The complete problem of equilibrium of an elastic beam with free lateral surface can be formulated in the following way: Determine the components of stress τ_{ij} and the displacements u_i that, in the region τ

[1] The development of several sections of this chapter follows along the lines of the prize-winning work by N. I. Muskhelishvili, Nekotoriye Osnovniye Zadachi Matematicheskoi Teorii Uprugosti.

occupied by the beam, satisfy the systems of equations

$$(29.1)\quad \begin{cases} \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = -F_x, \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = -F_y, \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zy}}{\partial y} + \dfrac{\partial \tau_{zz}}{\partial z} = -F_z, \end{cases}$$

$$(29.2)\quad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{1}{E}[\tau_{xx} - \sigma(\tau_{yy} + \tau_{zz})], \\ \dfrac{\partial v}{\partial y} = \dfrac{1}{E}[\tau_{yy} - \sigma(\tau_{zz} + \tau_{xx})], \\ \dfrac{\partial w}{\partial z} = \dfrac{1}{E}[\tau_{zz} - \sigma(\tau_{xx} + \tau_{yy})], \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = \dfrac{2(1+\sigma)}{E}\tau_{xy}, \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} = \dfrac{2(1+\sigma)}{E}\tau_{yz}, \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} = \dfrac{2(1+\sigma)}{E}\tau_{zx}, \end{cases}$$

and the boundary conditions

(29.3) $\tau_{zx}, \tau_{zy}, \tau_{zz},$ prescribed functions of x and y on the bases $z = 0$, $z = l$,

$$(29.4)\quad \begin{cases} \tau_{xx}\nu_x + \tau_{xy}\nu_y = 0, \\ \tau_{yx}\nu_x + \tau_{yy}\nu_y = 0, \\ \tau_{zx}\nu_x + \tau_{zy}\nu_y = 0, \end{cases} \quad \text{on the lateral surface of the cylinder.}$$

The functions τ_{ij}, naturally, must satisfy the Beltrami-Michell compatibility equations (24.15).

The problem, formulated with this degree of generality, presents formidable complications because of the difficulty of fulfilling the boundary conditions (29.3). In fact, the generality of formulation of the boundary conditions (29.3) is quite unnecessary from the practical point of view, since the actual distribution of applied stresses on the ends of the cylinder is rarely, if ever, known. A designer knows, more or less accurately, the resultant force **T** and the resultant moment **M** acting on the ends of the beam, and quite often the nature of the distribution of stresses over the ends of the beam, which give rise to the force **T** and the moment **M**, is a matter of indifference. On the other hand, if one accepts the principle of Saint-Venant and considers a beam whose length is large in comparison with the linear dimensions of its cross section, then the actual distribution of stresses over the ends has no appreciable influence on the character of the solution in portions of the beam sufficiently far removed from the ends. That is, one is free to prescribe any distribution

of stresses at the end of the cylinder so long as the resultant forces and moments reduce to those given in the formulation of the problem. This principle will be applied throughout our discussion of beams. This means that the mathematical solution obtained will give, near the ends of the beam, either (1) the exact solution of the physical problem in which the applied stresses are distributed in the way specified by the solution or (2) the approximate description of the physical situation in which the system of external forces and moments is statically equivalent to that assumed by the solution but is distributed in some different manner.

One need be concerned with one of the bases only for the specification of the resultant force **T** and of the resultant moment **M** on the base $z = l$ requires that the resultant force acting on the base $z = 0$ be $-\mathbf{T}$ and that the resultant moment acting on the same base be so chosen as to satisfy the condition of static equilibrium. Let the point O' of intersection of the z-axis with the base $z = l$ be the center of gravity of the base, and suppose that a force **T** and a couple **M** are applied at O'. The force **T** can be resolved into two components, one in the direction of the z-axis and the other in the plane of the base $z = l$. The component of force T_z in the direction of the z-axis will be responsible for tension or compression, while the other component T_B, lying in the plane of the base, will produce bending of the beam. The couple **M**, acting on the end of the beam, can likewise be decomposed into two couples, the moment of one of which is directed along the z-axis and hence will be responsible for twisting of the cylinder, while the moment of the other lies in the plane $z = l$ and will produce bending.

Thus, our problem can be solved, by utilizing the principle of superposition, if we succeed in solving the following four elementary problems:

1. Extension of a cylinder by longitudinal forces applied at the ends.
2. Bending of a cylinder by couples whose moments lie in the planes of the bases of the cylinder.
3. Torsion of a cylinder by couples whose moments are normal to the bases of the cylinder.
4. Flexure of a cylinder by a transverse force applied at one end of the cylinder, while on the other end there act a force equal in magnitude but oppositely directed to the transverse force and also a couple of such magnitude as to equilibrate the moment produced by the transverse forces.

Our general plan of attack upon the four elementary problems listed above is that of the Saint-Venant *semi-inverse method of solution*. This consists in making certain assumptions about the components of stress, strain, or displacement and yet leaving enough freedom in the quantities involved to satisfy the conditions of equilibrium and compatibility. In applying the semi-inverse method to problems on beams, we shall make one general assumption about the stress distribution in any beam; further assumptions regarding the stresses or the displacements will be introduced

in the solution of each problem. These assumptions will be justified when it is shown that they lead in each case to a solution that satisfies the conditions of equilibrium and compatibility. Then the proof in Sec. 27 of the uniqueness of solution of the general boundary-value problems of linear elasticity assures us that the solution obtained is unique.

Now, if we visualize the beam as made up of long filaments parallel to the axis of the cylinder, then it is sensible to assume that the action of forces and couples in the foregoing four problems may give rise to shearing stresses in the direction of the z-axis. These stresses act on the sides of the filaments and produce no stresses on the lateral surface of the filaments in the direction perpendicular to their lengths. Thus, let us assume tentatively that the system of stresses in all four problems is such that

$$\tau_{xx} = \tau_{yy} = \tau_{yx} = 0,$$

and let us investigate the consequences of this assumption.[1]

PROBLEMS

1. Show that if the stress components $\tau_{xx}, \tau_{xy}, \tau_{yy}$ and the body forces F_i vanish, then $\dfrac{\partial^2\tau_{zz}}{\partial x^2} = \dfrac{\partial^2\tau_{zz}}{\partial y^2} = \dfrac{\partial^2\tau_{zz}}{\partial z^2} = 0$; that is, the stress component τ_{zz} is linear in x, in y, and in z. Write out the most general form of the function in this case.

2. Integrate the differential equations of equilibrium $\tau_{ij,j} + F_i = 0$ throughout the volume of an elastic solid, apply the Divergence Theorem, and show that the equations of static equilibrium

$$\int_\Sigma \overset{\nu}{T}_i \, d\sigma + \int_\tau F_i \, d\tau = 0$$

are satisfied and hence that the resultant force on the body vanishes.

3. Show with the help of the Divergence Theorem that if the following differential equations of equilibrium

$$\tau_{yj,j} + F_y = 0, \qquad \tau_{zj,j} + F_z = 0, \qquad (j = x, y, z)$$

[1] One may equally proceed by assuming that the distribution on the cross section of the stress constituting each component of the resultant force and couple is the same at all sections. This is equivalent to assuming that for Probs. 1 to 3, stated on p. 93, we have

$$\frac{\partial\tau_{zx}}{\partial z} = \frac{\partial\tau_{zy}}{\partial z} = \frac{\partial\tau_{zz}}{\partial z} = 0,$$

while for Prob. 4 it is assumed that

$$\frac{\partial\tau_{zx}}{\partial z} = \frac{\partial\tau_{zy}}{\partial z} = \frac{\partial^2\tau_{zz}}{\partial z^2} = 0.$$

See W. Voigt, *Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, vol. 34 (1887), p. 53, and J. N. Goodier, *Philosophical Magazine*, ser. 7, vol. 23 (1937), p. 186.

are satisfied, then the following equation of static equilibrium also holds:

$$\int_{\Sigma} (y\overset{\nu}{T}_z - z\overset{\nu}{T}_y)\, d\sigma + \int_{\tau} (yF_z - zF_y)\, d\tau = 0,$$

thus expressing the vanishing of the x-component of the resultant moment on the body.

30. Extension of Beams by Longitudinal Forces. Let a force $\hat{T}$, directed along the z-axis, be applied at the center of gravity of the area a of the cross section of the base $z = l$ of the cylinder. If the stresses giving rise to the force T are assumed to be uniformly distributed, then

$$\left.\begin{aligned} \tau_{zz} &= \frac{T}{a} = p \text{ (a const)}, \\ \tau_{zx} &= \tau_{zy} = 0, \end{aligned}\right\} \qquad \text{on } z = l.$$

If we assume

$$\tau_{zz} = p, \qquad \tau_{xx} = \tau_{yy} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0,$$

throughout the cylinder, then the equilibrium equations (29.1) and (29.4) are obviously satisfied.[1] The Beltrami-Michell compatibility equations are also satisfied, since the components of the stress tensor are constants. The displacements u_i can be readily calculated. Thus, from (29.2),

$$\frac{\partial u}{\partial x} = -\frac{\sigma}{E}p, \qquad \frac{\partial v}{\partial y} = -\frac{\sigma}{E}p, \qquad \frac{\partial w}{\partial z} = \frac{p}{E},$$
$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0,$$

and since the right-hand members of these equations are constants, one is justified in assuming that the solutions are linear functions of x, y, and z. A simple calculation gives

$$u = -\frac{\sigma p}{E}x, \qquad v = -\frac{\sigma p}{E}y, \qquad w = \frac{p}{E}z,$$

if one neglects the terms representing rigid motion of the beam as a whole. Of course one could obtain the displacements by making use of the general formula (10.6). This problem has already been discussed in Sec. **23**.

PROBLEMS

1. Consider a bar of length l in., area of cross section a sq in., Young's modulus E lb per sq in., and stretched by a force of T lb applied at each end. Use both Eq. (26.12), $W = \frac{1}{2}\tau_{ij}e_{ij}$, and Eq. (27.1),

$$2\int_{\tau} W\, d\tau = \int_{\tau} F_i u_i\, d\tau + \int_{\sigma} \overset{\nu}{T}_i u_i\, d\sigma,$$

[1] The body forces F_i are assumed to vanish. The extension of a beam by gravitational forces is considered in the next section. The combined effect of extension of beams by longitudinal and gravitational forces can be obtained by applying the principle of superposition.

to show that the strain-energy density W and the total strain energy $U = \int_\tau W\, d\tau$ stored in the bar are given by

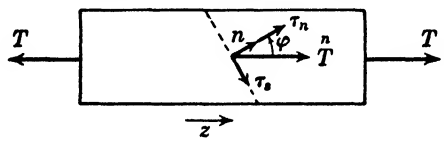
$$W = \frac{T^2}{2a^2E} \quad \text{in.-lb per cu in.},$$
$$U = \frac{T^2 l}{2aE} \quad \text{in.-lb.}$$

2. Find the greatest amount of strain energy per unit volume that can be stored in a steel bar under tensile forces T without producing permanent set. Take the elastic limit to be 30×10^3 lb per sq in. and Young's modulus as 30×10^6 lb per sq in.

3. In Prob. 1 take $l = 10$ in., $a = 2$ sq in., $T = 50 \times 10^3$ lb, $E = 30 \times 10^6$ lb per sq in. Find the strain-energy density W, and show numerically that the total strain energy U is one-half the product of the force T by the elongation of the rod.

4. Two gage marks 1 in. apart are made along the axis of a steel bar 10 in. long and of 2 sq in. cross-sectional area. The bar is then subjected to a tensile force of 50,000 lb. Find the stress, strain, elongation between gage marks, and total elongation of the bar. What is the total change of volume of the bar? What is the change in the cross-sectional area of the bar? Take $E = 30 \times 10^6$ lb per sq in., $\sigma = 0.3$.

5. Consider a beam stretched by a tensile force T applied at each end. The magnitude of the stress vector acting on a section with normal $\mathbf{n}$ is $\overset{n}{T} = T/(a \sec \varphi)$



$= \tau_{zz} \cos \varphi$, where a is the area of the cross section. Resolve $\overset{n}{\mathbf{T}}$ into normal and shear stresses τ_n, τ_s, and show that

$$\tau_n = \tau_{zz} \cos^2 \varphi, \qquad \tau_s = \tau_{zz} \sin \varphi \cos \varphi.$$

Derive these results also from the formulas of Sec. 19*b*. Show that the maximum normal stress is τ_{zz} (at $\varphi = 0$) and the maximum shear stress is $\frac{1}{2}\tau_{zz}$ (at $\varphi = 45°$). Compare this with the theorem of Sec. 18. What are the inclinations of the cross sections on which the shear and normal stresses are equal in magnitude?

6. Find the maximum shear stress in the beam of Prob. 4. What is the normal stress on the planes on which the shear stress is a maximum?

7. Consider a rod under uniform longitudinal stress $\tau_{zz} = p$. Let the rod be so constrained that there is no lateral contraction in the x-direction ($e_{xx} = 0$), while the rod is free to contract laterally in the y-direction. Define the effective Young's modulus by $E' = \tau_{zz}/e_{zz}$ and the effective Poisson's ratio by $\sigma' = -e_{yy}/e_{zz}$, and show that, owing to the lateral constraint, one has

$$E' = \frac{E}{1 - \sigma^2}, \qquad \sigma' = \frac{\sigma}{1 - \sigma}.$$

What is the range of possible values for E'? For σ'?

8. Let the rod in the preceding problem be so constrained as to prevent any lateral contraction. Show that the effective Young's modulus has the value

$$E' = \frac{1 - \sigma}{(1 - 2\sigma)(1 + \sigma)} E.$$

What is the effective Poisson's ratio?

31. Beam Stretched by Its Own Weight. Before proceeding to the problem of bending of beams, we shall discuss one example of a problem requiring a consideration of the body force.

Let a beam of length l, shown in Fig. 16, be supported in a suitable manner at its upper base, and assume that the force of gravity, directed downward, is the only external force acting on the beam. If the xy-plane of the coordinate system is chosen to coincide with the lower base of the beam before deformation takes place and if the positive direction of the z-axis is vertically upward, then the stress components τ_{ij} satisfy the system of Eqs. (29.1) with $F_x = F_y = 0$ and $F_z = -\rho g$, where ρ is the density of the beam. The stresses acting on each cross section of the beam are produced by the weight of the lower part of the beam, and we shall suppose that the stresses are distributed uniformly. Thus, we assume the system of stresses

$$\tau_{zz} = \rho g z, \qquad \tau_{xx} = \tau_{yy} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0,$$

which obviously satisfies the equations of equilibrium and the compatibility equations (24.15). The conditions (29.4) that no forces are applied to the lateral surface of the beam are likewise fulfilled. There are no tractions applied at the lower end; hence all components of stress vanish there, while at the upper end we have $\tau_{zz} = \rho g l$, which is directed vertically upward. Thus, the assumed distribution of stress requires that the upper end of the cylinder be supported in such a way as to yield a uniform distribution of stress.

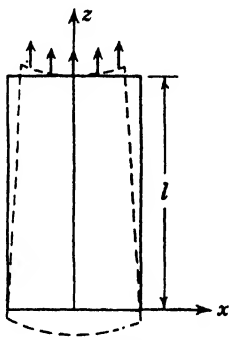


FIG. 16

In order to determine the displacements u_i, we note the relations (29.2), which yield,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\sigma \rho g z}{E}, \qquad \frac{\partial w}{\partial z} = \frac{\rho g z}{E}, \tag{31.1}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0. \tag{31.2}$$

Integrating the latter of Eqs. (31.1) gives

$$w = \frac{\rho g z^2}{2E} + w_0(x, y),$$

where w_0 is a function of x and y alone, and it follows from the last two of Eqs. (31.2) that

$$\frac{\partial u}{\partial z} = -\frac{\partial w_0}{\partial x}, \qquad \text{and} \qquad \frac{\partial v}{\partial z} = -\frac{\partial w_0}{\partial y}$$

Hence

$$u = -z\frac{\partial w_0}{\partial x} + u_0(x, y), \qquad \text{and} \qquad v = -z\frac{\partial w_0}{\partial y} + v_0(x, y),$$

where u_0 and v_0 involve x and y only. Substituting the values of u and v just found in the first two of Eqs. (31.1) gives

$$(31.3)\qquad \frac{\partial u_0}{\partial x} = 0, \qquad \frac{\partial v_0}{\partial y} = 0, \qquad \frac{\partial^2 w_0}{\partial x^2} = \frac{\sigma\rho g}{E}, \qquad \frac{\partial^2 w_0}{\partial y^2} = \frac{\sigma\rho g}{E},$$

while the substitution of the same values in the first of Eqs. (31.2) yields

$$(31.4)\qquad \frac{\partial^2 w_0}{\partial x\,\partial y} = 0, \qquad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0.$$

It is clear from the first two of the differential equations (31.3) that

$$u_0 = F(y), \qquad \text{and} \qquad v_0 = G(x),$$

where F is a function of y alone, while G is a function of x. The functions F and G, as follows from the second of Eqs. (31.4), satisfy the equation

$$\frac{dF(y)}{dy} + \frac{dG(x)}{dx} = 0,$$

and this requires that $\dfrac{dF}{dy} = a$, $\dfrac{dG}{dx} = -a$, where a is a constant. Thus,

$$u_0 = F(y) = ay + b, \qquad \text{and} \qquad v_0 = G(x) = -ax + c.$$

The integration of the equations on w_0 is equally easy, and one finds

$$w_0 = \frac{\sigma\rho g}{2E}(x^2 + y^2) + a'x + b'y + c',$$

where a', b', and c' are constants.

Thus, the complete expression for the displacements is

$$\begin{aligned} u &= -\frac{\sigma\rho g}{E} zx - a'z + ay + b, \\ v &= -\frac{\sigma\rho g}{E} zy - b'z - ax + c, \\ w &= \frac{\rho g}{2E}(z^2 + \sigma x^2 + \sigma y^2) + a'x + b'y + c'. \end{aligned}$$

The linear part of the solution represents rigid body displacement.[1] If we prevent the point $(0, 0, l)$ from being displaced, then $u = v = w = 0$ for $x = 0$, $y = 0$, $z = l$. To prevent the possibility of rotation about the z-axis, we fix an element of area in the xz-plane and passing through the point $(0, 0, l)$; then $\dfrac{\partial v}{\partial x} = 0$ at $(0, 0, l)$. In order to eliminate rotation about the axes through $(0, 0, l)$ that are parallel to the x- and y-axes, we

[1] See Prob. 4 at the end of this section. We demand, in effect, that the rotation components $\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right)$ vanish at $(0, 0, l)$.

fix an element of the z-axis; then $\frac{\partial u}{\partial z} = 0$, $\frac{\partial v}{\partial z} = 0$, at $(0, 0, l)$. These six conditions enable us to eliminate the six constants a, b, c and a', b', c'. An elementary calculation shows that the displacement, in this case, is given by

$$(31.5)\qquad \begin{cases} u = -\dfrac{\sigma\rho g}{E} zx, & v = -\dfrac{\sigma\rho g}{E} zy, \\ w = \dfrac{\rho g}{2E}(z^2 + \sigma x^2 + \sigma y^2 - l^2). \end{cases}$$

It is seen from this solution that points on the z-axis are displaced vertically according to the law

$$w = -\frac{\rho g}{2E}(l^2 - z^2).$$

All other points of the beam have both vertical and horizontal displacements on account of the contraction in the transverse direction. The shape of the beam, after deformation, is indicated by the dotted lines in Fig. 16. Any cross section of the beam is shrunk laterally by an amount proportional to the distance from the lower end and is distorted into a paraboloid of revolution. This can be seen by noting that, for a cross section $z = c$,

$$z' = c + w = c + \frac{\rho g(c^2 - l^2)}{2E} + \frac{\sigma\rho g}{2E}(x^2 + y^2).$$

The upper base of the cylinder is warped upward (see Fig. 16) because of the assumed uniform distribution of the stress component τ_{zz} over that face and the fixing of the point $(0, 0, l)$.

REFERENCES FOR COLLATERAL READING

A. E. H Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Sec. 86.

S. Timoshenko and J. N. Goodier: Theory of Elasticity, McGraw-Hill Book Company, Inc., New York, Sec. 86

PROBLEMS

1. Discuss the solution of the elastostatic problem for the case where

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -p + \rho g z, \qquad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0.$$

This state of stress corresponds to that found in a body immersed in a fluid whose density is the same as that of the body, where p is the pressure of the fluid at the level of the origin of coordinates.

2. Determine the displacements in a cylinder of length $2l$ and of density ρ when suspended in a fluid of density ρ'. Let the pressure of the fluid at the level of the center of gravity of the cylinder be p. Choose the origin of the coordinate system at the

center of gravity of the cylinder, and let the z-axis be vertical. *Hint:* Assume a system of stresses

$$\tau_{xx} = \tau_{yy} = -p + \rho' gz, \qquad \tau_{zz} = -p + (\rho - \rho')gl + \rho gz,$$
$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0.$$

3. Obtain the solution given in (31.5) from the general solution (10.6).

4. Show from Eq. (7.5) that the displacement components

$$\begin{aligned} u &= \qquad - ry + qz + a, \\ v &= rx \qquad - pz + b, \\ w &= -qx + py \qquad + c \end{aligned}$$

represent an (infinitesimal) rotation (p, q, r) and a translation (a, b, c).

5. Show that some of the results of Sec. 31 on a beam stretched by its own weight may be obtained readily by the procedure sketched below (used in strength of materials theory). As before, the stress on the faces of an element of cross-sectional area a and length dz is given by $\tau_{zz} = \rho gaz/a = \rho gz$. The elongation of this element is $\rho gz\, dz/E$. Integrate this expression over the length of the beam, and compare the result with that obtained from Sec. 31. Show that the total elongation in a beam stretched by its own weight W is the same as that produced by a load $\frac{1}{2}W$ applied at the end of the beam (with weight neglected).

32. Bending of Beams by Terminal Couples. In order to free the semi-inverse method of solution from elements of mystery that a beginner feels are involved in the usual statement: "Assume the system of stress defined by . . . ," we shall give first an intuitive picture of the probable state of affairs in a beam bent by a pair of couples applied at its ends. This picture will be of aid to us later on because it will bring into sharp focus the limitations of the approximate engineering theory of beams.

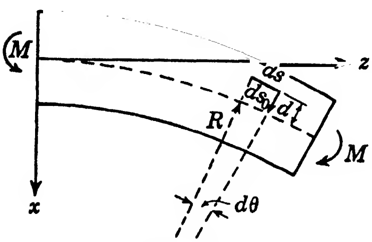


FIG. 17

Let a pair of couples of magnitude M be applied to the ends of a beam as shown in Fig. 17. It is clear that the longitudinal filaments of which the beam may be thought to be composed will be contracted on the face of the beam toward the center of curvature, and those on the opposite face will be extended. We shall call the line passing through the centroids of the cross sections of the beam the *central line*. If we assume that the central line of the beam, indicated in the figure by a dotted curve, is unaltered in length, and if plane sections of the beam normal to the central line are assumed to remain plane and normal to the deformed central line, then it is easy to see that the magnitude of extension (or contraction) of the longitudinal filaments is given by the formula

$$e = \frac{d}{R}.$$

In this relation, d is the distance of the filament from the central plane drawn through the central line at right angles to the plane of the couple (the xz-plane in Fig. 17), and R is the radius of curvature of the central line. Now the length ds_0 of the portion of central filament subtended by an angle $d\theta$ is $ds_0 = R\,d\theta$, while the length of the element ds subtended by the same angle $d\theta$ and at a distance d from the central plane is

$$ds = (R + d)\,d\theta.$$

Hence the extension e is given by

$$e = \frac{ds - ds_0}{ds_0} = \frac{(R + d)\,d\theta - R\,d\theta}{R\,d\theta} = \frac{d}{R}.$$

This extension d/R of the longitudinal filaments can be thought to be produced by a longitudinal stress τ, which, from the third of Eqs. (29.2), is

$$\tau = \frac{E}{R}\,d.$$

Obviously, τ denotes tension if the point in question is above the central line and compression if it is below. We choose the z-axis to coincide with the central line of the beam and take the x- and y-axes along the principal axes of inertia of the cross section A. From this choice of axes and from the definition of the central line as the line of centroids of the sections, we have

$$\int_A x\,d\sigma = \int_A y\,d\sigma = \int_A xy\,d\sigma = 0.$$

It follows that the distribution of stress in any section will be characterized by the formula

$$\tau_{zz} = -\frac{E}{R}\,x,$$

where the negative sign arises from our convention in regard to the signs of tensile and compressive stresses.

We shall now verify that the boundary conditions on the ends of the beam are satisfied, namely, that the resultant force and moment acting on the bases (or on any other cross section of the cylinder) reduce to a moment about the y-axis. The resultant force $\mathbf{T}$ acting on any section A has the components

$$T_x = \int_A \tau_{zx}\,d\sigma = 0, \qquad T_y = \int_A \tau_{zy}\,d\sigma = 0,$$
$$T_z = \int_A \tau_{zz}\,d\sigma = -\frac{E}{R}\int_A x\,d\sigma = 0.$$

The resultant moment about the x-axis is

$$M_x = \int_A y\tau_{zz}\,d\sigma = -\frac{E}{R}\int_A xy\,d\sigma = 0,$$

while the moment about the y-axis is given by

$$M_y = -\int_A x\tau_{zz}\, d\sigma = \frac{E}{R}\int_A x^2\, d\sigma = \frac{EI_y}{R},$$

where I_y is the moment of inertia of the cross section about the y-axis.

Thus, the curvature of the central line of a beam bent by a couple c magnitude M is[1]

$$R = \frac{EI}{M}. \tag{32.1}$$

Formula (32.1), connecting the curvature of the central line with the bending moment, is called the *Bernoulli-Euler law*. It will recur when we come to consider this problem in a rigorous way.

It appears from the foregoing discussion that the stress in a beam giving rise to a couple **M** is a longitudinal stress of magnitude

$$\tau_{zz} = -\frac{M}{I}x = -\frac{E}{R}x.$$

Under the action of the tensile stress τ_{zz}, the cross section of the beam will be deformed, and the amount of the transverse contraction (or extension), from the definition of Poisson's ratio σ (see Sec. 23), is

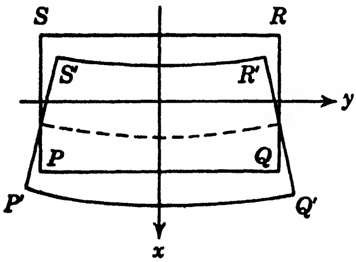


FIG. 18

$$\frac{\sigma x}{R} = \frac{\sigma M x}{EI}.$$

If the beam was initially of rectangular cross section $PQRS$ (Fig. 18), then, as will be shown in the rigorous discussion of this problem [see (32.10)], the parts RS and PQ of the boundary are each bent into a parabola whose radius of curvature is approximately R/σ. The *neutral plane* of the beam (that is, the plane in which there is no extension) and the faces of the beam that were originally parallel to the yz-plane are deformed into saddle-shaped, or anticlastic, surfaces.

The experimental measurement of the principal curvatures of the anticlastic surfaces provides a method of determining Poisson's ratio[2] σ, while the measurement of the radius of curvature of the central line serves to determine Young's modulus E.

It is clear from formula (32.1) that a beam with a large value of EI will

[1] The subscript y on I and M has been dropped, since no confusion is likely to arise here.

[2] Those interested in the experimental determination of σ are referred to S Timoshenko and J. N. Goodier, Theory of Elasticity, p 254, where further references on this subject will be found.

bend only slightly under the action of the couple **M**, and hence the magnitude of EI provides a measure of the rigidity of the beam. For this reason the constant EI is called the *modulus of flexural rigidity*. In order to increase the flexural rigidity of a beam, one must design it so as to make the moment of inertia I as large as possible. This is one of the reasons for making beams with cross sections in the shape of the letters I, T, Z, etc.

We are now ready to consider rigorously the problem of bending of a beam.

Assume the system of stresses

$$(32.2) \qquad \tau_{zz} = -\frac{M}{I}x, \qquad \tau_{xx} = \tau_{yy} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0,$$

and choose the axes of coordinates as before (see Fig. 19). Then, as shown above, the resultant force on any section vanishes, and the direction of the moment **M** of the couple is that of the y-axis. It is obvious that the equations of equilibrium throughout the interior and on the lateral surface of the cylinder are satisfied, as are the equations of compatibility.[1]

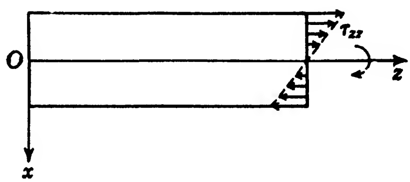


Fig. 19

Using (32.2) and the formulas (29.2), we find

$$(32.3) \qquad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\sigma M}{EI}x, & \dfrac{\partial v}{\partial y} = \dfrac{\sigma M}{EI}x, & \dfrac{\partial w}{\partial z} = -\dfrac{M}{EI}x, \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0, & \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} = 0, & \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} = 0. \end{cases}$$

The expressions for the displacements u_i can be obtained from the formulas (10.6) or by assuming u, v, and w to be functions of the second degree of x, y, and z with unknown coefficients and then determining the coefficients so as to satisfy Eqs. (32.3). We choose to integrate Eqs. (32.3) directly.

Thus, from the third of Eqs. (32.3), we obtain

$$w = -\frac{M}{EI}xz + w_0(x, y),$$

where w_0 is an unknown function of x and y. The fifth and sixth of Eqs. (32.3) give

$$\frac{\partial u}{\partial z} = \frac{M}{EI}z - \frac{\partial w_0}{\partial x}, \qquad \frac{\partial v}{\partial z} = -\frac{\partial w_0}{\partial y}.$$

[1] The body forces are assumed to vanish.

Hence

$$
(32.4) \qquad \begin{cases} u = \dfrac{M}{2EI} z^2 - z \dfrac{\partial w_0}{\partial x} + u_0(x, y), \\ v = -z \dfrac{\partial w_0}{\partial y} + v_0(x, y), \end{cases}
$$

where u_0 and v_0 are unknown functions of x and y. Substituting these values in the first two of Eqs. (32.3) gives

$$
(32.5) \qquad \begin{cases} -z \dfrac{\partial^2 w_0}{\partial x^2} + \dfrac{\partial u_0}{\partial x} = \dfrac{\sigma M}{EI} x, \\ -z \dfrac{\partial^2 w_0}{\partial y^2} + \dfrac{\partial v_0}{\partial y} = \dfrac{\sigma M}{EI} x. \end{cases}
$$

Since these equations are true for all values of z, it appears that

$$
(32.6) \qquad \frac{\partial^2 w_0}{\partial x^2} = 0, \qquad \frac{\partial^2 w_0}{\partial y^2} = 0,
$$

and it follows from the integration of (32.5) that

$$
u_0 = \frac{\sigma M}{2EI} x^2 + f_1(y), \qquad v_0 = \frac{\sigma M}{EI} xy + f_2(x).
$$

Inserting these expressions in (32.4) and substituting the resulting values of u and v in the fourth of Eqs. (32.3) gives the condition

$$
-2z \frac{\partial^2 w_0}{\partial x \, \partial y} + \frac{df_1(y)}{dy} + \frac{df_2(x)}{dx} + \frac{\sigma M}{EI} y = 0.
$$

Since the last three terms in this equation are independent of z, it follows that

$$
(32.7) \qquad \frac{\partial^2 w_0}{\partial x \, \partial y} = 0,
$$

and hence

$$
\frac{df_1(y)}{dy} + \frac{df_2(x)}{dx} + \frac{\sigma M}{EI} y = 0.
$$

Thus,

$$
(32.8) \qquad \frac{df_2}{dx} = -\alpha, \qquad \text{and} \qquad \frac{df_1}{dy} + \frac{\sigma M}{EI} y = \alpha,
$$

where α is a constant. We note from (32.6) and (32.7) that w_0 is a linear function of x and y, say

$$
w_0 = \beta x + \gamma y + c;
$$

furthermore, integrating Eqs. (32.8) gives

$$
f_2 = -\alpha x + b,
$$
$$
f_1 = -\frac{\sigma M}{2EI} y^2 + \alpha y + a,
$$

where b and a are arbitrary constants. Thus, the expressions for the displacements become

$$(32.9)\qquad \begin{cases} u = \dfrac{M}{2EI}(z^2 + \sigma x^2 - \sigma y^2) & + \alpha y - \beta z + a, \\ v = \dfrac{M}{EI}\sigma xy & - \alpha x \qquad - \gamma z + b, \\ w = -\dfrac{M}{EI}xz & + \beta x + \gamma y \qquad + c. \end{cases}$$

The constants of integration appearing in the solution can be determined from the mode of fixing the beam. We can determine them in the same way as was done in Sec. 31, namely, by fixing the centroid of the left-hand end of the beam at the origin and by fixing an element of the z-axis and an element of the xz-plane at the origin. These conditions ensure that there is no rigid body motion of translation or rotation about the origin. They can be formulated explicitly as follows:

$$u = v = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial x} = 0 \qquad \text{at } (0, 0, 0).$$

It follows from these relations that

$$\alpha = \beta = \gamma = a = b = c = 0.$$

The vanishing of the constants of integration also follows readily from Eqs. (3.5), from which it is seen that a, b, c represent a rigid body translation, while α, β, γ characterize a rigid body rotation about the origin. The solution can now be written in the form

$$(32.10)\qquad \begin{cases} u = \dfrac{M}{2EI}(z^2 + \sigma x^2 - \sigma y^2), \\ v = \dfrac{M}{EI}\sigma xy, \\ w = -\dfrac{M}{EI}xz. \end{cases}$$

It is clear from (32.10) that the filaments lying in the central plane $x = 0$ suffer no extensions; that is, the plane $x = 0$ is the neutral plane of the beam. The longitudinal filaments on one side of the central plane ($x > 0$) are contracted, whereas those on the other side ($x < 0$) are extended. Points which, prior to deformation, had the coordinates x_i go into points with coordinates x'_i, where $x'_i = x_i + u_i$. Hence the points on the z-axis (that is, the points on the central line of the beam) go into points

$$(32.11)\qquad x' = \frac{M}{2EI}z^2 = \frac{M}{2EI}z'^2, \qquad y' = 0, \qquad z' = z.$$

The *plane of bending* is defined to be the plane containing the deformed central line of the beam. Equation (32.11) shows that, in this example the plane of bending coincides with the plane ($y = 0$) of the couple **M**. The curve defined by (32.11) is a parabola whose radius of curvature R is given by the formula

$$\frac{1}{R} = \frac{\dfrac{d^2x'}{dz'^2}}{\left[1 + \left(\dfrac{dx'}{dz'}\right)^2\right]^{3/2}},$$

which is nearly equal to $\dfrac{d^2x'}{dz'^2}$ if $\dfrac{dx'}{dz'}$ is small. It follows from (32.11) that for small deflections

$$\frac{1}{R} = \frac{M}{EI},$$

which is the Bernoulli-Euler law, discovered earlier from rough geometrical considerations. This formula states that the magnitude of the bending moment **M** is proportional to the curvature of the central line of the beam. The Bernoulli-Euler law forms the point of departure for all considerations in the technical theory of beams.

Consider a cross section of the beam made by the plane $z = c$. After deformation, points in this cross section will lie in the plane

$$z' = c + w = c - \frac{M}{EI}xc = c\left(1 - \frac{x}{R}\right).$$

If the curvature is small, we can replace x by x' and obtain

$$z' = c\left(1 - \frac{x'}{R}\right),$$

which is the equation of a plane normal to the deformed central line. Hence the assumption that the normal sections remain plane after deformation (made at the beginning of this section) is valid.

In order to see how the cross sections of the beam are deformed, consider a beam of rectangular cross section. The sides $y = \pm b$ of the beam will go into

$$y' = \pm b + v = \pm b\left(1 + \frac{\sigma}{R}x\right),$$

and for small values of $1/R$ this is nearly the same as

$$y' = \pm b\left(1 + \frac{\sigma}{R}x'\right).$$

Thus, the vertical sides become inclined, as shown in Fig. 18. The points in the section $z = c$, which lie on the upper and lower faces $x = \pm a$ of the

beam, will go into points

$$x' = \pm a + u = \pm a + \frac{1}{2R}[c^2 + \sigma(a^2 - y^2)].$$

Hence, for small values of $1/R$, we have

$$x' = \pm a + \frac{1}{2R}[c^2 + \sigma(a^2 - y'^2)],$$

which is the equation of a parabola whose curvature at any point of the section is nearly σ/R.

It may be remarked in conclusion that, if the moment **M** of the couple is not directed along one of the principal axes of inertia of the section, then the couple can be resolved into two couples, each of which has moments directed along the principal axes. Then the foregoing considerations become applicable to each of the couples, and the solution of the problem can be obtained by superposition. It turns out that in this general case the plane of bending is also perpendicular to the neutral (undeformed) surface, although the plane of the couple does not necessarily coincide with the plane of bending.[1]

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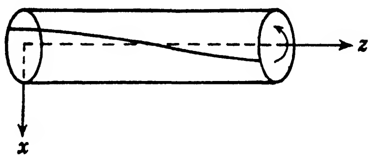
33. Torsion of a Circular Shaft. In the preceding section, we formed a physical picture of the distribution of stress in a beam bent by couples from a consideration of the extension of a longitudinal filament. In this section, we shall be guided by the displacements and shall deduce the stresses from the functions u_i.



Consider a circular cylinder, of length l, with one of its bases fixed in the xy-plane, while the other base (in the plane $z = l$) is acted upon by a couple whose moment lies along the z-axis. Under the action of the couple, the beam will be twisted, and the generators of the cylinder will be deformed into helical curves. On account of the symmetry of the cross section, it is reasonable to suppose that sections of the cylinder by

[1] See in this connection Secs. 52–61, dealing with the flexure problem.

planes normal to the z-axis will remain plane after deformation and that the action of the couple will merely rotate each section through some angle θ. The amount of rotation will clearly depend on the distance of the section from the base $z = 0$, and since the deformations are small, it is sensible to assume that the amount of rotation θ is proportional to the distance of the section from the fixed base. Thus,

$$\theta = \alpha z,$$

where α is the twist per unit length, that is, the relative angular displacement of a pair of cross sections that are unit distance apart.

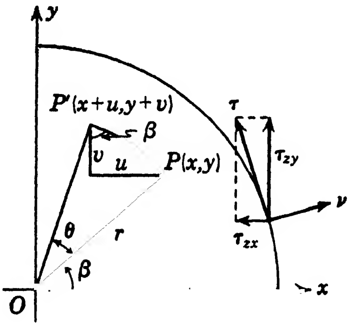


FIG. 20

If the cross sections of the cylinder remain plane after deformation, then the displacement w, along the z-axis, is zero. The displacements u and v are readily calculated. Thus, consider any point $P(x, y)$ in the circular cross section, which, before deformation, occupied the position shown in Fig. 20. After deformation, the point P will occupy a new position $P'(x + u, y + v)$. In terms of the angular displacement θ of the point P, we have

$$u = r \cos (\beta + \theta) - r \cos \beta = x(\cos \theta - 1) - y \sin \theta,$$
$$v = r \sin (\beta + \theta) - r \sin \beta = x \sin \theta + y(\cos \theta - 1),$$

where β is the angle between the radius vector r and the x-axis so that $x = r \cos \beta$, $y = r \sin \beta$. If the angle θ is small, we can write

$$u = -\theta y, \qquad v = \theta x.$$

Since $\theta = \alpha z$, we have for the displacements of any point with coordinates x, y, z

(33.1) $$u = -\alpha z y, \qquad v = \alpha z x, \qquad w = 0.$$

The system of stresses associated with the displacements (33.1) is given at once by the formulas (24.6). We thus have

(33.2) $$\tau_{zy} = \mu \alpha x, \qquad \tau_{zx} = -\mu \alpha y, \qquad \tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0,$$

which obviously satisfy the equations of equilibrium (with no body forces acting) and the equations of compatibility. The boundary conditions on the lateral surface are likewise satisfied. The first two of Eqs. (29.4) are identically satisfied, and the last one gives

$$\tau_{zx} \cos (x, \nu) + \tau_{zy} \cos (y, \nu) = -\mu \alpha y \cos (x, \nu) + \mu \alpha x \cos (y, \nu) \equiv 0,$$

since, for a circle of radius a, $\cos (x, \nu) = x/a$ and $\cos (y, \nu) = y/a$.

The only nonvanishing component of the couple **M** produced by the

distribution of stresses (33.2) over the end of the cylinder is M_z, which is easily calculated. Thus,

$$M_z = \iint (x\tau_{zy} - y\tau_{zx})\,dx\,dy$$
$$= \mu\alpha \iint (x^2 + y^2)\,dx\,dy = \mu\alpha I_0,$$

where $I_0 = \pi a^4/2$ is the polar moment of inertia of the circular cross section of radius a.

The resultant force acting on the end of the cylinder vanishes, and it follows from Saint-Venant's principle that whatever be the distribution of forces over the end of the cylinder that gives rise to the couple of magnitude M_z, the distribution of stress sufficiently far from the ends of the cylinder is essentially that specified by (33.2).

The stress vector[1]

$$\mathbf{T} = \mathbf{i}\tau_{zx} + \mathbf{j}\tau_{zy} + \mathbf{k}\tau_{zz} = \mu\alpha(-\mathbf{i}y + \mathbf{j}x),$$

acting at a point (x, y) on any cross section z-constant, lies in the plane of the section and is normal to the radius vector r joining the point (x, y) with the origin $(0, 0)$. The magnitude of $\mathbf{T}$ is

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \mu\alpha\sqrt{x^2 + y^2} = \mu\alpha r. \tag{33.3}$$

From this we see that the maximum stress is a tangential stress that acts on the boundary of the cylinder and has the magnitude $\mu\alpha a$, where a is the radius of the cylinder.

34. Torsion of Cylindrical Bars. Consider a cylindrical bar subjected to no body forces and free from external forces on its lateral surface. One end of the bar is fixed in the plane $z = 0$, while the other end, in the plane $z = l$, is twisted by a couple of magnitude M whose moment is directed along the axis of the bar.

Navier, being guided by Coulomb's solution of the torsion problem for a circular shaft, assumed that, in the general case of torsion of noncircular bars, the sections of the bar perpendicular to the z-axis will remain plane. This assumption led him to erroneous conclusions. The fact that the displacements characterized by formulas (33.1) cannot be valid for bars whose sections are not circular can be seen from the boundary conditions (29.4). A substitution of the stresses (33.2) in the third of the boundary conditions yields

$$-\mu\alpha y \cos(x, \nu) + \mu\alpha x \cos(y, \nu) = 0, \tag{34.1}$$

where ν, as always, denotes the exterior normal to the boundary C of the cross section R of the beam. But from Fig. 21 it is seen that

[1] We denote the unit base vectors along the x-, y-, and z-axes by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively.

$$(34.2)\qquad \begin{cases} \dfrac{dx}{ds} = \cos(x, s) = \sin(x, \nu) = -\cos(y, \nu), \\ \dfrac{dy}{ds} = \sin(x, s) = \cos(x, \nu), \end{cases}$$

so that, upon dividing out the nonvanishing factor $\mu\alpha$, Eq. (34.1) becomes

$$x\,dx + y\,dy = 0.$$

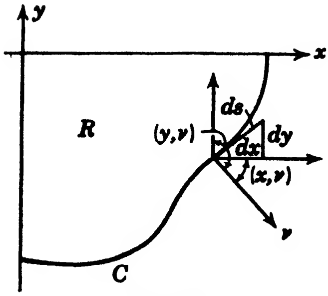


FIG 21

This is the differential equation of a family of circles. Thus, circular cylinders are the only bodies whose lateral surfaces can be expected to be free from applied external forces if the state of stress characterized by the formulas (33.2) obtains in the interior.

A natural modification of Navier's assumption is to suppose that, for cylinders other than circular ones, cross sections do not remain plane but are warped and that each section is warped in the same way. This leads us to assume displacements of the form

$$(34.3)\qquad u = -\alpha zy, \qquad v = \alpha zx, \qquad w = \alpha\varphi(x, y),$$

where $\varphi(x, y)$ is some function of x and y and α, as before, is the angle of twist per unit length of the bar. The function $\varphi(x, y)$ must be so determined as to satisfy the differential equations (29.1) and the boundary conditions (29.4).

A simple calculation of the stresses corresponding to the displacements (34.3) gives

$$(34.4)\qquad \begin{cases} \tau_{yz} = \mu\alpha\left(\dfrac{\partial\varphi}{\partial y} + x\right), \qquad \tau_{xz} = \mu\alpha\left(\dfrac{\partial\varphi}{\partial x} - y\right), \\ \tau_{xy} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0. \end{cases}$$

A substitution of these values in the equilibrium equations (29.1) shows that the equilibrium equations will be satisfied if $\varphi(x, y)$ satisfies the equation

$$(34.5)\qquad \nabla^2\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

throughout the cross section of the cylinder. Furthermore, if the system of stresses is to satisfy the boundary conditions (29.4) on the lateral surface of the cylinder, we see that

$$\left(\frac{\partial\varphi}{\partial x} - y\right)\cos(x, \nu) + \left(\frac{\partial\varphi}{\partial y} + x\right)\cos(y, \nu) = 0 \qquad \text{on } C,$$

where C is the boundary of the cross section R of the cylinder (Fig. 21).

But

$$\frac{\partial\varphi}{\partial x}\cos(x, \nu) + \frac{\partial\varphi}{\partial y}\cos(y, \nu) \equiv \frac{d\varphi}{d\nu},$$

so that the boundary condition can be written in the form

$$\frac{d\varphi}{d\nu} = y \cos(x, \nu) - x \cos(y, \nu) \qquad \text{on } C. \tag{34.6}$$

It follows from (34.5) that $\varphi(x, y)$ must be a harmonic function throughout the region R bounded by the curve C and that on the boundary C the normal derivative of $\varphi(x, y)$ must assume the value given by (34.6). Since the displacements are single-valued functions, it follows from (34.3) that $\varphi(x, y)$ must also be a single-valued function. Thus, the problem of determining the *torsion function* $\varphi(x, y)$ is a special case of the second boundary-value problem of Potential Theory. This latter problem is associated with the name of Neumann and consists in determining a function Φ that is harmonic in a given region and whose normal derivative is prescribed on the boundary of the region. We shall meet this problem again in our study of several problems of elasticity. At this time we shall simply remark that the harmonic function φ is determined by the boundary condition (34.6) to within an arbitrary constant.[1] The substitution of φ + constant in formulas (34.4) obviously does not alter the stresses, and it is clear from (34.3) that the addition of a constant to φ means a shift of the cylinder as a whole in the direction of the z-axis. Thus, the additive constant in the solution of the problem of Neumann is immaterial in our case.

The condition for the existence of a solution Φ of the problem of Neumann is that the integral of the normal derivative of the function Φ, calculated over the entire boundary C, vanish. This follows from the identity

$$\int_C \frac{d\Phi}{d\nu}\, ds = \iint_R \operatorname{div}(\nabla\Phi)\, d\sigma = \iint_R \nabla^2\Phi\, d\sigma$$

and from the fact that $\nabla^2\Phi = 0$. This condition is satisfied in our case, for [see (34.6)]

$$\begin{aligned} \int_C \frac{d\varphi}{d\nu}\, ds &= \int_C [y \cos(x, \nu) - x \cos(y, \nu)]\, ds \\ &= \int_C (y\, dy + x\, dx) = 0, \end{aligned}$$

since the integrand is the exact differential of the function $\frac{1}{2}(x^2 + y^2)$ + constant.

It is easy to show that the distribution of stresses given by Eqs. (34.4)

[1] See Sec. 42.

is equivalent to a torsional couple applied at the end $z = l$ of the cylinder and that the resultant force acting on the end of the cylinder vanishes. Now the resultant force in the x-direction is given by

$$\iint_R \tau_{zx}\,dx\,dy = \mu\alpha \iint_R \left(\frac{\partial\varphi}{\partial x} - y\right) dx\,dy,$$

and this can be written as

$$(34.7)\qquad \mu\alpha \iint_R \left\{\frac{\partial}{\partial x}\left[x\left(\frac{\partial\varphi}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\varphi}{\partial y} + x\right)\right]\right\} dx\,dy,$$

since φ satisfies the differential equation (34.5). Green's Theorem is directly applicable to the integral (34.7), and we get

$$(34.8)\qquad \mu\alpha \int_C x\left[\frac{d\varphi}{d\nu} - y\cos(x, \nu) + x\cos(y, \nu)\right] ds,$$

where the line integral is evaluated over the boundary C of the region R. The integral (34.8) vanishes on account of the boundary condition (34.6). It is shown in a similar way that

$$\iint_R \tau_{zy}\,dx\,dy = 0,$$

so that the resultant force acting on the end of the cylinder vanishes.

It remains to show that the system of stresses defined by Eqs. (34.4) is statically equivalent to a torsional couple. The resultant moment of the external forces applied to the end of the beam is

$$(34.9)\qquad \begin{aligned} M_z &= \iint_R (x\tau_{zy} - y\tau_{zx})\,dx\,dy \\ &= \mu\alpha \iint_R \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dx\,dy. \end{aligned}$$

The integral appearing in (34.9) depends on the torsion function φ and hence on the cross section R of the beam. Setting

$$(34.10)\qquad D = \mu \iint_R \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dx\,dy,$$

we have

$$(34.11)\qquad M = D\alpha.$$

The formula (34.11) shows that the twisting moment M is proportional to the angle α of twist per unit length, so that the constant D provides a measure of the rigidity of a beam subjected to torsion. For this reason the constant D (depending on the modulus of rigidity μ and on the shape of the cross section only) is called the *torsional rigidity* of the beam.

It follows from the foregoing that the torsion problem for a beam of any cross section is completely solved once the function $\varphi(x, y)$ is determined. For the torsional rigidity D is determined by φ from (34.10), and the moment M required to produce the angle α of twist per unit length can be calculated from (34.11).

In carrying out the foregoing calculations, no assumptions were made regarding the location of the origin O or concerning the orientation of the axes x, y. Inasmuch as the first two of the formulas (34.3) represent the infinitesimal rotation of any cross section of the beam as a whole about the z-axis, it may seem at first glance that a different choice of the axis of rotation parallel to the axis of z may yield a different solution of the problem. For instance, if the axis z' is chosen parallel to the z-axis, and if it intersects the xy-plane at some point (x_1, y_1), then the displacements u_1, v_1, and w_1 will be

$$u_1 = -\alpha z(y - y_1), \qquad v_1 = \alpha z(x - x_1), \qquad w_1 = \alpha\varphi_1(x, y),$$

and there is no a priori reason why the functions $\varphi_1(x, y)$ and $\varphi(x, y)$ should be identical.

Calculating stresses that correspond to displacements (u_1, v_1, w_1) yields

$$(34.12)\qquad \begin{cases} \tau_{zy} = \mu\alpha\left(\dfrac{\partial\varphi_1}{\partial y} + x - x_1\right), \\ \tau_{zx} = \mu\alpha\left(\dfrac{\partial\varphi_1}{\partial x} - y + y_1\right), \\ \tau_{xy} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0, \end{cases}$$

and the substitution of these values in the equations of equilibrium (29.1) shows that the function φ_1 likewise satisfies the equation

$$\nabla^2\varphi_1 \equiv \frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial y^2} = 0.$$

Moreover, the third of the boundary conditions (29.4) demands that

$$\left(\frac{\partial\varphi_1}{\partial x} + y_1\right)\cos(x, \nu) + \left(\frac{\partial\varphi_1}{\partial y} - x_1\right)\cos(y, \nu) = y\cos(x, \nu) - x\cos(y, \nu)$$

or

$$\frac{d}{d\nu}(\varphi_1 + y_1x - x_1y) = y\cos(x, \nu) - x\cos(y, \nu).$$

But the function $\varphi_1 + y_1x - x_1y$ is harmonic, and since it satisfies the same boundary condition as the function φ [see (34.6)], it follows from the uniqueness of solution of the problem of Neumann[1] that the two can

[1] See O. D. Kellogg, Foundations of Potential Theory, Chap. XI, Sec. 12.

differ only by a constant. Thus,

$$\varphi_1 = \varphi(x, y) - y_1 x + x_1 y + \text{const.}$$

A simple calculation making use of the formulas (34.12) shows that the system of stresses obtained by using the function $\varphi_1(x, y)$ is identical with that obtained by using the function $\varphi(x, y)$. It follows that the displacement in the two cases can differ only by a rigid body displacement. Thus, we see that the position of the origin of coordinates is immaterial in this problem.

We remark in conclusion that the formulation of the torsion problem given here is valid when R is a multiply connected region.

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35. Stress Function. Since the torsion function $\varphi(x, y)$ is harmonic in the region R representing the cross section of the beam, one can construct the analytic function[1] $\varphi + i\psi$, of complex variable $x + iy$, where $\psi(x, y)$ is the conjugate harmonic function, related to $\varphi(x, y)$ by the Cauchy-Riemann equations,

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{35.1}$$

Since the function $\varphi + i\psi$ is an analytic function of the complex variable $x + iy$, it is clear that the function $\psi(x, y)$ is determined by the formula

$$\psi(x, y) = \int_C \left(\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy\right) = \int_{P_0(x_0,y_0)}^{P(x,y)} \left(-\frac{\partial \varphi}{\partial y}\,dx + \frac{\partial \varphi}{\partial x}\,dy\right), \tag{35.2}$$

where the integral is evaluated over an arbitrary path joining some point $P_0(x_0, y_0)$ with an arbitrary point $P(x, y)$ belonging to the region R. If the region R is simply connected, the function $\psi(x, y)$ will be single-valued; in a multiply connected region, $\psi(x, y)$ may turn out to be multiple-valued. For the time being, we shall be concerned with simply connected regions, and the discussion in this section will be confined to such regions.

It is not difficult to phrase the torsion problem in terms of the conjugate function $\psi(x, y)$. Thus, noting the relations (34.2), one can write the

[1] Some basic results of the theory of analytic functions of a complex variable may be found in I. S. and E. S. Sokolnikoff, Higher Mathematics for Engineers and Physicists, Chap. X, pp. 440–491.

expression for the normal derivative $\frac{d\varphi}{d\nu}$ with the aid of the tangential derivatives $\frac{dx}{ds}$ and $\frac{dy}{ds}$, so that

$$\begin{aligned}\frac{d\varphi}{d\nu} &= \frac{\partial\varphi}{\partial x}\cos(x,\nu) + \frac{\partial\varphi}{\partial y}\cos(y,\nu)\\ &= \frac{\partial\varphi}{\partial x}\frac{dy}{ds} - \frac{\partial\varphi}{\partial y}\frac{dx}{ds}.\end{aligned}$$

Making use of the Cauchy-Riemann equations (35.1), we have

$$\frac{d\varphi}{d\nu} = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} \equiv \frac{d\psi}{ds}.$$

Moreover, the boundary condition (34.6) can be written as

$$\begin{aligned}\frac{d\varphi}{d\nu} &= y\cos(x,\nu) - x\cos(y,\nu)\\ &= x\frac{dx}{ds} + y\frac{dy}{ds} = \frac{1}{2}\frac{d}{ds}(x^2+y^2).\end{aligned}$$

Hence

$$\frac{d\psi}{ds} = \frac{1}{2}\frac{d}{ds}(x^2+y^2),$$

so that

(35.3) $$\psi = \tfrac{1}{2}(x^2+y^2) + \text{const} \qquad \text{on } C.$$

It will be recalled that the torsion function φ is determined to within a nonessential arbitrary constant; the derivatives of φ and hence those of ψ [see (35.1)] are determined uniquely, and the function ψ is determined by means of (35.2) to within a constant depending on the choice of $P_0(x_0, y_0)$. Accordingly, we are free to assign any value to the constant of integration in (35.3), since this choice will not affect stresses, and the two sets of displacements that correspond to two different choices of the arbitrary constant will differ from one another by a rigid body motion.

Thus, instead of solving a problem of Neumann, we can equally well solve a problem of Dirichlet by determining a function that is harmonic in a given region and which assumes prescribed values on the boundary of the region.

On account of the remarks just made, our problem consists in determining a function ψ that satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad \text{in } R,$$

and that satisfies the boundary condition

(35.4) $$\psi = \tfrac{1}{2}(x^2+y^2).$$

It is known that the solution of this problem is unique,[1] and there are

[1] See O. D. Kellogg, Foundations of Potential Theory, Chap. IX, Sec. 5.

general methods that permit one to construct solutions of the problem of Dirichlet. We shall consider some of them in detail in succeeding sections.

We shall now formulate the torsion problem in terms of the function Ψ, introduced by L. Prandtl,[1] which is defined as follows:

$$\Psi = \psi(x, y) - \tfrac{1}{2}(x^2 + y^2). \tag{35.5}$$

We have

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \psi}{\partial x} - x, \qquad \frac{\partial \Psi}{\partial y} = \frac{\partial \psi}{\partial y} - y,$$

and, upon recalling the formulas (34.4) and (35.1), it follows that

$$\tau_{zx} = \mu\alpha \frac{\partial \Psi}{\partial y}, \qquad \tau_{zy} = -\mu\alpha \frac{\partial \Psi}{\partial x}. \tag{35.6}$$

Since the stress components τ_{zx} and τ_{zy} are obtained from the function $\Psi(x, y)$ by differentiation, the latter is called the *stress function*. It is readily checked that the stress function Ψ satisfies Poisson's equation

$$\nabla^2\Psi \equiv \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -2 \qquad \text{in } R, \tag{35.7}$$

and on the boundary C of the region R [cf. (35.3) and (35.5)] assumes the value

$$\Psi = \text{const.}$$

Consider a family of curves, in the plane of the cross section of the beam, obtained by setting

$$\Psi(x, y) = \text{const.} \tag{35.8}$$

The slope $\dfrac{dy}{dx}$ of the tangent line to any curve of the family defined by (35.8) is determined from the formula

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0,$$

and, upon noting the relations (35.6), we obtain

$$\frac{dy}{dx} = \frac{\tau_{zy}}{\tau_{zx}}.$$

Thus, at each point of the curve $\Psi(x, y) = \text{const}$, the stress vector

$$\boldsymbol{\tau} = \mathbf{i}\tau_{zx} + \mathbf{j}\tau_{zy}$$

is directed along the tangent to the curve. The curves

$$\Psi(x, y) = \text{const}$$

[1] *Physikalische Zeitschrift*, vol. 4 (1903), pp. 758–770.

are called the *lines of shearing stress.* The magnitude τ of the tangential stress is

$$\tau = \sqrt{\tau_{zy}^2 + \tau_{zx}^2} = \mu\alpha\sqrt{\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2}.$$

Recalling that for a circular cylinder the magnitude of the tangential stress is given by

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \mu\alpha\sqrt{x^2 + y^2},$$

we see that in this case the maximum shearing stress occurs on the boundary of the section. It is not difficult to prove that in the general case the points at which maximum shearing stress occurs lie on the boundary C of the section, so that elastic failure of material in shear is to be expected on the lateral surface of the beam. In order to prove the assertion, we refer to a theorem.

THEOREM: *Let a function Φ of class C^2 and not identically equal to a constant satisfy the inequality $\nabla^2\Phi \geq 0$ in the region R; then this function attains its maximum on the boundary C of the region R.*

The proof of this theorem follows at once from the well-known property of *subharmonic* functions. It will be recalled that a function $\Phi(x, y)$ is called subharmonic in the region R if at every point (x_1, y_1) of the region

$$\Phi(x_1, y_1) \leq \frac{1}{2\pi r}\int_\gamma \Phi(x, y)\, ds, \tag{1}$$

where the integral is evaluated over the circle γ of sufficiently small radius r, with center at (x_1, y_1). Now, if it be assumed that the maximum value M of a subharmonic function $\Phi(x, y) \neq$ const is attained, not on the boundary C, but at some interior points of R, we arrive at a contradiction. For if S is a set of such interior points and Q is a frontier point of S, we have from (1)

$$M = \Phi(Q) \leq \frac{1}{2\pi r}\int_\gamma \Phi(x, y)\, ds, \tag{2}$$

where γ is a circle with center at Q and of radius r so small that γ is interior to R. But since γ is partly outside S, the mean value of Φ over γ is less than M, that is,

$$\frac{1}{2\pi r}\int_\gamma \Phi(x, y)\, ds < M$$

and this contradicts (2).

Since

$$\tau^2 = \mu^2\alpha^2\left[\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2\right]$$

a simple calculation, making use of (35.7), shows that

$$\nabla^2\tau^2 = 2\mu^2\alpha^2\left[\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\Psi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\Psi}{\partial y^2}\right)^2\right].$$

Thus $\nabla^2\tau^2$ is nonnegative, and therefore τ^2 is subharmonic in R. Accordingly, τ attains its maximum on the boundary of R.

Since the strength of the beam to resist torsion depends on the maximum shearing stress, practical rules for the design of beams carrying torsional loads are expressed in terms of the safe maximum shearing stress τ.

The formula (34.10) for the torsional rigidity D can be phrased in terms of the stress function[1] Ψ. The resulting expression is of great interest in deducing approximate solutions of the torsion problems by the membrane analogy discussed in Sec. 46.

We first recall the formula (34.11),

$$M = D\alpha,$$

where

$$M = \iint_R (x\tau_{zy} - y\tau_{zx})\,dx\,dy.$$

Since[2]

$$\tau_{zx} = \mu\alpha\frac{\partial\Psi}{\partial y}, \qquad \tau_{zy} = -\mu\alpha\frac{\partial\Psi}{\partial x},$$

we have

$$M = -\mu\alpha\iint_R\left(x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y}\right)dx\,dy.$$

so that

$$\begin{aligned}(35.9)\qquad D &= -\mu\iint_R\left(x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y}\right)dx\,dy\\ &= -\mu\iint_R\left[\frac{\partial(x\Psi)}{\partial x} + \frac{\partial(y\Psi)}{\partial y}\right]dx\,dy + 2\mu\iint_R\Psi\,dx\,dy.\end{aligned}$$

The first of the double integrals in the foregoing can be transformed by Green's Theorem so that (35.9) reads

$$D = -\mu\int_C\Psi[x\cos(x,\nu) + y\cos(y,\nu)]\,ds + 2\mu\iint_R\Psi\,dx\,dy.$$

But we can choose [see (35.4) and (35.5)]

$$\Psi = 0 \qquad \text{on } C,$$

[1] That Ψ attains its minimum values on the boundary follows from (35.7). For if Ψ were to take on its minimum at some interior point P, then $\Psi_x = \Psi_y = 0$, $\Psi_{xx} \geq 0$, $\Psi_{yy} \geq 0$ at P. But this is impossible, since $\Psi_{xx} + \Psi_{yy} = -2$ at P.

[2] See formulas (35.6).

and the foregoing expression becomes

$$D = 2\mu \iint_R \Psi \, dx \, dy. \tag{35.10}$$

It is obvious from (35.10) that the torsional rigidity of a beam whose cross section R is bounded by the contour C is twice the product of the shear modulus μ and the volume enclosed by the surface $z = \Psi(x, y)$ and the plane $z = 0$. We shall see in a later section that a homogeneous, uniformly stretched membrane subjected to a uniform pressure is distorted into a surface whose differential equation is of the same form as that for the stress function Ψ. The connection between the surface of the loaded membrane and the stress function Ψ is utilized in the experimental determination of the magnitude of stresses in cylinders whose cross sections are such as to make a mathematical determination of the torsion function very difficult.

Before proceeding to a consideration of specific examples, we note that our solution requires that the tangential stresses τ_{zx} and τ_{zy} be distributed over the ends of the beam in a manner specified by (34.4). In practical applications, this particular distribution of stress may not correspond to the actual physical situation, but, on the basis of Saint-Venant's principle, we can assert that, sufficiently far from the ends of the beam, the stress will depend on the magnitude of the couple M and will be quite independent of the mode of distribution of tractions over the ends of the beam.

We have seen that the torsion problem can be reduced to the problem of finding a function $\psi(x, y)$ that is harmonic in the region R and takes the values $\frac{1}{2}(x^2 + y^2)$ on the boundary C of R. Some special methods of solving the torsion problem will be considered in the following sections. In the next two sections, our plan of attack will be to consider a particular harmonic function ψ that contains some undetermined coefficients. These undetermined coefficients will be chosen in such a way that, on the boundary of a certain region, ψ takes on the values $\frac{1}{2}(x^2 + y^2)$. In Sec. 38, a solution in the form of an infinite series will be obtained for rectangular and triangular prisms. The general solution of the torsion problem for a beam of arbitrary solid cross section R is then given by mapping the region R upon the interior of a circle and then considering the solutions of the problems of Dirichlet and Neumann for the circular region.

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PROBLEMS

1. Consider a circular shaft of length l, radius a, and shear modulus μ, twisted by a couple M. Show that the greatest angle of twist θ and the maximum shear stress $T = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$ are given by

$$\theta_{\max} = \frac{2Ml}{\pi\mu a^4},$$
$$T_{\max} = \frac{2M}{\pi a^3}.$$

2. A steel shaft of circular cross section 2 in. in diameter and 5 ft long is twisted by end couples. Find the maximum twisting moment and angle of twist if the greatest shear stress is not to exceed 10,000 lb per sq in. Take $E = 30 \times 10^6$ lb per sq in., $\sigma = 0.3$.

3. The shaft of the preceding problem is not to be twisted more than 1°. What is the corresponding maximum shear stress?

4. Derive the expression

$$M_z = \frac{63,000}{n} H$$

for the torque M_z on a solid circular shaft transmitting H hp at a speed n rpm. *Hint:* Let the radius of the shaft (or pulley) be r in., and let $T = M_z/r$ be the tension in the belt. Calculate the work done in each minute against M_z. (1 hp = 33,000 ft-lb per min.)

5. Derive the expressions

$$\alpha = \frac{32M_z}{\mu\pi d^4} = \frac{2(\tau_z)_{\max}}{\mu d} = \frac{640,000}{\mu n d^4} H$$

for the twist per inch length α (radians) in a solid circular shaft of diameter d in., transmitting H hp at n rpm against a torque of M_z in.-lb.

6. How much torque can be transmitted by a solid circular shaft 3 in. in diameter if the allowable shear stress is 10,000 lb per sq in.? What is the angle of twist per foot of length? Use $\mu = 12 \times 10^6$ lb per sq in.

36. Torsion of Elliptical Cylinder. It was shown above that the solution of the torsion problem for a solid cylinder of arbitrary cross section is completely determined if one obtains the harmonic function ψ that on the boundary C of the cross section assumes the value

$$\psi = \tfrac{1}{2}(x^2 + y^2). \tag{36.1}$$

Consider the harmonic function

$$\psi = c^2(x^2 - y^2) + k^2, \tag{36.2}$$

where c and k are constants. The function defined by (36.2) will enable us to solve the torsion problem for some region R on the boundary of which (36.2) reduces to (36.1). Hence points of the boundary C of the

region R are determined by equating (36.1) and (36.2). Thus,

$$c^2(x^2 - y^2) + k^2 = \tfrac{1}{2}(x^2 + y^2),$$

or

$$(\tfrac{1}{2} - c^2)x^2 + (\tfrac{1}{2} + c^2)y^2 = k^2. \tag{36.3}$$

The curve defined by Eq. (36.3) is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

if we choose $c^2 < \frac{1}{2}$ and

$$a = \frac{k}{\sqrt{\frac{1}{2} - c^2}}, \qquad b = \frac{k}{\sqrt{\frac{1}{2} + c^2}}.$$

Then

$$c^2 = \frac{1}{2}\frac{a^2 - b^2}{a^2 + b^2}, \qquad k^2 = \frac{a^2b^2}{a^2 + b^2}.$$

Substituting the values of c and k in terms of a and b in (36.2), we obtain the solution of our boundary-value problem for an ellipse with semiaxes a and b, namely,

$$\psi = \frac{1}{2}\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) + \frac{a^2b^2}{a^2 + b^2}. \tag{36.4}$$

The components of stress (34.4) can be expressed directly in terms of the function ψ by noting the Cauchy-Riemann equations (35.1). Thus,

$$\tau_{zx} = \mu\alpha\left(\frac{\partial\psi}{\partial y} - y\right), \qquad \tau_{zy} = \mu\alpha\left(-\frac{\partial\psi}{\partial x} + x\right).$$

Hence

$$\tau_{zx} = \frac{-2\mu\alpha a^2 y}{a^2 + b^2}, \qquad \tau_{zy} = \frac{2\mu\alpha b^2 x}{a^2 + b^2}. \tag{36.5}$$

The torsion moment M is

$$\begin{aligned} M &= \iint_R (x\tau_{zy} - y\tau_{zx})\,dx\,dy \\ &= \frac{2\mu\alpha}{a^2 + b^2}\left(b^2\iint_R x^2\,dx\,dy + a^2\iint_R y^2\,dx\,dy\right) \\ &= \frac{2\mu\alpha}{a^2 + b^2}(a^2 I_x + b^2 I_y), \end{aligned}$$

where I_x and I_y are the moments of inertia of the elliptical section about the x- and y-axes. Recalling that

$$I_x = \frac{\pi ab^3}{4}, \qquad I_y = \frac{\pi a^3 b}{4},$$

we have

$$M = \frac{\pi\mu\alpha a^3 b^3}{a^2 + b^2},$$

so that the torsional rigidity

$$D = \frac{\pi\mu a^3 b^3}{a^2 + b^2}.$$

It was shown in the preceding section that the maximum shearing stress on any cross section occurs on the boundary of the section. The location of the points on the boundary at which the greatest stress τ_{max} occurs can be determined[1] by maximizing the expression for τ that has been obtained as a function of a single variable by utilizing the equation of the boundary C. In the case of an elliptical cylinder, the points of greatest shearing stress can be found easily from some simple geometrical considerations.

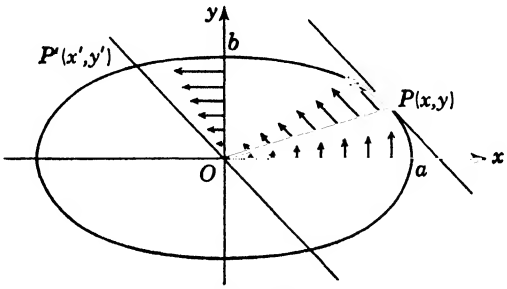

Fig. 22

Consider an elliptical section, shown in Fig. 22, and draw from the center of the ellipse a semidiameter OP to an arbitrary point $P(x, y)$ of the boundary. Since the diameter of the ellipse conjugate to the diameter through P is parallel to the tangent line[2] at $P(x, y)$, it follows that the conjugate semidiameter OP' intersects the curve at the point $P'(x', y')$, where

$$x' = -\frac{ay}{b}, \qquad y' = \frac{bx}{a}.$$

When the stresses at $P(x, y)$ are written in terms of the coordinates x', y' of the point P', we have

$$\tau_{zx} = \frac{2\mu\alpha ab}{a^2 + b^2}x', \qquad \tau_{zy} = \frac{2\mu\alpha ab}{a^2 + b^2}y',$$

so that the direction of stress at the point P is parallel to the conjugate semidiameter OP'. Furthermore, the magnitude τ of the tangential

[1] See Prob. 1 at the end of this section.

[2] See, for example, W. F. Osgood and W. C. Graustein, Plane and Solid Analytic Geometry, Chap. XIV.

stress at $P(x, y)$ is

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{yz}^2} = \frac{2\mu\alpha ab}{a^2 + b^2} \sqrt{x'^2 + y'^2} = \frac{2\mu\alpha ab}{a^2 + b^2} r',$$

where r' is the distance OP'. Since the conjugate semidiameter is of maximum length when the point P is at an extremity of the minor axis, it follows that

$$\tau_{max} = \frac{2\mu\alpha a^2 b}{a^2 + b^2}.$$

Thus, the maximum stress occurs at the extremities of the minor axis of the ellipse, contrary to an intuitive expectation that the maximum stress would be at the points of maximum curvature.

It is easy to verify that the conjugate harmonic function φ, apart from a nonessential constant, is[1]

$$\varphi = -\frac{a^2 - b^2}{a^2 + b^2} xy. \tag{36.6}$$

This function determines the warping of the cross sections of the cylinder, for the displacement along the z-axis is given by $w = \alpha\varphi(x, y)$. The contour lines, obtained by setting $\varphi = \text{const}$, are the hyperbolas shown in Fig. 23. The dotted lines indicate the portions of the section that become concave, and the solid those that become convex, when the cylinder is twisted by a couple in the directions shown in the figure by arrows.

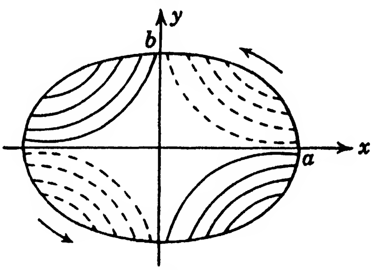

Fig 23

The lines of shearing stress are determined by drawing the contour lines for the surface $z = \Psi(x, y)$. Setting $\Psi(x, y) = \text{const}$ gives,[2] in this case, a family of concentric ellipses,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \frac{-c'(a^2 + b^2)}{a^2 b^2},$$

similar to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The displacement of the points of the cylinder is given at once by the formulas (34.3). The results obtained in Sec. 33 for a beam of circular cross section follow at once from the formulas of this section upon setting $b = a$.

PROBLEMS

1. Show that, in the torsion of an elliptical cylinder, the magnitude of the stress vector τ takes the following value on the boundary of the section $z = \text{const}$:

[1] See Eq. (36.1).

[2] See Prob. 3 at the end of this section.

$$\tau = 2\mu\alpha \frac{ab}{a^2 + b^2} \sqrt{a^2 - e^2x^2},$$
$$e = \frac{1}{a} \sqrt{a^2 - b^2}.$$

From this relation it follows that the maximum shearing stress occurs at the ends of the minor axis of the ellipse.

2. Derive the expression (36.6) from

$$\varphi(x, y) = \int_{P_0}^{P} \left(\frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy \right) + \text{const},$$

and evaluate the line integral over the path consisting of the straight-line segments from $P_0(x_0, y_0)$ to $Q(x, y_0)$ and from $Q(x, y_0)$ to $P(x, y)$.

3. Show that the stress function for an elliptical section can be written as

$$\Psi = \frac{-a^2b^2}{a^2 + b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$

and is thus proportional to the function appearing in the equation of the boundary of the section. The problem of determining the sections for which this proportionality holds has been treated by Leibenson.[1]

37. Simple Solutions of the Torsion Problem. Effect of Grooves. The method of solution of the torsion problem illustrated in the preceding section was used by Saint-Venant, who selected a number of simple polynomial solutions of the equation

$$\nabla^2 \psi = 0, \tag{37.1}$$

and determined the equation of the boundary of the cross section of the cylinder on which the function ψ reduces to $\frac{1}{2}(x^2 + y^2)$. Inasmuch as the real and imaginary parts of every analytic function of a complex variable $x + iy$ satisfy Eq. (37.1), we can build up a list of functions ψ and, by working, so to speak, backward, can determine the equations of the contours for which these functions ψ represent the solution of the torsion problem. For example, if we consider the function $(x + iy)^n$, then by choosing $n = 2$, we get two solutions, $x^2 - y^2$ and $2xy$, of Eq. (37.1). The first of these solutions was utilized in the preceding section to solve the torsion problem for an elliptical cylinder. If n is set equal to 3, we obtain the harmonic functions $x^3 - 3xy^2$ and $3x^2y - y^3$. Now consider the harmonic function

$$\psi = c(x^3 - 3xy^2) + k, \tag{37.2}$$

where c and k are constants. The function ψ determines the solution of the torsion problem for a cylinder whose cross section has the equation

$$c(x^3 - 3xy^2) + k = \tfrac{1}{2}(x^2 + y^2). \tag{37.3}$$

[1] L. Leibenson, "Über den Zusammenhang zwischen der Spannungsfunktion bei Torsion und der Konturgleichung eines Prismenquerschnittes," *Wissenschaftliche Berichte der Moskauer Universität*, vol. 2 (1934), pp. 99–102 (in Russian with a German summary).

By altering the values of the parameters in (37.3), we obtain various cross sections, some of which may be of technical interest. If we set $c = -1/6a$ and $k = 2a^2/3$, then (37.3) can be written in the factored form as

$$(x - a)(x - y\sqrt{3} + 2a)(x + y\sqrt{3} + 2a) = 0,$$

so that the boundary of the region is the equilateral triangle of altitude $3a$ (see Fig. 24).

Making use of the formulas (35.6), we find

$$\tau_{zx} = \frac{\mu\alpha}{a}\, y(x - a), \qquad \tau_{zy} = \frac{\mu\alpha}{2a}\,(x^2 + 2ax - y^2).$$

We see from these formulas that the x-component of the shearing stress vanishes along the x-axis, while the y-component becomes

$$(\tau_{zy})_{y=0} = \frac{\mu\alpha}{2a}\, x(x + 2a).$$

The distribution of stress along the x-axis is indicated in Fig. 25. The

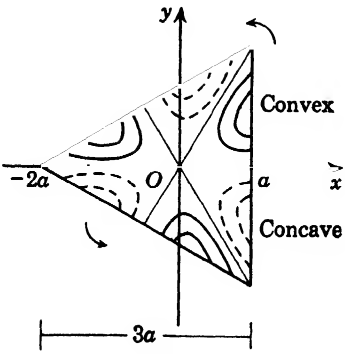

Fig. 24

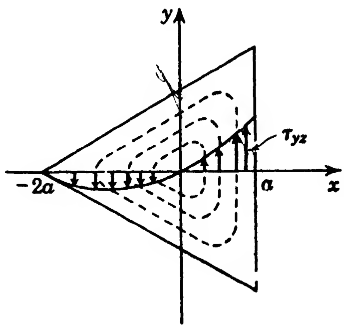

Fig. 25

shearing stress is a maximum at the mid-points of the sides of the triangle, and its value is

$$\tau_{max} = \tfrac{3}{2}a\mu\alpha.$$

The stress vanishes at the corners and at the origin O. The direction of the lines of shearing stress is along the curves

$$\psi - \tfrac{1}{2}(x^2 + y^2) = \text{const}$$

a few of which are indicated in Fig. 25 by dotted lines. It is easily checked, with the aid of (34.9), that the torsional couple has the magnitude

$$M = \tfrac{3}{5}\mu\alpha I_0,$$

where $I_0 = 3\sqrt{3}\, a^4$ is the polar moment of inertia of the triangle. The nature of the distortion of the initially plane sections is indicated in Fig.

24, where the contour lines of the surface $\varphi(x, y) \equiv (3x^2y - y^3)/6a = \text{const}$ are shown.

It appears from this example, and from that of the preceding section, that a circular shaft of the same cross-sectional area as an elliptical beam or a triangular prism has the greatest torsional rigidity.[1] One can also prove that, if the region is simply connected, then, for a given moment M and for a given cross-sectional area, the smallest maximum stress will be found in a circular beam. This is discussed further in Sec. 47 in connection with the torsion of beams with multiply connected cross sections.

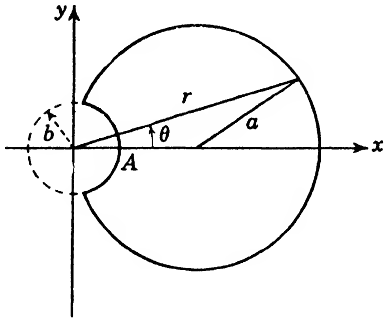

FIG 26

The effect of grooves or slots in the beam on the maximum shearing stress can be discussed in an elementary way by studying an example due to C. Weber.[2]

Consider a pair of harmonic functions,

$$x \quad \text{and} \quad \frac{x}{x^2 + y^2},$$

and introduce the polar coordinates defined by the equations $x = r \cos \theta$, $y = r \sin \theta$. We can construct a harmonic function ψ,

$$\psi = a\left(x - b^2 \frac{x}{x^2 + y^2}\right) + \frac{1}{2} b^2 = a\left(r \cos \theta - \frac{b^2 \cos \theta}{r}\right) + \frac{1}{2} b^2,$$

where a and b are constants.

On the boundary C of the cross section, ψ must reduce to

$$\psi = \tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}r^2,$$

so that the equation of the boundary for which the function ψ solves the torsion problem is

$$a\left(r \cos \theta - \frac{b^2 \cos \theta}{r}\right) + \frac{1}{2} b^2 = \frac{1}{2} r^2,$$

or

$$r^2 - b^2 - 2a(r^2 - b^2) \frac{\cos \theta}{r} = 0.$$

Factoring this expression gives

$$(r^2 - b^2)\left(1 - \frac{2a \cos \theta}{r}\right) = 0.$$

[1] See Prob. 1 at the end of this section.

[2] C. Weber, *Forschungsarbeiten, VDI*, No. 249 (1921).

Thus, the boundary is made up of two circles

$$r = b \qquad \text{and} \qquad r = 2a\cos\theta,$$

which are shown in Fig. 26.

Since the function ψ is known, one can easily calculate the stresses τ_{zx} and τ_{zy}. It turns out that the maximum shearing stress is at the point A and has the value[1]

$$\tau_{\max} \doteq 2\mu\alpha a,$$

which is twice as great as the peripheral stress in a circular shaft of radius a. This example indicates the importance of considering stresses in slots and keyways of shafts.

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PROBLEMS

1. Let D_0 be the torsional rigidity of a circular cylinder, D_e that of an elliptical cylinder, and D_t that of a beam whose cross section is an equilateral triangle. Show that for cross sections of equal areas

$$D_e = kD_0, \qquad D_t = \frac{2\pi\sqrt{3}}{15} D_0,$$

where

$$k = \frac{2ab}{a^2 + b^2} \leq 1,$$

and a, b are the semiaxes of the elliptical section.

2. Consider a circular shaft of radius a with a circular groove of radius b along a generator of the shaft (see Sec. 37). Show that on the groove the shearing stresses are

$$\begin{aligned}
\tau_{zx} &= \mu\alpha(2a\cos\theta - b)\sin\theta,\\
\tau_{zy} &= -\mu\alpha(2a\cos\theta - b)\cos\theta,\\
\tau &= \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \mu\alpha(2a\cos\theta - b),
\end{aligned}$$

while on the shaft we have

$$\begin{aligned}
\tau_{zx} &= \frac{\mu\alpha}{4a}(b^2 - 4a^2\cos^2\theta)\frac{\sin 2\theta}{\cos^2\theta},\\
\tau_{zy} &= -\frac{\mu\alpha}{4a}(b^2 - 4a^2\cos^2\theta)\frac{\cos 2\theta}{\cos^2\theta},\\
\tau &= \mu\alpha\left(a - \frac{b^2}{4a}\sec^2\theta\right).
\end{aligned}$$

[1] See Prob. 2 at the end of this section.

Find the magnitude of the shearing stress at the point (see Fig. 26) where the groove enters the shaft.

38. Torsion of a Rectangular Beam and of a Triangular Prism. Consider a beam of rectangular cross section, and let one side of the cross section, of length a, be parallel to the x-axis and that of length b be parallel to the y-axis. It will be supposed that $b \geq a$ and that the z-axis passes through the center of the cross section.

The torsion problem will be solved if we succeed in determining the function $\psi(x, y)$ that is harmonic in the region bounded by $x = \pm a/2$, $y = \pm b/2$ and that assumes on the boundary of the region the values $\frac{1}{2}(x^2 + y^2)$. In this case, the boundary conditions can be written as

$$\psi\left(\pm \frac{a}{2}, y\right) = \frac{a^2}{8} + \frac{y^2}{2}, \qquad \psi\left(x, \pm \frac{b}{2}\right) = \frac{b^2}{8} + \frac{x^2}{2}. \tag{38.1}$$

The boundary conditions (38.1) are somewhat complicated, and it will simplify our search for the function ψ if we introduce a function $f(x, y)$, defined by the formula

$$f(x, y) = \frac{\partial^2 \psi}{\partial x^2} + 1. \tag{38.2}$$

The function $f(x, y)$ is obviously harmonic. Since the function $\psi(x, y)$ satisfies the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0,$$

we can also write

$$f(x, y) = -\frac{\partial^2 \psi}{\partial y^2} + 1. \tag{38.3}$$

By differentiating Eqs. (38.1), we see that

$$\frac{\partial^2 \psi}{\partial y^2} = 1 \qquad \text{on } x = \pm \frac{a}{2},$$
$$\frac{\partial^2 \psi}{\partial x^2} = 1 \qquad \text{on } y = \pm \frac{b}{2},$$

and from (38.2) and (38.3) it follows that the boundary values of the harmonic function $f(x, y)$ are

$$\begin{cases} f(x, y) = 0 & \text{on } x = \pm \dfrac{a}{2}, \\[2ex] f(x, y) = 2 & \text{on } y = \pm \dfrac{b}{2}. \end{cases} \tag{38.4}$$

The function $f(x, y)$ satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{38.5}$$

and we seek a solution of this equation in the form of an infinite series

$$f(x, y) = \sum_{n=0}^{\infty} c_n X_n(x) Y_n(y),$$

where each term of the series satisfies the differential equation (38.5), and where $X_n(x)$ and $Y_n(y)$ are, respectively, functions of x alone and of y alone. Substituting $X_n(x)Y_n(y)$ in (38.5), and denoting the derivatives by primes, we get

$$X_n''(x)Y_n(y) + X_n(x)Y_n''(y) = 0,$$

or

$$\frac{X_n''(x)}{X_n(x)} = -\frac{Y_n''(y)}{Y_n(y)}.$$

Since the left-hand member of this expression is a function of x alone and the right-hand member depends only on y, the equality can be fulfilled only if each member is equal to a constant, say $-k_n^2$. We are thus led to a pair of ordinary differential equations

$$\frac{d^2X_n}{dx^2} + k_n^2 X_n = 0 \qquad \text{and} \qquad \frac{d^2Y_n}{dy^2} - k_n^2 Y_n = 0,$$

whose linearly independent solutions are

$$X_n = \begin{cases} \cos k_n x, \\ \sin k_n x, \end{cases} \qquad Y_n = \begin{cases} \cosh k_n y, \\ \sinh k_n y. \end{cases}$$

Since our solutions must satisfy the boundary conditions (38.4), we reject the terms involving the odd functions $\sin k_n x$ and $\sinh k_n y$, and choose the product $X_n Y_n$ of the form

$$\cos k_n x \cosh k_n y,$$

where

$$k_n = \frac{(2n + 1)\pi}{a}.$$

Thus, each term of the series

$$f(x, y) = \sum_{n=0}^{\infty} c_n \cos k_n x \cosh k_n y \tag{38.6}$$

satisfies the first of the boundary conditions (38.4), and it remains to satisfy the conditions on the edges $y = \pm b/2$. Substituting $y = \pm b/2$ in (38.6) yields the equation

$$2 = \sum_{n=0}^{\infty} c_n \cosh \frac{k_n b}{2} \cos k_n x, \tag{38.7}$$

from which it follows that the coefficients c_n can be formally determined by utilizing the scheme used in expanding functions in Fourier series. If we multiply both members of (38.7) by $\cos (2m + 1)\pi x/a$ and integrate term by term with respect to x between the limits $-a/2$ and $a/2$, then because of the orthogonal property of trigonometric functions, namely,

$$\int_{-a/2}^{a/2} \cos k_n x \cos k_m x \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{a}{2} & \text{if } m = n, \end{cases}$$

we get

$$\int_{-a/2}^{a/2} 2 \cos k_m x \, dx = \frac{a}{2} c_m \cosh \frac{k_m b}{2}.$$

Upon evaluating the integral, we see that[1]

$$c_m = \frac{8(-1)^m}{\pi(2m + 1)} \cdot \frac{1}{\cosh (k_m b/2)},$$

so that the formal solution is

$$f(x, y) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \frac{\cosh k_n y}{\cosh (k_n b/2)} \cos k_n x. \tag{38.8}$$

The stresses τ_{zx} and τ_{zy} are given by the formulas

$$\tau_{zx} = \mu\alpha \left(\frac{\partial \psi}{\partial y} - y \right), \qquad \tau_{zy} = \mu\alpha \left(-\frac{\partial \psi}{\partial x} + x \right), \tag{38.9}$$

and since

$$\frac{\partial^2 \psi}{\partial x^2} = f(x, y) - 1,$$

and

$$\frac{\partial^2 \psi}{\partial y^2} = -f(x, y) + 1,$$

we see that, in order to evaluate stresses, we must integrate the series (38.8) with respect to x and y. Integrating, and making use of the fact that $\tau_{zx} = 0$ on $x = \pm a/2$ and $\tau_{zy} = 0$ on $y = \pm b/2$, we obtain

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = -x + \dfrac{8a}{\pi^2} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} \frac{\cosh k_n y}{\cosh (k_n b/2)} \sin k_n x, \\ \dfrac{\partial \psi}{\partial y} = y - \dfrac{8a}{\pi^2} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} \frac{\sinh k_n y}{\cosh (k_n b/2)} \cos k_n x. \end{cases} \tag{38.10}$$

[1] These are the Fourier coefficients for $f(x) = 2$, $-a/2 < x < a/2$, and $f(x) = 0$, $-a < x < -a/2$ and $a/2 < x < a$.

Hence the stresses τ_{zx} and τ_{zy} can be calculated from the series

$$(38.11)\quad \begin{cases} \tau_{zx} = \dfrac{-8a\mu\alpha}{\pi^2} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)^2} \dfrac{\sinh k_n y}{\cosh (k_n b/2)} \cos k_n x, \\ \tau_{zy} = \mu\alpha \left[2x - \dfrac{8a}{\pi^2} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)^2} \dfrac{\cosh k_n y}{\cosh (k_n b/2)} \sin k_n x \right]. \end{cases}$$

The solutions (38.11) are formal, but the series converge so rapidly that there is no serious difficulty in justifying the term-by-term differentiation to show that the equilibrium equations are satisfied. The x-component of shear obviously vanishes when $y = 0$, while the y-component at the mid-point of the longer side is equal to

$$(38.12)\qquad \tau_{zy}\Big|_{\substack{x=a/2 \\ y=0}} = \mu\alpha a \left[1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \operatorname{sech} \frac{k_n b}{2} \right].$$

It is not difficult to prove that (38.12) gives the maximum value of the shearing stress, by taking note of the fact that the term $2x$ in the brackets of (38.11) dominates the series. Now in the most unfavorable case (for convergence) of a square beam ($b = a$),

$$(38.13)\quad \tau_{\max} = \mu\alpha a \left\{ 1 - \frac{8}{\pi^2} \left[\operatorname{sech} \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \operatorname{sech} (2n+1) \frac{\pi}{2} \right] \right\}.$$

But

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \operatorname{sech} (2n+1)\frac{\pi}{2} < \frac{1}{9} \sum_{n=1}^{\infty} \frac{2e^{-(2n+1)(\pi/2)}}{1 + e^{-(2n+1)\pi}}$$
$$< \frac{2}{9} \sum_{n=1}^{\infty} e^{-(2n+1)(\pi/2)} = \frac{2}{9} \frac{e^{-3\pi/2}}{1 - e^{-\pi}} = 0.002.$$

Since sech $(\pi/2) = 0.4$, it follows that the first term in the brackets in (38.13) gives the value of all the terms in the brackets with the accuracy of ½ per cent. Hence, for practical calculations, the value of $\tau_{\max}$ can be assumed to be given by the formula

$$\tau_{\max} \doteq \mu\alpha a \left(1 - \frac{8}{\pi^2} \operatorname{sech} \frac{\pi b}{2a} \right).$$

The twisting moment

$$M = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (x\tau_{zy} - y\tau_{zx})\, dx\, dy$$

is calculated by making use of the series (38.11). The result of the

calculations is

$$M = \frac{\mu\alpha ba^3}{6} + \frac{16\mu\alpha a^3 b}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} - \frac{64\mu\alpha a^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\tanh(k_n b/2)}{(2n+1)^5},$$

and since

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96},$$

we have the formula

$$M = \frac{\mu\alpha ba^3}{3} - \frac{64\mu\alpha a^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\tanh(k_n b/2)}{(2n+1)^5}. \tag{38.14}$$

Now the series in (38.14) can be written as

$$\tanh\frac{\pi b}{2a} + \sum_{n=1}^{\infty} \frac{\tanh(k_n b/2)}{(2n+1)^5},$$

and we note that $\sum_{n=1}^{\infty} \frac{\tanh(k_n b/2)}{(2n+1)^5}$ is less than

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^5} = 0.0046,$$

while $\tanh(\pi b/2a) \geq 0.917$. Thus, the first term of the series gives the value of the sum to within ½ per cent, and one can use, for practical purposes, the approximate formula

$$M \doteq \frac{\mu\alpha ba^3}{3} - \frac{64\mu\alpha a^4}{\pi^5} \tanh\frac{\pi b}{2a}.$$

Inasmuch as the partial derivatives $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ are known from (38.10), it is a straightforward matter to compute the torsion function φ. Noting formula (35.1), it is found that

$$\varphi(x, y) = xy - \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\sinh k_n y}{\cosh(k_n b/2)} \sin k_n x.$$

Accordingly, the displacement w is given by $w = \alpha\varphi(x, y)$. The contour lines of the surface $\varphi(x, y) = \text{const}$ for the case $b = a$ are shown in Fig. 27. The section is divided into eight triangular regions, which are warped as shown by the contour lines in Fig. 27. The function $\psi(x, y)$ can be determined by integrating Eqs. (38.10) and recalling the boundary condition $\psi(a/2, y) = \frac{1}{2}(x^2 + y^2) = a^2/8 + y^2/2$; the result is

$$\psi(x, y) = \frac{a^2}{4} + \frac{1}{2}(y^2 - x^2) - \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\frac{\cosh k_n y}{\cosh (k_n b/2)}\cos k_n x. \tag{38.15}$$

The solution of the torsion problem for a prism whose cross section is an isosceles right triangle (Fig. 28) can be obtained from the foregoing

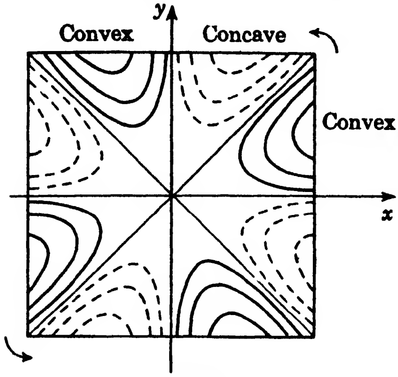

FIG. 27

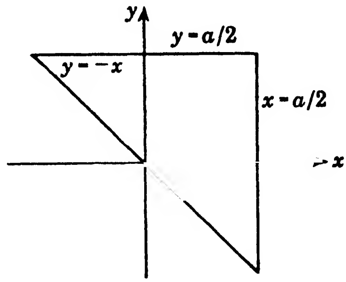

FIG 28

solution for a rectangular prism.[1] We construct the harmonic functions

$$\psi_1 = \frac{a^2}{4} + \frac{1}{2}(y^2 - x^2) - \left(x - \frac{a}{2}\right)\left(y - \frac{a}{2}\right) - \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\frac{\sinh k_n y}{\sinh (k_n a/2)}\cos k_n x,$$

and

$$\psi_2 = \frac{a^2}{4} + \frac{1}{2}(x^2 - y^2) - \left(x - \frac{a}{2}\right)\left(y - \frac{a}{2}\right) - \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\frac{\sinh k_n x}{\sinh (k_n a/2)}\cos k_n y.$$

A comparison with the expression (38.15) for ψ shows that the function ψ_1 reduces to $\frac{1}{2}(x^2 + y^2)$ on the sides $x = a/2$ and $y = a/2$. That ψ_2 also satisfies the boundary conditions on these sides can be shown either from considerations of symmetry or by direct calculation of the boundary values and by noting the expansion

$$\frac{a^2}{4} - x^2 = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\frac{(2n+1)\pi x}{a}.$$

[1] See B. G. Galerkin, *Bulletin de l'académie des sciences de Russie* (1919), p. 111, and G. Kolossoff, *Comptes rendus hebdomadaires des séances de l'académie des sciences*, Paris, vol. 178 (1924), p. 2057.

If we set $y = -x$ in ψ_1 and ψ_2 and add the results, we obtain $2x^2$. Thus, the harmonic function

$$\psi = \tfrac{1}{2}(\psi_1 + \psi_2)$$
$$= -xy + \frac{a}{2}(x + y) - \frac{4a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3 \sinh (k_n a/2)} (\sinh k_n y \cos k_n x + \sinh k_n x \cos k_n y)$$

reduces to $\tfrac{1}{2}(x^2 + y^2)$ on the boundary of the triangle bounded by the lines $x = a/2$, $y = a/2$, $y = -x$ and hence solves the torsion problem for the triangular prism. One can calculate the shearing stresses, in the manner indicated above, for the beam of rectangular cross section, and it is possible to show that the maximum shearing stress is at the middle of the hypotenuse.

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39. Complex Form of Fourier Series. The discussion of the torsion of a rectangular beam in the preceding section utilized the expansion of a certain function in a trigonometric series. We shall have occasion to make frequent use of Fourier series expansions, and it is the purpose of this section to recall some facts about Fourier series and to give a representation of Fourier series in complex form. Sufficient conditions for the expansion of an arbitrary function in a Fourier series are given by the following theorem.

THEOREM: *Let $f(\theta)$ be a real single-valued function defined arbitrarily in the interval $0 \leq \theta \leq 2\pi$, and outside this interval defined by the equation $f(\theta + 2\pi) = f(\theta)$. If $f(\theta)$ has at most a finite number of points of ordinary discontinuity and a finite number of maxima and minima in the interval $0 \leq \theta \leq 2\pi$, then it can be represented by the series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) \tag{39.1}$$

with

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt\, dt, \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt\, dt, \tag{39.2}$$

and the series converges at every point $\theta = \theta_0$ to the value

$$\tfrac{1}{2}[f(\theta_0+) + f(\theta_0-)].$$

The symbols $f(\theta_0+)$ and $f(\theta_0-)$ stand for the right- and left-hand limits of $f(\theta)$ as $\theta \to \theta_0$.

The restrictions imposed upon the function $f(\theta)$ in this theorem are known as the *Dirichlet conditions.*[1] We assume that the reader is familiar with this theorem.

If $f(\theta)$ not only satisfies the conditions of Dirichlet but is continuous in the closed interval $(0, 2\pi)$,[2] then one can show that the Fourier series converges *uniformly* in the closed interval $(0, 2\pi)$.

We also have the following theorem concerning the bounds on the coefficients in Fourier series:

THEOREM: *If the function $f(\theta)$ is periodic and is such that its pth derivative satisfies the conditions of Dirichlet in the interval* $(0, 2\pi)$, *then the Fourier coefficients for $f(\theta)$ satisfy the inequalities*

$$|a_n| < \frac{M}{n^{p+1}}, \qquad \textit{and} \qquad |b_n| < \frac{M}{n^{p+1}},$$

where M is a positive number independent of n.

An important conclusion follows directly from this theorem. Let the function $f(\theta)$ have the first derivative $f'(\theta)$, which satisfies the conditions of Dirichlet. Then the Fourier series for such a function has coefficients of order $1/n^2$, so that

$$|a_n \cos n\theta + b_n \sin n\theta| \le |a_n \cos n\theta| + |b_n \sin n\theta| \le |a_n| + |b_n| < \frac{M}{n^2},$$

where M is a positive number independent of n. Since the series of positive constants

$$\sum_{n=1}^{\infty} \frac{M}{n^2} = M \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, it follows from the Weierstrass M test that the Fourier series for a function whose first derivative satisfies the conditions of Dirichlet is absolutely and uniformly convergent and hence can be integrated term by term.[3]

Since the coefficients of the series obtained by differentiating the series term by term are of the order na_n and nb_n, it is clear that, in order to ensure the convergence of the derived series, it is sufficient to demand

[1] The restrictions imposed on the function $f(\theta)$ can be relaxed, and it is sufficient to demand that $f(\theta)$ be a function of bounded variation.

[2] In this case, the requirement of periodicity imposes the condition $f(0) = f(2\pi)$.

[3] As a matter of fact, every Fourier series can be integrated term by term.

that the second derivative $f''(\theta)$ fulfill the conditions of Dirichlet in the interval $(0, 2\pi)$.

The Fourier series (39.1) can be written in an equivalent form

$$(39.3)\qquad f(\theta) = c_0 + \sum_{k=1}^{\infty} c_k e^{ik\theta} + \sum_{k=1}^{\infty} c_{-k} e^{-ik\theta} \equiv \sum_{k=-\infty}^{\infty} c_k e^{ik\theta},$$

where

$$(39.4)\qquad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt, \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

In order to establish the identity of the representation (39.3) with (39.1), it is merely necessary to recall the Euler formula

$$e^{iu} = \cos u + i \sin u,$$

and verify that the formula (39.4) gives for $n > 0$

$$(39.5)\qquad c_n = \frac{a_n}{2} - i\frac{b_n}{2}, \qquad c_{-n} = \frac{a_n}{2} + i\frac{b_n}{2}, \qquad c_0 = \frac{a_0}{2}.$$

Then

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\theta} = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(\frac{a_k}{2} - i\frac{b_k}{2}\right)(\cos k\theta + i \sin k\theta) + \sum_{k=1}^{\infty}\left(\frac{a_k}{2} + i\frac{b_k}{2}\right)(\cos k\theta - i \sin k\theta)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta).$$

Let $f_1(\theta)$ and $f_2(\theta)$ be a pair of real functions, each of which can be expanded in Fourier series in the interval $(0, 2\pi)$, and form the complex function $f_1(\theta) + if_2(\theta)$. Then

$$(39.6)\qquad f_1(\theta) + if_2(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta},$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} [f_1(t) + if_2(t)]e^{-int}\,dt, \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

If we set

$$c_n = \gamma_n + i\delta_n,$$

where γ_n and δ_n are real numbers, then

$$
\begin{aligned}
f_1(\theta) + if_2(\theta) &= \sum_{k=-\infty}^{\infty} (\gamma_k + i\delta_k)(\cos k\theta + i \sin k\theta) \\
&= \sum_{k=-\infty}^{\infty} (\gamma_k \cos k\theta - \delta_k \sin k\theta) \\
&\qquad + i \sum_{k=-\infty}^{\infty} (\delta_k \cos k\theta + \gamma_k \sin k\theta) \\
&= \gamma_0 + \sum_{k=1}^{\infty} [(\gamma_k + \gamma_{-k}) \cos k\theta - (\delta_k - \delta_{-k}) \sin k\theta] + i\delta_0 \\
&\qquad + i \sum_{k=1}^{\infty} [(\delta_k + \delta_{-k}) \cos k\theta + (\gamma_k - \gamma_{-k}) \sin k\theta].
\end{aligned}
$$

Hence

$$
f_1(\theta) = \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta),
$$

$$
f_2(\theta) = \tfrac{1}{2}a_0' + \sum_{k=1}^{\infty} (a_k' \cos k\theta + b_k' \sin k\theta),
$$

where

$$
\begin{aligned}
&\tfrac{1}{2}a_0 = \gamma_0, \qquad a_k = \gamma_k + \gamma_{-k}, \qquad b_k = -\delta_k + \delta_{-k}, \\
&\tfrac{1}{2}a_0' = \delta_0, \qquad a_k' = \delta_k + \delta_{-k}, \qquad b_k' = \gamma_k - \gamma_{-k}, \\
&\qquad\qquad (k = 1, 2, 3, \ldots).
\end{aligned}
$$

It follows from these formulas that the representation of a complex function $f_1(\theta) + if_2(\theta)$ in a series of the type (39.3) is unique, since the representation of the functions $f_1(\theta)$ and $f_2(\theta)$ in series of the type (39.1) is unique.

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40. Summary of Some Results of the Complex Variable Theory. We shall need in our subsequent work some theorems from the theory of

functions of a complex variable. In this section, some of the more familiar results will be stated without proof, and the proofs of the less familiar ones will be outlined. A detailed discussion can be found in the reference books listed at the end of the next section.

. It will be recalled that a single-valued function

$$f(z) = u(x, y) + iv(x, y)$$

of a complex variable $z = x + iy$ is called *analytic*, or *holomorphic*, in a given region R if it possesses a unique derivative at every point of the region R. Points at which the function $f(z)$ ceases to have a derivative are termed the *singular* points of the analytic function. The necessary and sufficient conditions for the analyticity of the function $f(z)$ are given by the well-known Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{40.1}$$

where it is assumed that the partial derivatives involved are continuous functions of x and y. It is known that, if $f(z)$ is analytic in the region R, then not only do the first partial derivatives of u and v exist in R, but also those of all higher orders. It follows from this observation and from (40.1) that the real and imaginary parts of an analytic function satisfy the equation of Laplace; that is,

$$\nabla^2 u = 0, \qquad \nabla^2 v = 0.$$

The following theorem is basic to all considerations of the theory of analytic functions:

CAUCHY'S INTEGRAL THEOREM: *If $f(z)$ is continuous in the closed region*[1] *R bounded by a simple closed contour C, and if $f(z)$ is analytic at every interior point of R, then*

$$\int_C f(z)\, dz = 0.$$

This theorem can easily be extended to the case of multiply connected regions to yield another.

THEOREM: *If $f(z)$ is continuous in the closed, multiply connected region R bounded by the exterior simple contour C_0 and by the interior simple contours $C_1, C_2, \ldots, C_n$, then the integral of $f(z)$ over the exterior contour C_0 is equal to the sum of the integrals over the interior contours, whenever $f(z)$ is analytic in the interior of R. The integration over all the contours is performed in the same direction.*

The following numerical results are worth noting:

If n is an integer and $z = a$ is a fixed point that lies either within or

[1] The term "continuous in a closed region" is used to mean that the function is continuous up to and on the boundary.

without the simple closed contour C, then

$$\int_C (z - a)^n \, dz = 0 \qquad \text{if } n \neq -1.$$

If the point a is outside the contour C, then the truth of the formula follows from Cauchy's Theorem, whatever be the value of n; if it is within, then the result follows from elementary calculations. If the point a is within the contour, then an elementary calculation gives

$$\int_C \frac{dz}{z - a} = 2\pi i.$$

This latter formula, in conjunction with Cauchy's Integral Theorem, can be used to establish Cauchy's Integral Formula.

CAUCHY'S INTEGRAL FORMULA: *If $z = a$ is an interior point of the region R bounded by a contour C, then*

$$\frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{z - a} = f(a), \tag{40.2}$$

whenever $f(z)$ is continuous in the closed region R and analytic at every interior point of R.

If the variable of integration in (40.2) is denoted by ζ, and if z is any point interior to R, then (40.2) becomes

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta - z}. \tag{40.3}$$

Calculation of the derivative from the formula (40.3) yields

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2},$$

and, in general,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}.$$

The Integral Formula of Cauchy can be used to establish the fact that an analytic function $f(z)$ can be expanded in Taylor's series, so that

$$f(z) = f(a) + f'(a)(z - a) + \cdots + \frac{f^{(n)}(a)}{n!} (z - a)^n + \cdots .$$

This series converges to $f(z)$ at every point z interior to any circle γ that lies within the region R and whose center is at a. Moreover, the representation of $f(z)$ in Taylor's series is unique.

Consider now the region R bounded by two concentric circles C_1 and C_2, and let $z = a$ be the center of the circles. If $f(z)$ is continuous in the closed annular region formed by C_1 and C_2, and if it is analytic at every

interior point of the ring, then one can represent $f(z)$ by the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} b_k(z - a)^k,$$

where

$$b_k = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - a)^{k+1}}, \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

and where C is an arbitrary path drawn in R that encloses C_1. It is obvious that the series of Laurent reduces to Taylor's series whenever the function $f(z)$ is analytic throughout the region bounded by the circle C_1.

If $f(z)$ has a pole of order m at $z = a$, then the Laurent series about the point $z = a$ takes the form

$$f(z) = \frac{b_{-m}}{(z - a)^m} + \cdots + \frac{b_{-2}}{(z - a)^2} + \frac{b_{-1}}{z - a} + b_0 + b_1(z - a) + b_2(z - a)^2 + \cdots.$$

If we set $z - a = \zeta$ and integrate around a curve C enclosing $z = a$ and no other singularity of $f(z)$, then

$$\int_C f(z)\,dz = \int_C \left(\frac{b_{-m}}{\zeta^m} + \cdots + \frac{b_{-2}}{\zeta^2} + \frac{b_{-1}}{\zeta} + b_0 + b_1\zeta + b_2\zeta^2 + \cdots\right) d\zeta$$
$$= b_{-1}\int_C \frac{d\zeta}{\zeta} = 2\pi i b_{-1}.$$

The quantity b_{-1} is called the *residue* of $f(z)$ at the pole $z = a$. If

$$f(z) = \frac{g(z)}{(z - a)h(z)}$$

has a *simple* pole at $z = a$ ($m = 1$), then the residue at $z = a$ is $g(a)/h(a)$. In general, when C encloses n poles at $z = a_1$, $z = a_2$, . . . , $z = a_n$, the last equation is replaced by

$$\int_C f(z)\,dz = 2\pi i \times (\text{sum of residues at poles}).$$

If the Laurent expansion of $f(z)$ at each pole is known, then to evaluate

$$\frac{1}{2\pi i}\int_C f(z)\,dz$$

we have merely to add the coefficients of

$$\frac{1}{z - a_1}, \qquad \frac{1}{z - a_2}, \ldots$$

in the several expansions.

The evaluation of residues may often be simplified by observing that if $f(z)$ and $g(z)$ are analytic at $z = a$, and if $z - a$ is a nonrepeated factor

of $g(z)$, then the residue at a of $f(z)/g(z)$ is $f(a)/g'(a)$. The residue at a multiple pole can be found from the theorem that the residue at $z = a$ of $f(z)/(z - a)^n$ is $f^{(n-1)}(a)/(n - 1)!$; it is assumed that $f(z)$ is analytic at a and that n is a positive integer.

It should be noted that the function $f(\zeta)$ in the Formula (40.3) of Cauchy represents the values of an analytic function $f(z)$ on the boundary C of the region R. Now if we consider the integral

$$\Phi_1(z) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)}{\zeta - z}\, d\zeta, \tag{40.4}$$

where $F(\zeta)$ is *any continuous function* defined on the simple closed boundary C and z is interior to R, then this integral defines some function of z and it is easy to verify that $\Phi_1(z)$ has the derivative $\Phi_1'(z)$, which is given by the formula

$$\Phi_1'(z) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)\, d\zeta}{(\zeta - z)^2},$$

and, in general,

$$\Phi_1^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{F(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}.$$

Thus, the function $\Phi_1(z)$ is analytic for every value of z that is interior to the region R bounded by C. If z is some point exterior to the region R, then the integral

$$\Phi_2(z) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)\, d\zeta}{\zeta - z} \tag{40.5}$$

defines some function $\Phi_2(z)$, and it is easy to see that $\Phi_2(z)$ likewise has derivatives of all orders and hence is analytic. Thus, the integrals (40.4) and (40.5) of Cauchy's type define two analytic functions that in general will be distinct. The situation here is the same even when $F(\zeta)$ represents the boundary values of some analytic function $f(z)$. For, by (40.3), if z is interior to the contour, the value of the integral is precisely equal to $f(z)$, and if z is outside the contour, then the integral defines the function 0, since the integrand $f(\zeta)/(\zeta - z)$ is an analytic function of ζ throughout R. It should be observed that, as z tends to some definite point ζ on the contour from the interior of R and from the exterior, the difference between the two limiting values is $f(\zeta) - 0 = f(\zeta)$.

One can raise a similar question regarding the connection of the limiting values of the functions $\Phi_1(z)$ and $\Phi_2(z)$ with the *density function* $F(\zeta)$. If we place no restrictions on the function $F(\zeta)$ beyond continuity on the contour, then the problem becomes an exceedingly difficult one. If, however, some further restrictions on $F(\zeta)$ are imposed, then it is possible to establish a definite connection of the density function $F(\zeta)$ with $\lim\limits_{z \to \zeta} \Phi_1(z)$ and $\lim\limits_{z \to \zeta} \Phi_2(z)$.

This connection is provided by the formulas of Plemelj.[1] To state these formulas we need a definition.

DEFINITION: *A function $F(\zeta)$ is said to satisfy the Hölder condition (or the Lipschitz condition of order α) on a smooth curve C if for every pair of points (ζ_1, ζ_2) on C*

$$|F(\zeta_2) - F(\zeta_1)| \leq M|\zeta_2 - \zeta_1|^\alpha, \tag{40.6}$$

where M and α are positive constants.[2]

It is clear that the Hölder condition is less restrictive than the requirement that $F(\zeta)$ have a bounded derivative.

We can now state the Plemelj formulas.

THE PLEMELJ FORMULAS: *If the density function $F(\zeta)$ in the integral*

$$\Phi(z) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)}{\zeta - z} d\zeta$$

satisfies the Hölder condition on a smooth closed contour C, then the limits $\Phi^+(t)$ and $\Phi^-(t)$ as z approaches an arbitrary point t on C from the interior and exterior of C, respectively, are:

$$\begin{cases} \Phi^+(t) \equiv \dfrac{1}{2} F(t) + \dfrac{1}{2\pi i} \displaystyle\int_C \frac{F(\zeta)}{\zeta - t} d\zeta, \\ \Phi^-(t) \equiv -\dfrac{1}{2} F(t) + \dfrac{1}{2\pi i} \displaystyle\int_C \frac{F(\zeta)}{\zeta - t} d\zeta. \end{cases} \tag{40.7}$$

The improper integrals in (40.7) *are interpreted in the sense of Cauchy's principal values.*[3]

We shall make use of integrals of Cauchy's type to represent analytically some functions that are useful in the theory of elasticity. However, it must be noted that such representation is not unique, so that the same function can be represented by different integrals of Cauchy's type. As an illustration, consider a contour C that contains in the interior the point $z = 0$, and let us determine the analytic function that vanishes at every point of the region R enclosed by C. If we choose in (40.4) the density

[1] J. Plemelj, *Monatshefte für Mathematik und Physik*, vol. 19 (1908), pp. 205–210. A detailed discussion of these formulas under restrictions somewhat less severe than those made by Plemelj is contained in Chap. 2 of N. I. Muskhelishvili's Singular Integral Equations (1953).

[2] Usually α is restricted to lie in the interval $0 < \alpha \leq 1$, because for $\alpha > 1$ the condition (40.6) implies that $F'(\zeta) = 0$, so that $F(\zeta) = \text{const.}$

[3] If an arc L of length 2ϵ with $\zeta = t$ as the mid-point is deleted from C, then the integral $\int_{C-L} \frac{F(\zeta)}{\zeta - t} d\zeta$ over the remaining curve $C - L$ becomes proper and the principal value of $\int_C \frac{F(\zeta)}{\zeta - t} d\zeta$ is defined as $\lim_{\epsilon \to 0} \int_{C-L} \frac{F(\zeta)}{\zeta - t} d\zeta$.

function $F(\zeta) = 0$, then $\Phi_1(z) \equiv 0$. Also, if we choose $F(\zeta) = 1/\zeta$, then

$$\Phi_1(z) = \frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta(\zeta - z)} = 0$$

for every position of the point z in the region R.[1] Hence if we add this integral to an integral of Cauchy's type that defines an analytic function $\Psi(z)$, we shall obtain another integral of Cauchy's type that defines the same analytic function $\Psi(z)$. It follows from these remarks that no conclusion can be drawn concerning the equality of the density functions $F_1(\zeta)$ and $F_2(\zeta)$ from the equality of the two integrals

$$\frac{1}{2\pi i}\int_C \frac{F_1(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_C \frac{F_2(\zeta)}{\zeta - z}\,d\zeta$$

for all values of z in the interior of C. We shall see, however, that if some additional restrictions are imposed on the density functions and on the contour C, then the equality will obtain. This is the subject of the next section, which contains a discussion of the Theorem of Harnack.

41. Theorem of Harnack.[2] In considering the applications of the theory of functions of a complex variable to problems in elasticity, we shall most frequently deal with the region bounded by the unit circle, that is, the region $|z| \leq 1$. In order to avoid a possible misinterpretation of the formulas, we shall draw the unit circle in the complex ζ-plane, where $\zeta = \xi + i\eta$ (ξ and η being real). The boundary of the unit circle $|\zeta| \leq 1$ will be denoted by the letter γ, and the points on the boundary γ by $\sigma = e^{i\theta}$.[3] All functions of the argument θ will be assumed to be periodic, so that $f(\theta + 2\pi) = f(\theta)$.

THEOREM: *Let $f(\theta)$ and $\varphi(\theta)$ be continuous real functions of the argument θ (defined on the unit circle γ); if*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\theta)\,d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\theta)\,d\sigma}{\sigma - \zeta} \tag{41.1}$$

for all values of ζ interior to γ, then

$$f(\theta) \equiv \varphi(\theta).$$

If the point ζ is exterior to γ, and if the equality (41.1) *is true for all values*

[1] For $\dfrac{1}{\zeta(\zeta - z)} = \dfrac{1}{z(\zeta - z)} - \dfrac{1}{z\zeta}$; hence

$$\frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta(\zeta - z)} = \frac{1}{2\pi i z}\int_C \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i z}\int_C \frac{d\zeta}{\zeta} = \frac{1}{z} - \frac{1}{z} = 0.$$

[2] A less restrictive form of Harnack's Theorem is discussed in N. I. Muskhelishvili's Singular Integral Equations (1953), p. 64.

[3] The letter σ was used earlier for Poisson's ratio and in the expression $d\sigma$ for the element of area, but the distinction is so obvious that no complications should arise.

of ζ, then

$$f(\theta) = \varphi(\theta) + const.$$

We consider first the case when the point ζ is inside γ. It follows from equality (41.1) that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\theta) - \varphi(\theta)}{\sigma - \zeta}\,d\sigma \equiv \frac{1}{2\pi i}\int_\gamma \frac{F(\theta)}{\sigma - \zeta}\,d\sigma \equiv 0,$$

where $F(\theta) \equiv f(\theta) - \varphi(\theta)$, and we shall prove that $F(\theta) \equiv 0$.

Now since $|\zeta| < 1$, we have

$$\frac{1}{\sigma - \zeta} = \frac{1}{\sigma} + \frac{\zeta}{\sigma^2} + \frac{\zeta^2}{\sigma^3} + \cdots,$$

and

$$(41.2)\qquad \frac{1}{2\pi i}\int_\gamma \frac{F(\theta)}{\sigma - \zeta}\,d\sigma = \frac{1}{2\pi i}\int \sum_{n=0}^{\infty} \zeta^n \frac{F(\theta)\,d\sigma}{\sigma^{n+1}}$$
$$= \sum_{n=0}^{\infty} (a_n - ib_n)\zeta^n,$$

where [see (39.4) and (39.5)]

$$a_n - ib_n = \frac{1}{2\pi i}\int_\gamma F(\theta)\sigma^{-n-1}\,d\sigma = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{-in\theta}\,d\theta.$$

But (41.2) vanishes for all values of ζ; hence $a_n = b_n = 0$ ($n = 0, 1, 2, \ldots$). A reference to formula (39.4) shows that all Fourier coefficients of the function $F(\theta)$ vanish, and hence $F(\theta) \equiv 0$.

Consider now the case when $|\zeta| > 1$; then

$$\frac{1}{\sigma - \zeta} = -\frac{1}{\zeta} - \frac{\sigma}{\zeta^2} - \frac{\sigma^2}{\zeta^3} - \cdots$$

and

$$(41.3)\qquad \frac{1}{2\pi i}\int_\gamma \frac{F(\theta)}{\sigma - \zeta}\,d\sigma = -\frac{1}{2\pi i}\int_\gamma \sum_{n=1}^{\infty} \frac{\sigma^{n-1}F(\theta)\,d\sigma}{\zeta^n}$$
$$= -\sum_{n=1}^{\infty} \frac{a_n + ib_n}{\zeta^n},$$

where

$$a_n + ib_n = \frac{1}{2\pi i}\int_\gamma F(\theta)\sigma^{n-1}\,d\sigma$$
$$= \frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{in\theta}\,d\theta, \qquad (n = 1, 2, 3, \ldots).$$

Since (41.3) vanishes for all values of $|\zeta| > 1$, $a_n = b_n = 0$ ($n = 1, 2, 3, \ldots$). Thus, all Fourier coefficients of $F(\theta)$, with the possible excep-

tion of a_0, vanish, and hence

$$\varphi(\theta) = f(\theta) + \text{const.}$$

It follows from this proof that if the point ζ is outside γ, and if *in addition to the equality* (41.1) we have the equality

$$\frac{1}{2\pi i}\int_\gamma f(\theta)\,\frac{d\sigma}{\sigma} = \frac{1}{2\pi i}\int_\gamma \varphi(\theta)\,\frac{d\sigma}{\sigma},$$

then $f(\theta) = \varphi(\theta)$.

An important corollary follows from this theorem.

COROLLARY: *If we have four real continuous functions f_1, f_2, φ_1, φ_2 and the following simultaneous equalities for all values of ζ:*

$$\frac{1}{2\pi i}\int_\gamma \frac{f_1 + if_2}{\sigma - \zeta}\,d\sigma = \frac{1}{2\pi i}\int_\gamma \frac{\varphi_1 + i\varphi_2}{\sigma - \zeta}\,d\sigma,$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f_1 - if_2}{\sigma - \zeta}\,d\sigma = \frac{1}{2\pi i}\int_\gamma \frac{\varphi_1 - i\varphi_2}{\sigma - \zeta}\,d\sigma,$$

then

$$\varphi_1 = f_1, \qquad \varphi_2 = f_2, \qquad \textit{if } |\zeta| < 1,$$

and

$$\varphi_1 = f_1 + \textit{const}, \qquad \varphi_2 = f_2 + \textit{const}, \qquad \textit{if } |\zeta| > 1.$$

This corollary follows at once from Harnack's Theorem when we consider the results of adding and subtracting the equalities in question.

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42. Formulas of Schwarz and Poisson. We have already seen that the determination of the torsion function $\varphi(x, y)$ and its conjugate function $\psi(x, y)$ are special cases of the fundamental boundary-value problems of Potential Theory—the so-called problems of Dirichlet and Neumann. These problems occur also in other branches of applied mathematics, notably in hydrodynamics and in electrodynamics. While the solution of the two-dimensional problems of Dirichlet and Neumann for special types of boundaries is likely to present serious calculational difficulties, it is possible to write down general formulas for the case when the boundary of the region is a circle. We shall give a derivation of formulas associated with the names of Schwarz and Poisson that solve the problem of Dirichlet for a circular region.

Consider a region bounded by a circle, which we can take, without loss of generality, to be a unit circle with center at the origin. As in the preceding section, we denote the boundary of the circle $|\zeta| = 1$ by γ and any point on the boundary by $\sigma = e^{i\theta}$.

Let it be required to determine a harmonic function $u(\xi, \eta)$, which on the boundary of the circle γ assumes the values

$$u\Big|_{\gamma} = f(\theta), \tag{42.1}$$

where $f(\theta)$ is a continuous real function of θ. Denote the conjugate harmonic function by $v(\xi, \eta)$; the function $v(\xi, \eta)$ is determined to within an arbitrary constant from the knowledge of the function $u(\xi, \eta)$ [see (35.2)]. Then the function

$$F(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$$

is an analytic function of the complex variable $\zeta = \xi + i\eta$ for all values of ζ interior to $|\zeta| = 1$. If we assume that $F(\zeta)$ is continuous in the closed region $|\zeta| \leq 1$, then we can rewrite the boundary condition (42.1) in the form[1]

$$F(\sigma) + \bar{F}(\bar{\sigma}) = 2f(\theta) \qquad \text{on } \gamma. \tag{42.2}$$

If we multiply both members of Eq. (42.2) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma - \zeta}$, where ζ is any point interior to γ, and integrate over the circle γ, we obtain the formula

$$\frac{1}{2\pi i}\int_{\gamma}\frac{F(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\bar{F}(\bar{\sigma})}{\sigma - \zeta}\,d\sigma = \frac{1}{\pi i}\int_{\gamma}\frac{f(\theta)}{\sigma - \zeta}\,d\sigma, \tag{42.3}$$

which by the Theorem of Harnack is entirely equivalent to (42.2).

The first of the integrals in the left-hand member of (42.3), by Cauchy's Integral Formula, is equal to $F(\zeta)$, while the second is equal[2] to $\bar{F}(0)$. Let

[1] We define $\bar{F}(\zeta) = \overline{F(\bar{\zeta})}$ and $\bar{F}(\bar{\zeta}) = \overline{F(\zeta)}$. It is possible to prove that, if $f(\theta)$ satisfies Hölder's condition, then the function $F(\zeta)$ given by (42.6) will be continuous in the closed region $|\zeta| \leq 1$. We recall that a function $f(\theta)$ is said to satisfy Hölder's condition (or a Lipschitz condition) if, for any pair of values θ' and θ'' in the interval in question,

$$|f(\theta'') - f(\theta')| \leq M|\theta'' - \theta'|^{\alpha},$$

where M and α are positive constants. This condition is less restrictive than the requirement of the existence of a bounded derivative.

[2] Since $F(\zeta) = F(0) + F'(0)\zeta + \frac{1}{2!}F''(0)\zeta^2 + \cdots$, and since on $|\zeta| = 1$ $\bar{\zeta} = 1/\sigma$, $\bar{F}(\bar{\sigma}) = \bar{F}(0) + \bar{F}'(0)\frac{1}{\sigma} + \frac{1}{2!}\bar{F}''(0)\frac{1}{\sigma^2} + \cdots$ and term-by-term integration gives the desired result upon noting that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma}\frac{d\sigma}{\sigma^n(\sigma - \zeta)} &= 0, \qquad \text{if } n > 0,\\ &= 1, \qquad \text{if } n = 0.\end{aligned}$$

$\bar{F}(0) = a_0 - ib_0$; then (42.3) becomes

$$F(\zeta) = \frac{1}{\pi i}\int_\gamma \frac{f(\theta)}{\sigma - \zeta}\,d\sigma - a_0 + ib_0. \tag{42.4}$$

If we set $\zeta = 0$ in (42.4), we obtain

$$a_0 + ib_0 = \frac{1}{\pi i}\int_\gamma \frac{f(\theta)\,d\sigma}{\sigma} - a_0 + ib_0,$$

and hence

$$2a_0 = \frac{1}{\pi i}\int_\gamma \frac{f(\theta)\,d\sigma}{\sigma} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta. \tag{42.5}$$

The quantity b_0 is left undetermined, as one would expect, since the function $v(\xi, \eta)$ is determined to within an arbitrary real constant.

Inserting the value of a_0 from (42.5) in (42.4), we have

$$\begin{aligned} F(\zeta) &= \frac{1}{\pi i}\int_\gamma \frac{f(\theta)\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int_\gamma \frac{f(\theta)\,d\sigma}{\sigma} + ib_0 \\ &= \frac{1}{2\pi i}\int_\gamma f(\theta)\,\frac{\sigma + \zeta}{\sigma - \zeta}\frac{d\sigma}{\sigma} + ib_0, \end{aligned} \tag{42.6}$$

which is the desired formula of Schwarz.

Now if we substitute $\zeta = \rho e^{i\psi}$ and $\sigma = e^{i\theta}$ in (42.6) and separate the real and imaginary parts, we find

$$\mathcal{R}F(\zeta) \equiv u(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - \rho^2)f(\theta)\,d\theta}{1 - 2\rho\cos(\theta - \psi) + \rho^2}. \tag{42.7}$$

This is the integral of Poisson, which gives the solution of the problem of Dirichlet. It is possible to prove that (42.7) represents the solution of the problem of Dirichlet under the assumption that $f(\theta)$ is merely a piecewise continuous function.[1]

The discussion of this section was confined to the general solution of the first boundary-value problems of Potential Theory, when the boundary curve is a circle. It is possible to generalize the formulas obtained above so as to make them apply to any simply connected region. This is done by introducing a mapping function, and we proceed next to an outline of some basic notions that underlie the idea of conformal mapping of simply connected domains.

43. Conformal Mapping. Let the functional relationship $z = \omega(\zeta)$ set up a correspondence between the points $\zeta = \xi + i\eta$ of the complex ζ-plane and $z = x + iy$ of the complex z-plane. If $z = \omega(\zeta)$ is analytic in some region R of the ζ-plane, then the totality of values z belongs to some region R' of the z-plane and it is said that the region R is mapped

[1] See O. D. Kellogg, Foundations of Potential Theory, and G. C. Evans, The Logarithmic Potential, Chap. IV, for a discussion of the problem of Neumann

into the region R' by the mapping function $\omega(\zeta)$. If C is some curve drawn in the region R and the point ζ is allowed to move along C, then the corresponding point z will trace a curve C' in the z-plane and C' is called the map of C (Fig. 29).

The relationship between the curves C and C' is interesting. Consider a pair of points ζ and $\zeta + \Delta\zeta$ on C, and let the arc length between them be $\Delta s = \widehat{PQ}$. The corresponding points in the region R' are denoted by

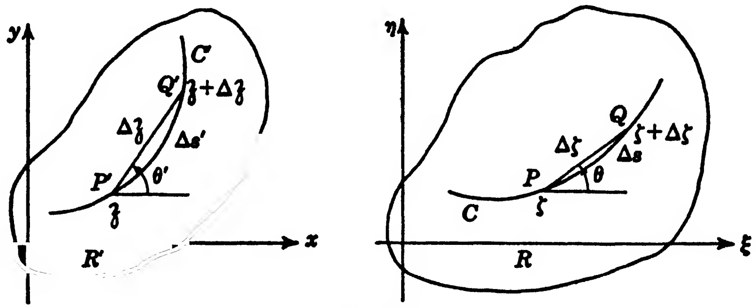

FIG. 29

z and $z + \Delta z$, and the distance between them, measured along the curve C', is $\Delta s' = \widehat{P'Q'}$. Since the ratio of the lengths of arc elements has the same limit as the ratio of the lengths of the corresponding chords,

$$\lim_{\Delta\zeta \to 0} \frac{\Delta s'}{\Delta s} = \lim_{\Delta\zeta \to 0} \left| \frac{\Delta z}{\Delta\zeta} \right| = \left| \frac{dz}{d\zeta} \right|.$$

Since $z = \omega(\zeta)$ is assumed to be analytic, $\dfrac{dz}{d\zeta}$ has a unique value independent of the manner in which $\Delta\zeta \to 0$ and it follows that the transformation causes elements of arc passing through P *in any direction* to experience a change in length whose magnitude is determined by the modulus of $\dfrac{dz}{d\zeta}$ calculated at P.

It will be shown next that the argument of $\dfrac{dz}{d\zeta}$ determines the orientation of the element of arc $\Delta s'$ relative to Δs. The argument of the complex number $\Delta\zeta$ is measured by the angle θ made by the chord PQ with the ξ-axis, while the argument of Δz is measured by the corresponding angle θ' between the x-axis and the chord $P'Q'$. Hence the difference between the angles θ' and θ is equal to

$$\arg \Delta z - \arg \Delta\zeta = \arg \frac{\Delta z}{\Delta\zeta}.$$

As $\Delta\zeta \to 0$, the vectors $\Delta\zeta$ and Δz tend to coincide with the tangents to C

at P and to C' at P', respectively, and hence[1] $\arg \frac{d\mathfrak{z}}{d\zeta}$ is the angle of rotation of the element of arc $\Delta s'$ relative to Δs. It follows immediately from this statement that, if C_1 and C_2 are two curves in the ζ-plane that intersect at an angle τ, then the corresponding curves C_1' and C_2' in the $\mathfrak{z}$-plane also intersect at an angle τ, since the tangents to these curves are rotated through the same angle. A transformation that preserves angles is called *conformal*, and thus one can state the following theorem.

THEOREM: *The mapping performed by an analytic function $\omega(\zeta)$ is conformal at all points of the ζ-plane where $\omega'(\zeta) \neq 0$.*

We shall be concerned, for the most part, with the mapping of simply connected regions, where the mapping is one-to-one and hence $\omega'(\zeta) \neq 0$. The regions R and R' may, however, be finite or infinite. It should be noted that if the region R is finite and R' is infinite, then the function $\omega(\zeta)$ must become infinite at some point a of the region R; otherwise we could not have a point in the region R that corresponds to the point at infinity in the region R'. It is possible to show that at such points the function $\omega(\zeta)$ has a simple pole, so that its structure in the neighborhood of the point is

$$\omega(\zeta) = \frac{c}{\zeta - a} + f(\zeta),$$

where c is a constant and $f(\zeta)$ is analytic at all points of the region R. Other types of singularities cannot be present, since the mapping is assumed to be one-to-one.

If both regions R and R' are infinite, and if the points at infinity correspond, then the function $\omega(\zeta)$ has the form

$$\omega(\zeta) = c\zeta + f(\zeta),$$

where c is a constant and $f(\zeta)$ is analytic in the infinite region R. We recall that a function is said to be analytic in an infinite region R if, for sufficiently large ζ, it has the structure

$$a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots + \frac{a_n}{\zeta^n} + \cdots .$$

Let there be given two arbitrary simply connected regions R and R', each of which is bounded by a simple closed contour that can be represented parametrically by[2]

$$\xi = \xi(t), \qquad \eta = \eta(t), \qquad (0 \leq t \leq t_1).$$

Is it possible to find the mapping function $\mathfrak{z} = \omega(\zeta)$ that will map the region R on R' conformally and in such a way that the mapping is con-

[1] Note that this statement assumes that $\frac{d\mathfrak{z}}{d\zeta} \neq 0$ at P.

[2] Since the curves are closed, we must have $\xi(0) = \xi(t_1)$, $\eta(0) = \eta(t_1)$. The term *simple contour* is used to mean rectifiable contour.

tinuous up to and including the boundaries C and C' of the regions R and R'? The answer to this question was given in the affirmative by C. Caratheodory[1] in 1913, and it is known that the mapping function $\omega(\zeta)$ is determined uniquely if we specify the correspondence of two arbitrarily chosen points ζ_0 and $\mathfrak{z}_0$ of the regions R and R' and the directions of arbitrarily chosen linear elements passing through these points.[2]

We can assume without loss of generality[3] that the region R in the ζ-plane is bounded by a unit circle $|\zeta| = 1$, and it is clear that, if the region R' is mapped by the function $\mathfrak{z} = \omega(\zeta)$ on the unit circle $|\zeta| \leqq 1$, then the function

$$\mathfrak{z} = \omega\left(\frac{1}{\zeta}\right)$$

maps the region R' on an infinite plane with a circular hole. In general, it will be found convenient to map finite, simply connected regions on the unit circle $|\zeta| \leq 1$ and infinite regions on the portion of the ζ-plane defined by the equation $|\zeta| \geq 1$.

In regard to the mapping of multiply connected regions, we shall make a few general remarks. It can be shown that a doubly connected region R' can be mapped on a circular ring but that the radii of the circles making up the ring cannot be chosen arbitrarily. It is obvious that in general one can map in one-to-one manner only regions of like connectivity. The condition of like connectivity, however, is not sufficient for the existence of a mapping function.

The affirmative answer to the question of the existence of a function $\mathfrak{z} = \omega(\zeta)$ that maps conformally a given region in the $\mathfrak{z}$-plane on another given region in the ζ-plane helps little in the matter of the actual construction of mapping functions for specified regions.[4] However. there

[1] *Mathematische Annalen*, vol. 73, pp. 305–320.

[2] This theorem is associated with the name of Riemann, who proved the existence of $\omega(\zeta)$ under conditions that are less general than those enunciated above.

[3] If the regions R_1 and R_2 in the planes $\mathfrak{z}_1$ and $\mathfrak{z}_2$, respectively, are mapped on the unit circle in the ζ-plane by functions $\mathfrak{z}_1 = \omega_1(\zeta)$ and $\mathfrak{z}_2 = \omega_2(\zeta)$, then the region R_1 is mapped on the region R_2 by a transformation $\mathfrak{z}_1 = \Omega(\mathfrak{z}_2)$, obtained by eliminating ζ from $\mathfrak{z}_1 = \omega_1(\zeta)$ and $\mathfrak{z}_2 = \omega_2(\zeta)$.

[4] A systematic account of several practical methods for constructing conformal maps is contained in L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis (in Russian) (1952), pp. 374–563. See also Zeev Nehari, Conformal Mapping (1952). A bibliography of numerical methods in conformal mapping was compiled by W. Seidel, in Construction and Application of Conformal Maps (1952) pp. 269–280. A method for approximate conformal mapping of polygonal regions on a unit circle was proposed by I. S. Hara, *Dopovidi Akademii Nauk Ukrainskoĭ RSR* (1953), pp. 289-293 (in Ukrainian).

An excellent account of the underlying theory is given by G. M. Goluzin, Geometric Theory of Functions of a Complex Variable (1952) (in Russian).

A brief catalogue of useful conformal maps was compiled by H. Kober, Dictionary of Conformal Representation (1952).

are some explicit formulas that permit one to construct mapping functions for certain classes of regions. If, for example, the region R' is that bounded by a rectilinear polygon of n sides, then the function $\omega(\zeta)$ that maps the interior of the polygon on the unit circle $|\zeta| \leq 1$ has the form

$$\mathfrak{z} = A \int_0^\zeta (\zeta - \zeta_1)^{\alpha_1 - 1}(\zeta - \zeta_2)^{\alpha_2 - 1} \cdots (\zeta - \zeta_n)^{\alpha_n - 1}\, d\zeta + B, \tag{43.1}$$

where ζ_i are the points on the boundary γ of the unit circle that correspond to the vertices of the polygon in the $\mathfrak{z}$-plane, and the numbers $\alpha_i\pi$ are the interior angles at the vertices of the polygon.[1]

The formula (43.1) was derived by Schwarz and Christoffel[2] and is known as the *Schwarz-Christoffel transformation.*

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44. Solution of the Torsion Problem by Means of Conformal Mapping. Let the torsion function $\varphi(x, y)$ (Secs. 34, 35) be combined with its conjugate function $\psi(x, y)$ to form the function $F(\mathfrak{z}) = \varphi(x, y) + i\psi(x, y)$, where $\mathfrak{z} = x + iy$. The function $F(\mathfrak{z})$ is analytic throughout the region R representing the cross section of the beam. If the region R is simply connected, we can map it conformally on a unit circle in the ζ-plane. Let

$$\mathfrak{z} = \omega(\zeta) \tag{44.1}$$

be the function that maps the region R on the unit circle $|\zeta| \leq 1$. The function $F(\mathfrak{z})$ can be expressed in terms of the variable ζ, so that

$$\varphi + i\psi = F[\omega(\zeta)] = f(\zeta), \tag{44.2}$$

where $f(\zeta)$ is analytic in the interior of the circle $|\zeta| = 1$.

[1] The formula is usually phrased in terms of mapping of the polygon on the half plane, but the transformation that maps the unit circle on the half plane does not alter the form of (43.1).

[2] For derivation of this formula see H. A. Schwarz, *Gesammelte Abhandlungen*, vol. 2, pp. 65–83; E. B. Christoffel, *Annali di matematica pura ed applicata*, vol. 1 (1867), pp. 95–103, vol. 4 (1871), pp. 1–9. For a detailed discussion of the Schwarz-Christoffel transformation and of the Schwarz reflection principle, see Zeev Nehari, Conformal Mapping (1952), pp. 173–198.

It will be recalled [see (35.4)] that the function ψ satisfies on the boundary C of the region R the condition

$$\psi = \tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}z\bar{z};$$

hence the imaginary part of the function $f(\zeta)$ defined by (44.2) must satisfy the condition

$$\psi = \tfrac{1}{2}\omega(\zeta)\bar{\omega}(\bar{\zeta}) \qquad \text{on } \gamma, \tag{44.3}$$

where γ is the boundary of the circle $|\zeta| = 1$.

Thus, the torsion problem will be solved if we succeed in determining the real part ψ of the analytic function

$$\frac{1}{i} f(\zeta) = \psi - i\varphi,$$

which on the boundary γ of the unit circle $|\zeta| = 1$ assumes the values

$$\psi = \tfrac{1}{2}\omega(\zeta)\bar{\omega}(\bar{\zeta}).$$

But this is a special case of the problem treated in Sec. 42, and a reference to (42.4) shows that

$$\frac{1}{i} f(\zeta) = \frac{1}{\pi i}\int_\gamma \frac{1}{2}\frac{\omega(\sigma)\bar{\omega}(\bar{\sigma})}{\sigma - \zeta}\, d\sigma - a_0 + ib_0$$

or

$$f(\zeta) = \frac{1}{2\pi}\int_\gamma \frac{\omega(\sigma)\bar{\omega}(\bar{\sigma})}{\sigma - \zeta}\, d\sigma + \text{const.} \tag{44.4}$$

Noting that, on the boundary γ of the unit circle $|\zeta| = 1$, $\sigma = e^{i\theta}$ and hence $\bar{\sigma} = e^{-i\theta} = 1/\sigma$, one sees that the integral (44.4) can be written as

$$f(\zeta) = \varphi + i\psi = \frac{1}{2\pi}\int_\gamma \frac{\omega(\sigma)\bar{\omega}(1/\sigma)}{\sigma - \zeta}\, d\sigma + \text{const.} \tag{44.5}$$

The formula (44.5) gives us at once the torsion function φ and its conjugate ψ, so that the solution of the torsion problem is reduced to quadratures. If the numerator $\omega(\sigma)\bar{\omega}(1/\sigma)$, of the integrand, happens to be a rational function of σ, then the integral can be evaluated with the aid of the theorems on residues.

It is not difficult to express[1] the torsional rigidity D directly in terms of the function $f(\zeta)$. From (34.10)

$$\begin{aligned} D &= \mu \iint_R (x^2 + y^2)\, dx\, dy + \mu \iint_R \left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dx\, dy \\ &\equiv \mu I_0 + \mu D_0, \end{aligned} \tag{44.6}$$

[1] The calculations leading to formulas (44.7), (44.8), and (44.10) are due to N. I. Muskhelishvili. See, for example, his paper "Sur le problème de torsion des cylindres élastiques isotropes," *Atti della reale accademia nazionale dei Lincei*, ser. 6, vol. 9 (1929), pp. 295–300.

where I_0 is the polar moment of inertia of the area bounded by C and

$$D_0 = \iint_R \left(x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x}\right) dx\, dy = \iint_R \left[\frac{\partial}{\partial y}(x\varphi) - \frac{\partial}{\partial x}(y\varphi)\right] dx\, dy.$$

An application of Green's formula to this integral gives

$$D_0 = -\int_C \varphi(x\, dx + y\, dy) = -\int_C \varphi d \frac{1}{2} r^2,$$

where $r^2 = x^2 + y^2$. But on the contour C

$$r^2 = \mathfrak{z}\bar{\mathfrak{z}} = \omega(\sigma)\bar{\omega}(\bar{\sigma}) \qquad \text{and} \qquad \varphi = \tfrac{1}{2}[f(\sigma) + \bar{f}(\bar{\sigma})];$$

hence

$$D_0 = -\tfrac{1}{4} \int_\gamma [f(\sigma) + \bar{f}(\bar{\sigma})]\, d[\omega(\sigma)\bar{\omega}(\bar{\sigma})]. \tag{44.7}$$

Also

$$I_0 = \iint_R (x^2 + y^2)\, dx\, dy = \iint_R \left[\frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial x}(xy^2)\right] dx\, dy$$

$$= -\int_C xy(x\, dx - y\, dy).$$

But

$$x = \frac{\mathfrak{z} + \bar{\mathfrak{z}}}{2}, \qquad y = \frac{\mathfrak{z} - \bar{\mathfrak{z}}}{2i},$$

and we find that

$$I_0 = -\frac{1}{8i} \int_C (\mathfrak{z}^2 - \bar{\mathfrak{z}}^2)(\mathfrak{z}\, d\mathfrak{z} + \bar{\mathfrak{z}}\, d\bar{\mathfrak{z}}).$$

But

$$\int_C \mathfrak{z}^3\, d\mathfrak{z} = 0, \qquad \int_C \bar{\mathfrak{z}}^3\, d\bar{\mathfrak{z}} = 0,$$

and

$$\int_C \mathfrak{z}^2\bar{\mathfrak{z}}\, d\bar{\mathfrak{z}} = \int_C \mathfrak{z}^2\, d(\tfrac{1}{2}\bar{\mathfrak{z}}^2) = -\int_C \bar{\mathfrak{z}}^2\mathfrak{z}\, d\mathfrak{z},$$

where we make use of integration by parts. Hence we can write the polar moment of inertia I_0 in the form

$$I_0 = -\frac{i}{4} \int_C \bar{\mathfrak{z}}^2\mathfrak{z}\, d\mathfrak{z} = -\frac{i}{4} \int_\gamma [\bar{\omega}(\bar{\sigma})]^2 \omega(\sigma)\, d\omega(\sigma). \tag{44.8}$$

If $\omega(\sigma)$ is a rational function, then the integrands of (44.7) and (44.8) can be easily evaluated with the aid of theorems on residues and the expression for the torsional rigidity D can be obtained in closed form.

We may note that the shear components τ_{zx} and τ_{zy} of the stress tensor can likewise be expressed directly in terms of the functions $F(\mathfrak{z})$ and $f(\sigma)$.

It follows from the formulas (34.4) and (35.1) that

$$\tau_{zx} - i\tau_{zy} = \mu\alpha\left(\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y} - y - ix\right)$$
$$= \mu\alpha\left[\frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} - i(x - iy)\right].$$

But

$$\frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{dF}{d\mathfrak{z}},$$

and we get

(44.9) $$\tau_{zx} - i\tau_{zy} = \mu\alpha[F'(\mathfrak{z}) - i\bar{\mathfrak{z}}].$$

Since $F(\mathfrak{z}) = F[\omega(\zeta)] = f(\zeta)$, we have

$$F'(\mathfrak{z}) = f'(\zeta)\frac{d\zeta}{d\mathfrak{z}} = f'(\zeta)\frac{1}{\omega'(\zeta)}.$$

Hence (44.9) becomes

(44.10) $$\tau_{zx} - i\tau_{zy} = \mu\alpha\left[\frac{f'(\zeta)}{\omega'(\zeta)} - i\bar{\omega}(\bar{\zeta})\right].$$

This formula is extremely useful in calculating the components of shear.

If the mapping function is written in the form

$$\mathfrak{z} = \omega(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n,$$

then it is not difficult to give a formal solution of the torsion problem in terms of the coefficients a_n. We have

(44.11) $$\omega(\sigma)\bar{\omega}\left(\frac{1}{\sigma}\right) = \sum_{m=0}^{\infty} a_m\sigma^m \sum_{n=0}^{\infty} \bar{a}_n\sigma^{-n}$$
$$= \sum_{n=0}^{\infty} b_n\sigma^n + \sum_{n=1}^{\infty} \bar{b}_n\sigma^{-n},$$

where

(44.12) $$b_n = \sum_{r=0}^{\infty} a_{n+r}\bar{a}_r.$$

Upon inserting the expression (44.11) in (44.5), it is seen that

$$f(\zeta) = \frac{1}{2\pi}\int_\gamma \left[\sum_{n=0}^{\infty} b_n\sigma^n + \sum_{n=1}^{\infty} \bar{b}_n\sigma^{-n}\right]\frac{d\sigma}{\sigma - \zeta}$$

or

(44.13) $$f(\zeta) = \varphi + i\psi = i\sum_{0}^{\infty} b_n\zeta^n,$$

where Cauchy's Integral Formula and Eq. (41.2) have been used. The expression (44.13) for the complex stress function $\varphi + i\psi$ was derived by R. M. Morris[1] by a different method and was used to obtain formal solutions of the problem of torsion for those cases in which the complex constants a_n are known.

A formal expression involving the constants a_n can be given for the torsional rigidity $D = \mu(I_0 + D_0)$ [see (44.6)]. Equation (44.8) for the moment of inertia I_0 can be written as

$$I_0 = -\frac{i}{4}\int_\gamma \left[\omega(\sigma)\bar{\omega}\left(\frac{1}{\sigma}\right)\right]\bar{\omega}\left(\frac{1}{\sigma}\right) d\omega(\sigma).$$

Now, from $\omega(\sigma) = \sum_{n=0}^{\infty} a_n\sigma^n$, it follows that

$$\bar{\omega}\left(\frac{1}{\sigma}\right) d\omega(\sigma) = i \sum_{n=-\infty}^{\infty} c_n\sigma^n \, d\theta,$$

where

$$(44.14) \qquad \begin{cases} c_n = \displaystyle\sum_{r=0}^{\infty} (n+r)a_{n+r}\bar{a}_r, \\ c_{-n} = \displaystyle\sum_{r=0}^{\infty} r\bar{a}_{n+r}a_r, \qquad (n = 0, 1, 2, \ldots). \end{cases}$$

From this relation and from (44.11), we get

$$I_0 = \frac{1}{4}\int_0^{2\pi}\left(\sum_{m=0}^{\infty} b_m\sigma^m + \sum_{m=1}^{\infty} \bar{b}_m\sigma^{-m}\right)\left(\sum_{n=0}^{\infty} c_n\sigma^n + \sum_{n=1}^{\infty} c_{-n}\sigma^{-n}\right) d\theta.$$

Since $\sigma = e^{i\theta}$, we see that the integral of every term involving σ^n $(n \neq 0)$ vanishes and we are left with

$$I_0 = \frac{\pi}{2}\left[b_0c_0 + \sum_{n=1}^{\infty} (b_nc_{-n} + \bar{b}_nc_n)\right].$$

Similarly we can write

$$\begin{aligned} D_0 &= \frac{1}{4}\int_0^{2\pi}\left(\sum_{m=0}^{\infty} b_m\sigma^m - \sum_{m=0}^{\infty} b_m\sigma^{-m}\right)\left(\sum_{n=1}^{\infty} nb_n\sigma^n - \sum_{n=1}^{\infty} n\bar{b}_n\sigma^{-n}\right) d\theta \\ &= -\pi \sum_{n=1}^{\infty} nb_n\bar{b}_n. \end{aligned}$$

[1] R. M. Morris, "The Internal Problems of Two-dimensional Potential Theory," *Mathematische Annalen*, vol. 116 (1939), pp. 374–400; vol. 117 (1939), pp. 31–38.

Combining the expressions for I_0 and D_0, we get finally

(44.15) $$D = \frac{\mu\pi}{2}\left[b_0c_0 + \sum_{n=1}^{\infty}(c_n\bar{b}_n + c_{-n}b_n - 2nb_n\bar{b}_n)\right].$$

As an illustration of the application of the foregoing procedure, we consider a beam whose cross section is the cardioid

$$r = 2c(1 + \cos\alpha)$$

($r^2 = x^2 + y^2$, $\tan\alpha = y/x$). It is readily verified that in this case a suitable mapping function is

$$z = c(1 - \zeta)^2,$$

so that the only nonvanishing coefficients a_n are

$$a_0 = c, \qquad a_1 = -2c, \qquad a_2 = c.$$

The nonvanishing constants b_n and c_n are easily found to be

$$\begin{aligned} &b_0 = 6c^2, \qquad b_1 = -4c^2, \qquad b_2 = c^2, \\ c_{-1} = -2c^2, \qquad &c_0 = 6c^2, \qquad c_1 = -6c^2, \qquad c_2 = 2c^2. \end{aligned}$$

The complex stress function is (see also Sec. 58)

(44.16) $$f(\zeta) = \varphi + i\psi = i\sum_{n=0}^{\infty} b_n\zeta^n = ic^2(6 - 4\zeta + \zeta^2),$$

while

(44.17) $$D = 17\mu\pi c^4.$$

It should be noted that the method outlined above is readily applicable whenever the mapping function $\omega(\zeta)$ for the region R is a polynomial or whenever the mapping can be approximated with sufficient accuracy by a polynomial. If the mapping function is known as a power series in ζ, then a formal solution can be given in terms of the coefficients a_n of the mapping function. If the mapping function is known in a closed form, then it may be easier to proceed directly from Eq. (44.5) rather than expand $\omega(\zeta)$ in a power series and then deal with the resulting infinite series (44.12) to (44.15). The reader will verify that formula (44.4) can be used in this case to obtain the result (44.16) with no calculational effort.

The problem of the cardioid is of some interest inasmuch as it indicates an approximate behavior of a checked beam.

PROBLEM

Obtain the solution of the torsion problem for a cardioid by utilizing formula (44.4), and thus verify (44.14).

45. Applications of Conformal Mapping. This section contains several illustrations of the application of the foregoing theory to the solution of the torsion problem.

First we consider a cylinder whose cross section R is bounded by the inverse of an ellipse with respect to its center. When the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is inverted with respect to its center, the point (x, y) is carried to the point (x', y'), which is such that

$$r^2(r')^2 \equiv (x^2 + y^2)[(x')^2 + (y')^2] = 1.$$

The resulting curve C (Fig. 30) is given in terms of the parameter u by the equations

$$\frac{x}{x^2 + y^2} = \frac{1}{c} \cosh k \cos u,$$

$$\frac{y}{x^2 + y^2} = \frac{1}{c} \sinh k \sin u,$$

or

$$(45.1) \qquad x + iy = c \sec (u + ik)$$

with $c = 1/\sqrt{a^2 - b^2}$, $\tanh k = b/a$.

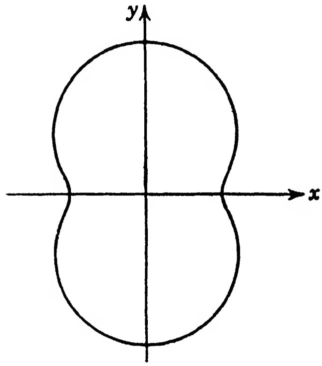
FIG. 30

Equation (45.1) can evidently be written as

$$(45.2) \qquad z = c \sec (w + ik), \qquad v = 0,$$

with $z = x + iy$, $w = u + iv$. If we put $\zeta = e^{iw}$ in (45.2), we see that the resulting function

$$(45.3) \qquad z = \omega(\zeta) \equiv \frac{2ce^k\zeta}{\zeta^2 + e^{2k}}, \qquad c > 0, \qquad k > 0,$$

maps the cross section R of the cylinder upon the interior of the unit circle $|\zeta| \leq 1$.

In the preceding section, expressions were derived that give a formal solution of the torsion problem when the mapping function is expanded in an infinite series

$$z = \omega(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n = \sum_{n=0}^{\infty} a_n e^{inw}.$$

Such expressions, given by R. M. Morris, were used by T. J. Higgins,[1] who observed that in this case

[1] "The Torsion of a Prism with Cross Section the Inverse of an Ellipse," *Journal of Applied Physics*, vol. 13 (1942), pp. 457–459.

$$a_n = \begin{cases} 0, & \text{if } n = 0, 2, 4, \dots, \\ 2c(-1)^{(n-1)/2}e^{-nk}, & \text{if } n = 1, 3, 5, \dots. \end{cases}$$

The infinite series entering into (44.13) and (44.15) were then summed and the torsion function and twisting moment obtained in closed form. However, since $\omega(\zeta)$ is a rational function of ζ, it is simpler to proceed directly[1] from (44.5), (44.7), and (44.8).

When $\omega(\zeta)$ from (45.3) is inserted in (44.5), it is seen that

$$\begin{aligned} f(\zeta) &= \varphi + i\psi \\ &= \frac{2c^2}{\pi}\int_\gamma \frac{\sigma^2\,d\sigma}{(\sigma^2 + e^{2k})(\sigma^2 + e^{-2k})(\sigma - \zeta)} + \text{const} \\ &= -4c^2 i(R_1 + R_2) + \text{const}, \end{aligned} \tag{45.4}$$

where R_1 and R_2 are the residues of the integrand at $\sigma = ie^k$ and $\sigma = -ie^k$, respectively. We have

$$R_1 = \left[\frac{\sigma^2}{(\sigma + ie^k)(\sigma^2 + e^{-2k})(\sigma - \zeta)}\right]_{\sigma = ie^k} = \frac{ie^k}{4(\zeta - ie^k)\sinh 2k},$$
$$R_2 = \frac{-ie^k}{4(\zeta + ie^k)\sinh 2k},$$

and hence

$$f(\zeta) = \varphi + i\psi = c^2 \operatorname{csch} 2k \tan(w + ik),$$

where the constant in (45.4) has been taken equal to $-ic^2 \operatorname{csch} 2k$.

The shearing stresses may be found, either from the relations

$$\tau_{xz} = \mu\alpha\frac{\partial\Psi}{\partial y}, \qquad \tau_{yz} = -\mu\alpha\frac{\partial\Psi}{\partial x}, \qquad \Psi = \psi - \frac{1}{2}(x^2 + y^2),$$

or from Eq. (44.10), to be

$$\tau_{zx} = -2\mu\alpha c \sin u\left[\frac{\operatorname{csch} 2k \cosh(v + k)}{\cos 2u - \cosh 2(v + k)} + \frac{\sinh(v + k)}{\cos 2u + \cosh 2(v + k)}\right],$$
$$\tau_{zy} = -2\mu\alpha c \cos u\left[\frac{\operatorname{csch} 2k \sinh(v + k)}{\cos 2u - \cosh 2(v + k)} - \frac{\cosh(v + k)}{\cos 2u + \cosh 2(v + k)}\right].$$

Equation (44.8) for the moment of inertia I_0 takes the form

$$\begin{aligned} I_0 &= 4c^4 i\int_\gamma \frac{\sigma^3(\sigma^2 - e^{2k})}{(\sigma^2 + e^{2k})^3(\sigma^2 + e^{-2k})^2}\,d\sigma \\ &= -8\pi c^4(R_3 + R_4), \end{aligned}$$

where R_3 and R_4 are the residues of the integrand at $\sigma = ie^{-k}$ and $\sigma = -ie^{-k}$, respectively. The residues are

[1] See I. S. Sokolnikoff and R. D. Specht, "Two Dimensional Boundary Value Problems in Potential Theory," *Journal of Applied Physics*, vol. 14 (1943), pp. 91–95.

$$R_3 = \frac{d}{d\sigma}\left[\frac{\sigma^3(\sigma^2 - e^{2k})}{(\sigma^2 + e^{2k})^3(\sigma + ie^{-k})^2}\right]_{\sigma = ie^{-k}}$$
$$= \frac{-\operatorname{csch}^4 2k(2 + \cosh 4k)}{16} = R_4,$$

and therefore

$$I_0 = \pi c^4(2 + \cosh 4k) \operatorname{csch}^4 2k.$$

Similarly from (44.7) we get

$$D_0 = -i4c^4 \operatorname{csch} 2k \int_\gamma \frac{\sigma(1 - \sigma^4)^2}{(\sigma^2 + e^{2k})^3(\sigma^2 + e^{-2k})^3}\, d\sigma$$
$$= 8\pi c^4 \operatorname{csch} 2k(R_5 + R_6),$$

in which R_5 and R_6 are the residues at $\sigma = ie^{-k}$ and $\sigma = -ie^{-k}$. We find that

$$R_5 = \frac{1}{2}\frac{d^2}{d\sigma^2}\left[\frac{\sigma(1 - \sigma^4)^2}{(\sigma^2 + e^{2k})^3(\sigma + ie^{-k})^3}\right]_{\sigma = ie^{-k}}$$
$$= -\tfrac{1}{8} \operatorname{csch}^3 2k = R_6,$$

and hence

$$D_0 = -2\pi c^4 \operatorname{csch}^4 2k.$$

The twisting moment is given by

$$M = \mu\alpha(I_0 + D_0)$$
$$= \mu\alpha\pi c^4(2 \operatorname{csch}^2 2k + \operatorname{csch}^4 2k).$$

The curve resulting from inversion of the ellipse with respect to its focus is called an *elliptic limaçon.* The torsion problem for a cylinder with elliptic-limaçon cross section was treated by Stevenson and Holl and Rock,[1] and the corresponding problem for a hyperbolic limaçon by Lin, Whitehead, and Yang.[2] Methods of solution used by these investigators differ somewhat from those presented here.

As an example of the type of calculations required when the mapping function is not rational, consider the map of the unit circle obtained with the aid of

$$(45.5) \qquad z = \omega(\zeta) \equiv a\sqrt{1 + \zeta}, \qquad \text{where } a > 0.$$

We shall deal with that branch of the multiple-valued function $\sqrt{1 + \zeta}$ that gives $+1$ for $\zeta = 0$.

[1] A. C. Stevenson, *Proceedings of the London Mathematical Society* 2, vol. 45 (1939), p. 126.

D. L. Holl and D. H. Rock, *Zeitschrift für angewandte Mathematik und Mechanik,* vol. 19 (1939), p. 141.

[2] T. C. Lin and L. G. Whitehead, *University of Washington, Engineering Experiment Station Series, Bulletin* 118 (1951), pp. 108–111.

T. C. Lin and H. T Yang, *University of Washington, Engineering Experiment Station Series, Bulletin* 118 (1951), pp. 112–119.

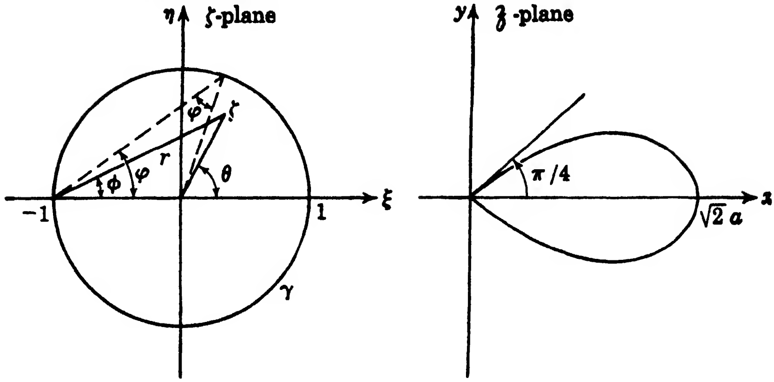

Fig. 31

Then from Fig. 31

$$\mathfrak{z} = a\sqrt{r}\,e^{i\varphi/2}, \qquad \left(-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\right).$$

When ζ moves along the circle γ,

$$\varphi = \tfrac{1}{2}\theta, \qquad (-\pi \leq \theta \leq \pi),$$

and

$$r = 2\cos\tfrac{1}{2}\theta.$$

Hence

$$\mathfrak{z} = a\sqrt{2\cos\tfrac{1}{2}\theta}\;e^{i\theta/4}, \qquad (-\pi \leq \theta \leq \pi).$$

Let R and ψ denote the modulus and the argument of $\mathfrak{z}$; then

$$Re^{i\psi} = a\sqrt{2\cos\tfrac{1}{2}\theta}\;e^{i\theta/4}.$$

Hence

$$R = a\sqrt{2\cos\frac{1}{2}\theta}, \qquad \psi = \frac{\theta}{4},$$

and it follows that

$$R = a\sqrt{2\cos 2\psi}.$$

Thus, the map of the unit circle is one loop of the lemniscate shown in Fig. 31.

Substituting from (45.5) in the formula (44.5), we have

$$(45.6) \quad f(\zeta) = \frac{1}{2\pi}\int_\gamma \frac{a^2\sqrt{1+\sigma}\sqrt{1+\dfrac{1}{\sigma}}}{\sigma-\zeta}\,d\sigma = \frac{a^2}{2\pi}\int_\gamma \frac{1+\sigma}{\sqrt{\sigma}}\,\frac{d\sigma}{\sigma-\zeta}.$$

Since the sign of the square root must be chosen positive, we can write $\sqrt{\sigma} = e^{i\theta/2}$.

If we cut the negative axis as shown in Fig. 32, then the integrand of (45.6) will be a single-valued function in the simply connected region

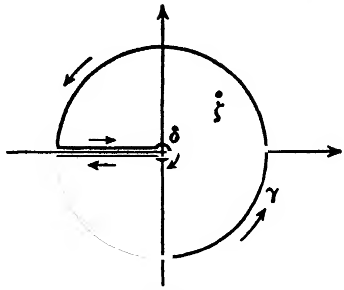

FIG. 32

indicated in the figure and the only singularity of the integrand is the pole at $\sigma = \zeta$. Hence

$$\frac{1}{2\pi i}\left[\int_\gamma g(\sigma, \zeta)\, d\sigma + \int_{-1}^{0} g(\sigma, \zeta)\, d\sigma + \int_\delta g(\sigma, \zeta)\, d\sigma + \int_0^{-1} g(\sigma, \zeta)\, d\sigma\right] = R,$$

where

$$g(\sigma, \zeta) \equiv \frac{1 + \sigma}{\sqrt{\sigma}\,(\sigma - \zeta)},$$

R is the residue of $g(\sigma, \zeta)$ at $\sigma = \zeta$, and δ is a small circle about the origin. But the residue R at $\sigma = \zeta$ is obviously

$$R = \frac{1 + \zeta}{\sqrt{\zeta}}.$$

Thus,

$$\frac{1}{2\pi i}\int_\gamma g(\sigma, \zeta)\, d\sigma = -\frac{1}{2\pi i}\left[\int_{-1}^{0} g(\sigma, \zeta)\, d\sigma + \int_0^{-1} g(\sigma, \zeta)\, d\sigma + \int_\delta g(\sigma, \zeta)\, d\sigma\right] + R$$

$$= -\frac{1}{\pi}\int_0^1 \frac{1 - t}{\sqrt{t}}\,\frac{dt}{t + \zeta} + \frac{1 + \zeta}{\sqrt{\zeta}},$$

where we have dropped the integral over the small circle δ, since it vanishes when the radius of δ tends to zero, and where the integrals over the portion of the real axis between 0 and -1 are combined by making an obvious change of variable and by noting the difference in sign of the function $\sqrt{\sigma}$ on the upper and lower banks of the cut. Integrating and dropping the nonessential additive constant, we have finally

$$f(\zeta) = \frac{a^2}{\pi}\,\frac{1 + \zeta}{\sqrt{\zeta}} \log \frac{1 + i\sqrt{\zeta}}{1 - i\sqrt{\zeta}},$$

where

$$\log \frac{1 + i\sqrt{\zeta}}{1 - i\sqrt{\zeta}} = 2i\sqrt{\zeta}\left(1 - \frac{\zeta}{3} + \frac{\zeta^2}{5} - \cdots\right).$$

The function $f(\zeta)$ solves the torsion problem for a beam whose cross section is one loop of the lemniscate. The calculation of stresses presents no serious difficulty and is left as an exercise to the reader.

As a third example of this general method of attack upon problems of torsion, consider the case of a cylinder whose cross section is bounded by two circular arcs.[1]

Consider a region R of the complex z-plane bounded by two circular arcs C_1 and C_2 making an angle $\alpha \neq 0$ at their points of intersection $z = z_1$ and $z = z_2$ (Fig. 33*a*). It is obvious that the transformation

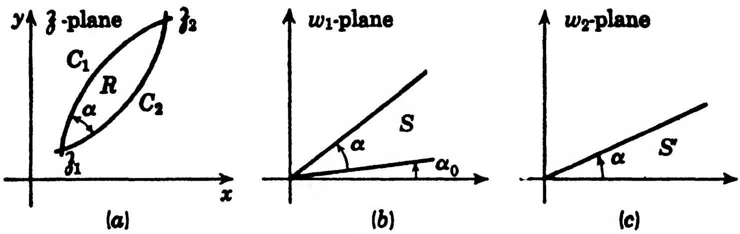

FIG. 33

$w_1 = (z - z_1)/(z - z_2)$ maps the point z_1 on the origin of the w_1-plane, the point z_2 on the point at infinity, and the region R on the infinite sector S shown in Fig. 33*b*. The sector S is rotated through an angle α_0 to bring one radius into coincidence with the real axis of the w_2-plane by the transformation $w_2 = e^{-i\alpha_0}w_1$. If the transformation $w_3 = w_2^{\pi/\alpha}$ is applied, then the domain S' is mapped on the upper half plane of the complex w_3-plane. Finally, this upper half plane is carried into the unit circle in the ζ-plane by the mapping function $\zeta = (i - w_3)/(1 - iw_3)$. If these successive transformations are combined, one obtains the mapping function

$$\zeta = \frac{Ci(z - z_1)^{\pi/\alpha} + (z - z_2)^{\pi/\alpha}}{Ci(z - z_1)^{\pi/\alpha} - (z - z_2)^{\pi/\alpha}},$$

which effects conformal mapping of the region R on the unit circle in the complex ζ-plane. The choice of the constants C, z_1, and z_2 is uniquely determined by the geometrical configuration and the scale used in mapping the region R on the unit circle. For $\alpha = \pi/n$, where n is an integer, the mapping function becomes rational, a fact that greatly simplifies the evaluation of (44.5). To illustrate the procedure, it will suffice to consider two important special cases when the region R is one of the following:

1. A lune formed by two circular arcs of equal radius and intersecting at right angles;
2. A semicircle.

[1] The discussion that follows is taken from the paper "Torsion of Regions Bounded by Circular Arcs," by I. S. and E. S. Sokolnikoff, *Bulletin of the American Mathematical Society*, vol. 44 (1938), pp. 384–387.

It is easily verified that the mapping functions appropriate to the regions defined in (1) and (2) are, respectively,

$$\zeta = \frac{2z}{z^2 + 1}$$

and

$$\zeta = \frac{(z + 1)^2 - i(z - 1)^2}{(z + 1)^2 + i(z - 1)^2}.$$

The solution of the torsion problem for the region described in (1) above is given next.

From

$$\zeta = \frac{2z}{z^2 + 1},$$

it follows that, on the boundary of R, $z = [1 - (1 - \sigma^2)^{1/2}]/\sigma$, where the appropriate branch of the square root is determined from the observation that the imaginary part of z is positive whenever θ in $\sigma = e^{i\theta}$ lies between 0 and π. Then the numerator of the integrand in (44.5) is

$$\omega(\sigma)\bar{\omega}\left(\frac{1}{\sigma}\right) = [1 - (1 - \sigma^2)^{1/2}]\left[1 - \left(1 - \frac{1}{\sigma^2}\right)^{1/2}\right].$$

Substituting this expression in (44.5) and evaluating the resulting integrals, we get

$$f(\zeta) = -i(1 - \zeta^2)^{1/2} - \frac{i(1 - \zeta^2)}{\pi\zeta}\log\frac{1 - \zeta}{1 + \zeta} + \text{const},$$

or if we return to the z-plane with the aid of the mapping function,

$$F(z) = i\,\frac{z^2 - 1}{z^2 + 1} - \frac{i(1 - z^2)^2}{\pi z(1 + z^2)}\log\frac{1 - z}{1 + z} + \text{const}.$$

The imaginary part of $F(z)$, which is the desired solution, is

$$\begin{aligned}\psi(x, y) = \{\pi(x^2 + y^2)[(x^2 + y^2 + 1)^2 - 4y^2]\}^{-1}\{\pi(x^2 + y^2)&\\ \cdot[(x^2 + y^2)^2 - 1]&\\ + x(x^2 + y^2 + 1)[4y^2 - (x^2 + y^2 - 1)^2]\log S&\\ + y(x^2 + y^2 - 1)[(x^2 + y^2 + 1)^2 + 4x^2]\beta\},&\end{aligned}$$

where

$$S = \frac{[(1 - x^2 - y^2)^2 + 4y^2]^{1/2}}{(1 + x)^2 + y^2} \quad \text{and} \quad \beta = \tan^{-1}\frac{-2y}{1 - x^2 - y^2}.$$

A simple calculation shows that, on the boundary of the lune formed by $x^2 + (y \pm 1)^2 = 2$, $\psi(x, y)$ reduces to

$$\psi = \tfrac{1}{2}(x^2 + y^2) + \text{const},$$

as it should.

An analogous calculation gives the solution of the torsion problem for the semicircular region. We obtain

$$f(\zeta) = \frac{i2^{1/2}(1+i\zeta)(1-\zeta^2)^{1/2}}{(i+\zeta)^2} + \frac{4}{\pi(i+\zeta)} + \frac{2}{\pi}\frac{(1-\zeta^2)}{(i+\zeta)^2}\log\frac{i(1-\zeta)}{1+\zeta} + \text{const},$$

or, if we return to the z-plane with the aid of the mapping function,

$$F(z) = \frac{1}{2\pi}\left[\pi i z^2 + \frac{2(z^2+1)}{z} + \frac{(z^2-1)^2}{z^2}\log\frac{1-z}{1+z} + \text{const}\right].$$

This result agrees with that obtained by Greenhill by an entirely different method.[1]

The torsion problem for a lens-shaped prism whose cross section is formed by the arcs of two circles of different radii was solved in bipolar coordinates with the aid of Fourier integrals by Uflyand.[2]

The examples discussed in this section illustrate the remarkable ease with which the torsion problem can be solved when the mapping function $\omega(\zeta)$ has a simple form. The method of solution illustrated above can also be used to solve the torsion problem for beams of polygonal cross section when the mapping function obtained with the aid of the Schwarz-Christoffel formula (43.1) is not too unwieldy.[3]

Even when the mapping function is known, it may prove advantageous to use some other method, as was done in Sec. 38 in the study of torsion of rectangular beams, where series of orthogonal functions were employed. The series method of solving the torsion problem for polygonal beams with cross sections made up of rectangular components was used effectively by Arutyunyan and Aleksandryan and Gulkanyan[4] to obtain an exact solution of the torsion problem for a beam of finite L, T, and channel sections. The same method was used by Abramyan to solve the torsion

[1] A. G. Greenhill, "Fluid Motion in a Rotating Quadrantal Cylinder," *Messenger of Mathematics*, vol. 8 (1879), p. 89; "On the Motion of a Frictionless Liquid in a Rotating Sector," *Messenger of Mathematics*, vol. 10 (1881), p. 83.

See also A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, p. 319.

[2] Ya. S. Uflyand, *Doklady Akademii Nauk SSSR*, vol. 68 (1949), pp. 17–20.

[3] For the uses of the Schwarz-Christoffel formula in the torsion problem see:

E. Trefftz, "Über die Torsion prismatischer Stabe von polygonalen Querschnitt," *Mathematische Annalen*, vol. 82 (1921), pp. 97–112.

I. S. Sokolnikoff, "On a Solution of Laplace's Equation with an Application to the Torsion Problem for a Polygon with Reentrant Angles," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 719–732.

B. R. Seth, "On the General Solution of a Class of Physical Problems," *Philosophical Magazine* (7), vol. 20 (1935), pp. 632–640.

P. F. Kufarev, "Torsion and Bending of Members of Polygonal Sections," *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* New Ser., vol. 1 (1937), pp. 43–76.

[4] N. Kh. Arutyunyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 13 (1949), pp. 107–112; E. A. Aleksandryan and N. O. Gulkanyan, *Akademiya Nauk Armyan. SSR, Izvestiya, Phys. Mat. Nauki*, vol. 6 (1953), pp. 37–51.

problem for a beam with finite cruciform cross section.[1] A variant of this method was employed by Abramyan and Arutyunyan to solve the torsion problem for a beam with special trapezoidal cross section.[2]

Conformal mapping has been used to provide solutions of the torsion and flexure problems for circular beams with one or two slits extending from the ends of a diameter.[3] A solution of the torsion problem for a rectangular beam containing cracks was given by Gulkanyan.[4] The stress concentration in twisted prismatic rods whose cross sections have reentrant angles was recently considered by Pivovarov.[5]

Problem

Analyze the behavior of the shearing stresses in the first illustration of Sec. 45 when k approaches zero. The cross section in this case differs little from the figure consisting of the pair of tangent circles. *Hint:* Write the transformation (45.3) in the form

$$\omega(\zeta) = \frac{\varkappa\zeta}{\zeta^2 + a^2}, \qquad \varkappa > 0, \qquad a > 1,$$

and deduce

$$f(\zeta) = \frac{ia^2\varkappa^2}{(a^4 - 1)(\zeta^2 + a^2)}.$$

Use formula (44.10) to obtain τ_{zx} and τ_{zy}, and let $a \to 1$.

46. Membrane and Other Analogies. It is clear from the discussion given in Secs. 38 and 45 that a rigorous solution of the torsion problem for beams whose cross sections are in the shape of the letters I, U, L, T, etc., is likely to prove extremely vexing. While there are some rigorous solutions of the torsion problem for beams of polygonal cross section,[6] the resultant formulas are too involved to be of immediate value to a practical designing engineer, who requires some simple, reasonably accurate formulas. To meet this need, a variety of approximate formulas have been developed for the torsion constants of sections whose components

[1] B. L. Abramyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 13 (1949), pp. 551–556.

[2] B. L. Abramyan and N. Kh. Arutyunyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 15 (1951), pp. 97–102.

[3] W. M. Shepherd, *Proceedings of the Royal Society (London) (A)*, vol. 138 (1932), pp. 607–634; vol. 154 (1936), pp. 500–509.

A. C. Stevenson, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 237 (1938), pp. 161–229.

L. A. Wigglesworth, *Proceedings of the London Mathematical Society* (2), vol. 47 (1940), pp. 20–37.

[4] N. O. Gulkanyan, *Akademiya Nauk Armyan. SSR, Izvestiya, Phys. Mat. Nauki*, vol. 5 (1952), pp. 67–96. See also O. M. Sapondzyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 13 (1949), pp. 501–512 (in Russian); and W. Nowacki, *Arch. Mech Stos.*, vol. 5 (1953), pp. 21–46 (in Polish).

[5] A. M. Pivovarov, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 17 (1953), pp 253–260.

[6] See references in Sec. 45.

are rectangles. Many such formulas are based on the mathematical analogy between the torsion problem and the behavior of a stretched elastic membrane subjected to a uniform excess of pressure on one side.[1] A reference to this analogy was made in Sec. 35, and we proceed to discuss it here in detail.

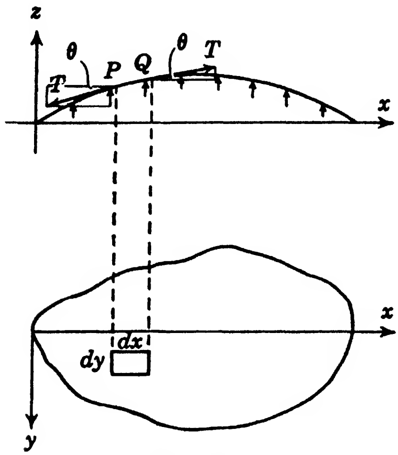

Fig. 34

Let a very thin homogeneous membrane, such as a soap film, be stretched under a uniform tension T per unit length over an opening made in a rigid plate. The opening in the plate is assumed to have the same shape as the cross section of the beam subjected to torsion, and the membrane is supposed to be fixed at the edge of the opening. If p is the pressure per unit area of the membrane, and if the membrane is in equilibrium, then the force $p\,dx\,dy$, acting on an element of area $dx\,dy$ (Fig. 34), must be balanced by the resultant of the vertical components of the tensile stresses acting on the boundary of the element of area. Now the resultant of the vertical components of the tensile forces acting on the edges dy is

$$\begin{aligned}(T\,dy\sin\theta)_Q - (T\,dy\sin\theta)_P &\doteq \left(T\,dy\frac{\partial z}{\partial x}\right)_Q - \left(T\,dy\frac{\partial z}{\partial x}\right)_P \\ &\doteq \left(T\,dy\frac{\partial z}{\partial x}\right)_P + \frac{\partial}{\partial x}\left(T\,dy\frac{\partial z}{\partial x}\right)_P dx - \left(T\,dy\frac{\partial z}{\partial x}\right)_P \\ &= T\frac{\partial^2 z}{\partial x^2}\,dx\,dy,\end{aligned}$$

where it is assumed that the deflection z is small. Similarly, the resultant of the vertical components acting on the edges dx is

$$T\frac{\partial^2 z}{\partial y^2}\,dx\,dy.$$

[1] A detailed discussion of the procedure employed in deriving some approximate formulas for sections whose components are rectangles (as well as for some tubular sections) is given in the *National Advisory Committee for Aeronautics Report* 334, by G. W. Trayer and H. W. March, entitled "The Torsion of Members Having Sections Common in Aircraft Construction." This report contains an extensive bibliography and a comparison of their formulas with those obtained by other investigators. A description of the experimental procedure used in studying torsion of beams with the aid of soap films is given by Trayer and March and also by A. A. Griffith and G. I. Taylor in the *Advisory Committee on Aeronautics Technical Report*, Great Britain (1917–1918). A brief account of the procedure employed by Griffith and Taylor is found in S. Timoshenko and J. N. Goodier, Theory of Elasticity, Sec. 99.

Hence the equation of equilibrium of the element is

$$p\,dx\,dy + T\frac{\partial^2 z}{\partial x^2}\,dx\,dy + T\frac{\partial^2 z}{\partial y^2}\,dx\,dy = 0,$$

and we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-p}{T}. \tag{46.1}$$

This equation must be solved subject to the condition $z = 0$ on the edge of the opening. If we substitute in (46.1)

$$z = \frac{p}{2T}\Psi, \tag{46.2}$$

the equation becomes

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -2, \tag{46.3}$$

subject to the condition that

$$\Psi = 0 \qquad \text{on the boundary.} \tag{46.4}$$

Equation (46.3) is identical with that obtained in Sec. 35 for the stress function Ψ, and it is clear from the discussion there that the slope of the membrane at any point is proportional to the magnitude of the shearing stress

$$\tau = \mu\alpha\sqrt{\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2} = \mu\alpha\frac{d\Psi}{d\nu}$$

at the corresponding point of the section subjected to torsion. The contour lines z = const of the membrane correspond to the lines of shearing stress $\Psi(x, y)$ = const. Recalling that the torsional rigidity of the beam is given by Eq. (35.10),

$$D = 2\mu\iint_R \Psi\,dx\,dy,$$

it becomes clear that the volume between the plane of the opening $z = 0$ and the surface of the membrane is proportional to the torsional rigidity D of the section. Since the contour lines of the membrane can be mapped out, and the slope at each point and the volume under the membrane can be determined, one can secure the desired information concerning the lines of shearing stress and the torsional rigidity of the beam from experimental measurements.

A consideration of the equation of the unloaded membrane,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

which is so supported at the edges that

$$z = \tfrac{1}{2}(x^2 + y^2) \qquad \text{on the boundary,} \tag{46.5}$$

shows that one can also determine experimentally the function $\psi(x, y)$ [see (35.3)] from a study of an unloaded soap film stretched so that the

heights of the membrane over the contour of the section have the values given by (46.5).

The membrane analogy has been used in an interesting way by Timoshenko to discuss an approximate behavior of a beam of narrow, rectangular cross section and in analyzing the stress concentration near fillets in channel sections and I beams. It is interesting to note that the maximum shearing stress in a narrow beam of thickness c is twice as great as in a circular shaft of diameter c and subjected to the same twist. The details of the calculations and further discussion will be found in Timoshenko and Goodier's Theory of Elasticity, Secs. 93 and 94.

The technique of measuring the ordinates of the membrane has been discussed by Thiel,[1] who used stereoscopic photography, while Reichenbächer[2] has decsribed an optical device for the automatic plotting of the contour lines of the membrane. The soap film has been replaced by a paraffin surface by Kopf and Weber,[3] and by the interface between two immiscible liquids by Piccard and Baes[4] and by Sunatani, Matuyama, and Hatamura.[5] L. Föppl[6] and Deutler[7] have discussed the form of the membrane analogy in which the film is under zero resultant pressure and its boundary has variable height.

The boundary-value problems of torsion can also be interpreted in terms of various hydrodynamical analogies. These are discussed briefly in Timoshenko and Goodier's Theory of Elasticity, Sec. 100, where several references are given. To these may be added a paper by Den Hartog and McGivern[8] in which experimental technique is described.

The analogy between the torsion of a cylinder and the potential of a plane electric field affords another way of obtaining experimental solu-

[1] A. Thiel, "Photogrammetrisches Verfahren zur versuchsmässigen Lösung von Torsionsaufgaben (nach einem Seifenhautgleichnis von L. Föppl)," *Ingenieur Archiv*, vol. 5 (1934), pp. 417–429.

[2] H. Reichenbächer, "Selbsttatige Ausmessung von Seifenhautmodellen (Anwendung auf das Torsionsproblem)," *Ingenieur Archiv*, vol. 7 (1936), pp. 257–272.

[3] E. Kopf and E. Weber, "Verfahren zur Ermittlung der Torsionsbeanspruchung mittels Membranmodell," *Zeitschrift des Vereines deutscher Ingenieure*, vol. 78 (1934), pp. 913–914.

[4] A. Piccard and L. Baes, "Mode expérimental nouveau relatif à l'application des surfaces à courbure constante à la solution du problème de la torsion des barres prismatiques," *Proceedings of the Second International Congress for Applied Mechanics, Zürich* (1927), pp. 195–199.

[5] Chidô Sunatani, Tokuzo Matuyama, and Motomune Hatamura, "The Solution of Torsion Problems by Means of a Liquid Surface," *Technical Reports of the Tôhoku Imperial University*, vol. 12 (1937), pp. 374–396.

[6] L. Föppl, "Eine Ergänzung des Prandtlschen Seifenhaut-Gleichnisses zur Torsion," *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 15 (1935), pp. 37–40.

[7] H. Deutler, "Zur versuchsmässigen Lösung von Torsionsaufgaben mit Hilfe des Seifenhautgleichnisses," *Ingenieur Archiv*, vol. 9 (1938), pp. 280–282.

[8] J. P. Den Hartog and J. G. McGivern, "On the Hydrodynamic Analogy of Torsion," *Journal of Applied Mechanics*, vol. 2 (1935), pp. A46–A48.

tions of the torsion problem. This is described in Sec. 7, Chap. III, of Technische Dynamik by C. B. Biezeno and R. Grammel and in a paper by H. Cranz.[1]

The equation for current flow in a conductor of variable thickness is identical with that describing the torsion of a shaft of varying circular section.[2] This analogy, which yields a practical method for studying stress concentration in the neighborhood of fillets or grooves in shafts under torsion, is described in Sec. 104 of Timoshenko and Goodier's Theory of Elasticity and in papers by Thum and Bautz,[3] Jacobsen,[4] and Salet.[5]

A discussion of several analogic methods of approximate solution of Saint-Venant's torsion problem, including extensive bibliographical references, is contained in two papers by T. J. Higgins in *Proceedings of the Society for Experimental Stress Analysis*, vol. 2 (1945), pp. 17–27, vol. 3 (1945), pp. 94–101.

47. Torsion of Hollow Beams. The discussion of the torsion problem has been confined thus far to solid beams, so that the region of the cross section has been simply connected. Hollow or tubular beams are of considerable technical importance, and it is necessary to extend the formulation of the torsion problem so as to include multiply connected regions.

Let it be assumed that a beam has several longitudinal cavities so that the boundary of the cross section of the beam is made up of several simple closed contours. Denote the exterior contour by C_0, and let $C_1, C_2, \ldots, C_n$ be the simple closed contours lying entirely within the contour C_0 (Fig. 35). The contours $C_1, C_2, \ldots, C_n$ correspond to the cavities of the beam. The discussion in Sec. 34 that led to the formulation of the differential equation (34.5) is valid in this case, and we have the differential equation

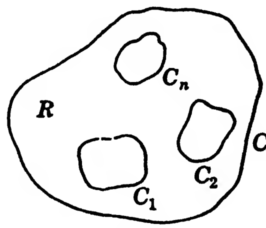

Fig. 35

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \text{in } R, \tag{47.1}$$

[1] H. Cranz, "Experimentelle Lösung von Torsionsaufgaben," *Ingenieur Archiv*, vol. 4 (1933), pp. 506–509.

[2] See Sec. 49.

[3] A. Thum and W. Bautz, "Die Ermittlung von Spannungsspitzen in verdrehbeanspruchten Wellen durch ein elektrisches Modell," *Zeitschrift des Vereines deutscher Ingenieure*, vol. 78 (1934), pp. 17–19.

[4] L. S. Jacobsen, "Torsional Stresses in Shafts Having Grooves or Fillets," *Journal of Applied Mechanics*, vol. 2 (1935), pp. A154–A155.

[5] G. Salet, "Détermination des pointes de tension dans les arbres de révolution soumis à torsion au moyen d'un modèle électrique," *Bulletin de l'association technique maritime et aéronatique*, vol. 40 (1936), pp. 341–350, 351.

where R is the multiply connected region interior to C_0 and exterior to $C_1, C_2, \ldots, C_n$. Since the longitudinal cavities and the outer surface are free from external loads, we have, as shown in Sec. 34, the boundary conditions

$$\text{(47.2)} \quad \frac{d\varphi}{d\nu} = y \cos(x, \nu) - x \cos(y, \nu) \quad \text{on } C_i, \quad (i = 0, 1, 2, \ldots, n).$$

These boundary conditions, as shown in Sec. 35, can be expressed in terms of the conjugate function ψ as

$$\frac{d\psi}{ds} = \frac{d}{ds}\frac{x^2 + y^2}{2} \qquad \text{on } C_i, \qquad (i = 0, 1, 2, \ldots, n),$$

and the integration along each contour C_i yields

$$\text{(47.3)} \quad \psi = \tfrac{1}{2}(x^2 + y^2) + k_i \qquad \text{on } C_i, \qquad (i = 0, 1, 2, \ldots, n),$$

where the k_i are the integration constants. The value of one of these constants, say k_0, can be specified arbitrarily,[1] but the remaining n constants k_i must be determined so that the function

$$\varphi(x, y) = \int_{P_0(x_0, y_0)}^{P(x, y)} \left(\frac{\partial \varphi}{\partial x}\, dx + \frac{\partial \varphi}{\partial y}\, dy\right),$$

which can be written as

$$\text{(47.4)} \qquad \varphi(x, y) = \int_{P_0(x_0, y_0)}^{P(x, y)} \left(\frac{\partial \psi}{\partial y}\, dx - \frac{\partial \psi}{\partial x}\, dy\right),$$

is single-valued[2] throughout the region R.

If the region R is simply connected, the only requirement that the integral of the form

$$\text{(47.5)} \qquad F(x, y) = \int_{P_0(x_0, y_0)}^{P(x, y)} [M(x, y)\, dx + N(x, y)\, dy]$$

define a single-valued function $F(x, y)$ is that $M(x, y)$ and $N(x, y)$ be of class C^1 in R and that throughout the region R

$$\text{(47.6)} \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

But if $F(x, y)$, defined by (47.5), is to be single-valued in a multiply connected domain, then in addition to (47.6) we must demand that the integrals

$$\int_{C_i} (M\, dx + N\, dy)$$

vanish when evaluated over each interior contour forming the boundary of R. Since ψ in (47.4) is a harmonic function, the condition (47.6) is,

[1] See remarks in the paragraph following Eq. (35.3).

[2] We recall that the displacement $w = \alpha\varphi(x, y)$, and w is a single-valued function.

clearly, satisfied and $\varphi(x, y)$ will be single-valued in R if, and only if,

$$\int_{C_i} \left(\frac{\partial \psi}{\partial y}\, dx - \frac{\partial \psi}{\partial x}\, dy\right) = 0, \qquad (i = 1, 2, \ldots, n). \tag{47.7}$$

Thus the constants k_i $(i = 1, 2, \ldots, n)$ in (47.3) must be chosen so that the solution of the Dirichlet problem

$$\begin{cases} \nabla^2\psi = 0 & \text{in } R, \\ \psi = \tfrac{1}{2}(x^2 + y^2) + k_i & \text{on } C_i \qquad (i = 0, 1, 2, \ldots, n) \end{cases} \tag{47.8}$$

satisfies the set of n conditions (47.7). The value of k_0, as we have already remarked, can be assigned arbitrarily.

If the problem is rephrased in terms of the Prandtl stress function

$$\Psi \equiv \psi(x, y) - \tfrac{1}{2}(x^2 + y^2),$$

the system (47.8) leads to the new system,

$$\begin{cases} \nabla^2\Psi = -2 & \text{in } R, \\ \Psi = k_i & \text{on } C_i, \qquad (i = 0, 1, 2, \ldots, n) \end{cases} \tag{47.9}$$

and the definition of Ψ yields,

$$\frac{\partial \psi}{\partial x} = \frac{\partial \Psi}{\partial x} + x, \qquad \frac{\partial \psi}{\partial y} = \frac{\partial \Psi}{\partial y} + y.$$

Accordingly, the set of conditions (47.7) becomes,

$$\int_{C_i} \left(\frac{\partial \Psi}{\partial y}\, dx - \frac{\partial \Psi}{\partial x}\, dy\right) + \int_{C_i} (y\, dx - x\, dy) = 0.$$

The second of the line integrals in this formula is numerically equal to twice the area A_i enclosed by C_i, and the first can be written as

$$\int_{C_i} \left(\frac{\partial \Psi}{\partial y}\frac{dx}{ds} - \frac{\partial \Psi}{\partial x}\frac{dy}{ds}\right) ds = \int_{C_i} \frac{d\Psi}{d\nu}\, ds.$$

Thus, the set of formulas (47.7) is equivalent to the set

$$\oint_{C_i} \frac{d\Psi}{d\nu}\, ds = -2A_i \qquad (i = 1, 2, \ldots, n). \tag{47.10}$$

The formula (34.10) for the calculation of the torsional rigidity D is still available, and we have

$$\begin{aligned} D &= \mu \iint_R \left(x^2 + y^2 + x\frac{\partial \varphi}{\partial y} - y\frac{\partial \varphi}{\partial x}\right) dx\, dy \\ &= \mu \iint_R -\left(x\frac{\partial \Psi}{\partial x} + y\frac{\partial \Psi}{\partial y}\right) dx\, dy, \end{aligned} \tag{47.11}$$

where we make use of the relations

$$\frac{\partial \varphi}{\partial x} = y + \frac{\partial \Psi}{\partial y} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = -\left(x + \frac{\partial \Psi}{\partial x}\right),$$

obtained in Sec. 35. The integration in (47.11) is performed over the multiply connected region R. The right-hand member of (47.11) can be rewritten as

$$\begin{aligned} D &= \mu \iint_R \left[2\Psi - \frac{\partial}{\partial x}(x\Psi) - \frac{\partial}{\partial y}(y\Psi)\right] dx\, dy \\ &= 2\mu \iint_R \Psi\, dx\, dy + \mu \int_C \Psi(y\, dx - x\, dy), \end{aligned}$$

where we make use of Green's Theorem, and the subscript C on the line integral means that the integration is to be performed in appropriate directions over all the contours C_i $(i = 0, 1, 2, \ldots, n)$. Now if we choose the value of Ψ over the contour C_0 to be zero (that is, $k_0 = 0$) and note the boundary conditions in (47.9), we have

$$D = 2\mu \iint_R \Psi\, dx\, dy + \mu \sum_{i=1}^{n} k_i \int_{C_i} (y\, dx - x\, dy)$$

But

$$\oint_{C_i} (y\, dx - x\, dy) = 2 \iint_{A_i} dx\, dy = 2A_i,$$

where A_i is the area enclosed by the contour C_i, and we have

$$D = 2\mu \iint_R \Psi\, dx\, dy + \sum_{i=1}^{n} 2\mu k_i A_i.$$

The expression for the twisting moment M is

$$M = 2\mu\alpha\left(\iint_R \Psi\, dx\, dy + \sum_{i=1}^{n} k_i A_i\right). \tag{47.12}$$

It will be recalled that the curves $\Psi(x, y) = \text{const}$ determine the lines of shearing stress (see Sec. 35), and it follows from the boundary conditions in (47.9) that one can obtain a solution for the torsion problem of a hollow shaft from the solution of the torsion problem of a solid shaft by deleting the portion of material contained within the curve $\Psi(x, y) = \text{const}$. Thus, in the discussion of the problem of torsion for an elliptic cylinder in Sec. 36, it was shown that the lines of shearing stress are similar ellipses, concentric with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

representing the cross section of the cylinder. Accordingly, if we delete

the portion of material contained within the elliptical cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = (1 - k)^2, \qquad (0 < k < 1), \tag{47.13}$$

then the stress function Ψ for an elliptical beam of semi-axes a and b will have a constant value over the curve (47.13) and the same function Ψ will thus solve the torsion problem for a hollow beam bounded by similar elliptical cylinders.

The lines of shearing stress for a beam of circular cross section are circles concentric with the outer boundary, and it follows at once that the formulas contained in Sec. 33 are applicable to hollow circular shafts. In particular, the torsional rigidity D is

$$D = \frac{\mu\pi}{2}(a^4 - a_0^4), \tag{47.14}$$

where a_0 is the radius of the inner circle and a is that of the outer one.

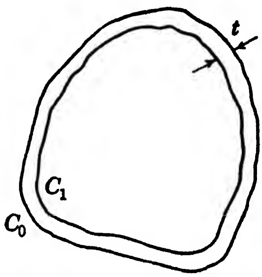

FIG. 36

Some important approximate formulas that are applicable to thin tubes follow readily. While it is not the purpose of this volume to deal with approximate engineering formulas, we make a brief reference to their development. Let a thin tubular section of thickness t be bounded by an exterior contour C_0 and an interior contour C_1 (Fig. 36).

If the tube is thin, we can assume that Ψ varies linearly along the thickness. Then

$$\iint_R \Psi \, dx \, dy \doteq \tfrac{1}{2}k_1 \iint_R dx \, dy = \tfrac{1}{2}k_1\bar{A},$$

if we take $\Psi = 0$ on C_0, $\Psi = k_1$ on C_1 and represent the cross-sectional area of the tube by $\bar{A}$. The shearing stress τ_s at any point in the cross section is,

$$\begin{aligned} \tau_s = \sqrt{\tau_{sx}^2 + \tau_{sy}^2} &= \mu\alpha\sqrt{\Psi_x^2 + \Psi_y^2} \\ &= \mu\alpha\frac{d\Psi}{d\nu}, \end{aligned}$$

and, in view of our assumption that Ψ varies linearly with thickness,

$$\left|\frac{d\Psi}{d\nu}\right| = \left|\frac{\Psi_0 - \Psi_1}{t}\right| = \frac{k_1}{t}.$$

Thus, approximately,

$$\tau_s = \frac{\mu\alpha k_1}{t}.$$

To determine k_1, we use (47.10), which yields

$$k_1 \int_{C_1} \frac{ds}{t} = 2A_1,$$

so that

$$k_1 = \frac{2A_1t}{l},$$

where l is the length of C_1. The twisting moment M is given by (47.12), and we find,

$$M = \frac{4\mu\alpha t}{l} A_1\left(\frac{1}{2}\bar{A} + A_1\right).$$

Since the tube is assumed to be thin, $\frac{1}{2}A \ll A_1$, so that

$$M \doteq \frac{4\mu\alpha t A_1^2}{l},$$

and

$$\tau_s = \mu\alpha\frac{k_1}{t} = \frac{2\mu\alpha A_1}{l}.$$

In his memoir on torsion, Saint-Venant conjectured that the torsional rigidity of a solid beam of given cross-sectional area increases as the moment of inertia of the cross section decreases. Since circular area has the least polar moment of inertia of all simply connected regions of given area, it seems plausible that for a given twisting moment M and cross-sectional area A_1 the smallest maximum stress will be found in a circular beam. The proof that the circular beam indeed has the greatest torsional rigidity of all solid beams of given cross-sectional area was supplied only recently by Polya.[1] The Saint-Venant conjecture has been generalized by Polya and Weinstein,[2] who proved that, of all multiply connected cross sections with given area and with given joint area of the holes, the ring bounded by two concentric circles has the maximum torsional rigidity.

Explicit solutions of the torsion problems for beams with multiply connected cross sections are not numerous. Greenhill[3] obtained by an indirect method a solution of the torsion problem for the hollow cylinder whose cross section is bounded by confocal ellipses, and Macdonald[4] used a similar technique to solve the problem for a hollow beam whose cross section is the region bounded by two eccentric circles. Weinel[5] reconsidered this problem with the aid of bipolar coordinates. A simpler solution, utilizing the mapping of an eccentric ring on a circular ring, was obtained by Vekua and Rukhadze.[6] Conformal mapping of the doubly

[1] G. Polya, *Quarterly of Applied Mathematics*, vol. 6 (1948), pp. 267–277.

[2] G. Polya and A. Weinstein, *Annals of Mathematics*, vol. 52 (1950), pp. 154–163.

[3] See Prob. 2 at the end of this section.

[4] H. M. Macdonald, *Proceedings of the Cambridge Philosophical Society*, vol. 8 (1893), pp. 62–68.

[5] E. Weinel, *Ingenieur Archiv*, vol. 3 (1932), pp. 67–75.

[6] I. N. Vekua and A. K. Rukhadze, *Izvestiya (Bulletin) Akademiya Nauk SSSR*, No. 3 (1933), pp. 167–178. These authors consider a more general problem of torsion of a

connected region on a circular ring was also used by Bartels[1] to give an independent treatment of this problem with the aid of an integral formula, similar to that of Villat,[2] for the solution of the Dirichlet problem in the annular ring.

An effective use of conformal mapping and singular integral equations was made by Sherman in deducing a number of useful approximate solutions of the torsion problem for beams weakened by longitudinal cavities. The available solutions include the following types of regions:

1. A region whose exterior boundary is a circle and whose interior boundary an ellipse with coincident center.[3]

2. A region bounded externally by a circle and internally by a square with rounded corners and with coincident centers.[4]

3. A triply connected region corresponding to the cross section of a circular beam weakened by two longitudinal circular cavities.[5] Some aspects of this problem have been considered previously[6] by Goluzin and Chih Bing Ling.

A method of solution of the Dirichlet problem for multiply connected domains in the series of orthogonal functions has been proposed by Bergman.[7]

A solution of the Saint-Venant torsion and flexure problems for a rectangular beam with rectangular longitudinal cavity was obtained by Abramyan.[8] Variational methods were used by Arutiunyan[9] to solve the torsion problem for the isotropic and orthotropic rods in the form of

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circular beam reinforced by an eccentric circular core made of different material. A summary of this paper is contained in N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity (1949), pp. 546–551.

[1] R. C. F. Bartels, Torsion of Hollow Cylinders, *Transactions of the American Mathematical Socieiy*, vol. 53 (1943), pp. 1–13.

[2] H. Villat, *Rendiconti del circolo matematico di Palermo*, vol. 33 (1912), p. 147.

[3] D. I. Sherman, *Doklady Akademiya Nauk SSSR*, New Series, vol. 69 (1948), pp. 499–502; D. I. Sherman and M. Z. Narodetsky, *Inzhenernyi Sbornik*, vol. 6 (1950); D. I. Sherman, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 17 (1953), pp. 470–476.

[4] D. I. Sherman, *Izvestiya (Bulletin) Akademiya Nauk SSSR*, Technical Series (1951), pp. 969–995.

[5] R. D. Stepanov and D. I. Sherman, *Inzhenernyi Sbornik*, vol. 11 (1952), pp. 127–150.

[6] G. M. Goluzin, *Matematicheski Sbornik*, vol. 41 (1934), No. 2; C. B. Ling, *Quarterly of Applied Mathematics*, vol. 5 (1947).

[7] S. Bergman, "The Kernel Function and Conformal Mapping," *American Mathematical Society Mathematical Surveys* 5 (1950).

[8] B. L. Abramyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 14 (1950), pp. 265–276. In this paper the problem is reduced to the solution of the system of linear ordinary differential equations of the second order with constant coefficients, and the solution is given in the form of infinite series.

[9] N. Kh. Arutiunyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 11 (1947), pp. 543–546.

an elliptical sector,[1] *or in the shape of an elliptical ring bounded by two similar ellipses slit from their common center along the radii. It is interesting to note that the torsional rigidity of an isotropic elliptical tube whose thickness is two-tenths of the major axis, when slit along the major axis, is approximately one-seventieth that of the solid tube.*

Since solutions of torsion problems for beams with multiply connected cross sections are not easy to obtain, considerable attention has been devoted to the problem of obtaining bounds for the torsional rigidity without first calculating the torsion function. The estimates of such bounds are based on a study of the properties of Dirichlet's integrals in the calculus of variations. Useful bounds have been obtained by Topolyanski, Weinstein, Diaz, and others.[2] Weinberger[3] recently computed bounds for the torsional rigidity of circular, triangular, hexagonal, and square beams with circular, rectangular, and triangular longitudinal cavities. The numerical values of torsional rigidities for various solid beams are recorded in a monograph by Polya and Szego.[4]

The application of the membrane analogy to obtain experimental solutions of the torsion problem for slit tubes and hollow beams is described by Timoshenko[5] and Biezeno and Grammel.[6] This method was applied to the torsion problem of an eccentric circular annulus by Engelmann.[7] Neményi[8] has discussed the use of numerical and experimental (membrane) methods in the torsion problem for beams of multiply connected cross section.

PROBLEMS

1. Find the conjugate torsion functions φ, ψ, the stress function Ψ, and the constant k entering into (47.9) for a hollow circular shaft. Derive the expression (47.14) for the torsional rigidity of the shaft from both (34.10) and (47.12).

[1] A solution of Saint-Venant's torsion problem for the circular sector is recorded on pp. 278–279 of S. Timoshenko and J. N. Goodier's Theory of Elasticity.

[2] D. B. Topolyanski, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 11 (1947), pp. 551–554.

A. Weinstein, *American Mathematical Society Proceedings of Symposia in Applied Mathematics*, vol. 3 (1950), pp. 141–161.

J. B Diaz, *Seminario Matematico de Barcelona Collectaneia Mathematica*, vol. 4 (1951), pp. 1–50.

[3] H. F. Weinberger, *Journal of Mathematics and Physics*, vol. 32 (1953), pp. 54–62.

[4] G. Polya and G. Szego, Isoperimetric Inequalities in Mathematical Physics (1950).

[5] S. Timoshenko and J. N. Goodier, Theory of Elasticity, Sec. 101. See also, W. Nowacki, *Arch. Mech. Stos.*, vol. 5 (1953), pp. 21–24 (in Polish).

[6] Technische Dynamik, Chap. III, Sec. 26, pp. 199–201.

[7] Fritz Engelmann, "Verdrehung von Stäben mit Einseitig-ring-förmigem Querschnitt," *Forschung auf dem Gebiete des Ingenieurwesens*, vol. 6 (1935), pp. 146–154.

[8] P. Neményi, "Lösung des Torsionsproblems für Stäbe mit mehrfach zusammenhängendem Quershnitt," *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 1 (1921), pp. 364–367.

2. If $z = a \cosh \zeta$, where $z = x + iy$ and $\zeta = \xi + i\eta$, show that the family of curves ξ = const defines a set of confocal ellipses. Hence verify that the function

$$\psi = \frac{a^2}{4} \frac{\sinh 2(\xi_0 - \xi) \sinh 2(\xi - \xi_1)}{\sinh 2(\xi_0 - \xi_1)} \cos 2\eta,$$

solves the torsion problem for a tube whose cross section is bounded by confocal ellipses $\xi = \xi_0$ and $\xi = \xi_1$. This result was deduced by A. G. Greenhill, *Quarterly Journal of Mathematics*, Oxford Series, vol. 16 (1879), pp. 227–256.

3. Compute the torsional rigidity of a hollow shaft whose cross section is bounded by two similar ellipses.

4. Use Green's formula to show that

$$\begin{aligned} -\iint_R (x\varphi_y - y\varphi_x)\,dx\,dy &= \int_C \varphi \left(\frac{d\varphi}{d\nu}\right) ds \\ &= \iint_R (\varphi_x^2 + \varphi_y^2)\,dx\,dy, \end{aligned}$$

if φ is the torsion function of Sec. 34. Hence show that formula (34.10) can be written as

$$D = \mu \iint_R (x^2 + y^2)\,dx\,dy - \mu \iint_R (\varphi_x^2 + \varphi_y^2)\,dx\,dy.$$

Conclude from this that $D \leq \mu \iint_R (x^2 + y^2)\,dx\,dy$; that is, the torsional rigidity of R is never greater than μI_0, where I_0 is the polar moment of inertia of R.

5. Use results in the preceding problem and the fact that the Dirichlet integral $\iint_R (\varphi_x^2 + \varphi_y^2)\,dx\,dy$ vanishes if, and only if, φ is a constant, to prove that $D = \mu I_0$ only when R is a circle or a concentric circular ring.

48. Curvilinear Coordinates. The possibility of obtaining a simple solution of a given boundary-value problem often crucially depends upon the choice of that coordinate system in which the boundary conditions assume a simple form. In dealing with axially symmetric bodies, for example, it is usually advisable to phrase the problem in spherical or cylindrical coordinates. In some problems the shape of the boundary may suggest the use of ellipsoidal coordinates, in others toroidal coordinates are indicated, and so on. The object of this section is to deduce expressions for the components of stress and strain tensors and to record the field equations of linear elasticity in orthogonal curvilinear coordinates.[1]

Consider a set of three independent functions of the cartesian variables x, y, z,

$$\alpha_i = \alpha_i(x, y, z), \qquad (i = 1, 2, 3). \tag{48.1}$$

[1] The calculations in this section are far less general and more laborious than they would have been if the apparatus of tensor calculus were at our disposal. A concise general tensorial derivation of the basic equations of linear and nonlinear mechanics of continuous media is contained in I. S. Sokolnikoff's Tensor Analysis, pp. 290–319

The intersections of the surfaces

$$\alpha_i(x, y, z) = \text{const}, \qquad (i = 1, 2, 3)$$

pair by pair determine the coordinate lines of our curvilinear coordinate system, and the intersection of coordinate lines determines a point that will be labeled $(\alpha_1, \alpha_2, \alpha_3)$. It is assumed that the coordinate system (α_i) is orthogonal, so that the element of arc ds has the form

$$ds^2 = \sum_{i=1}^{3} g_{ii}\, d\alpha_i^2, \tag{48.2}$$

where g_{ii} are the metric coefficients that can be calculated[1] from (48.1).

Let P_0 and P be two neighboring points in an unstrained medium, and let these points take the positions P_0' and P' after deformation. We shall confine our discussion to infinitesimal deformations[2] and shall represent the displacements in the directions normal to the coordinate surfaces $\alpha_1, \alpha_2, \alpha_3$ by u_1, u_2, u_3, respectively. The curvilinear coordinates of the points P_0 and P are α_i and $\alpha_i + d\alpha_i$, respectively. The coordinates of the points P_0' and P' will be denoted by $\alpha_i + \xi_i$ and $\alpha_i + \xi_i + d\alpha_i + d\xi_i$. Then it follows from (48.2) that

$$u_1 = \sqrt{g_{11}}\, \xi_1, \qquad u_2 = \sqrt{g_{22}}\, \xi_2, \qquad u_3 = \sqrt{g_{33}}\, \xi_3.$$

Now the length of the element of arc ds joining the points P_0 and P is given by

$$ds^2 = \sum_{i=1}^{3} g_{ii}(\alpha_1, \alpha_2, \alpha_3)\, d\alpha_i^2, \tag{48.3}$$

while the length of the same element in the deformed state is given by

$$(ds')^2 = \sum_{i=1}^{3} g_{ii}(\alpha_1 + \xi_1, \alpha_2 + \xi_2, \alpha_3 + \xi_3)(d\alpha_i + d\xi_i)^2. \tag{48.4}$$

But

$$g_{ii}(\alpha_1 + \xi_1, \alpha_2 + \xi_2, \alpha_3 + \xi_3) = g_{ii}(\alpha_1, \alpha_2, \alpha_3) + \sum_{j=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_j} \xi_j$$

to the order of approximation contemplated by the linear theory, and

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2\, d\alpha_i\, d\xi_i + d\xi_i^2 \doteq (d\alpha_i)^2 + 2 \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j\, d\alpha_i.$$

[1] See Prob. 1 at the end of this section. All summations in the main parts of this section will be indicated by a summation sign.

[2] See Sec 7

Hence (48.4) can be written as

$$(48.5) \qquad (ds')^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} G_{ij}\, d\alpha_i\, d\alpha_j,$$

where

$$(48.6) \qquad G_{ij} = \delta_{ij}\left(g_{ii} + \sum_{k=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right) + g_{ii}\frac{\partial \xi_i}{\partial \alpha_j} + g_{jj}\frac{\partial \xi_j}{\partial \alpha_i},$$

and where we neglect the terms involving the products of ξ_j and $\frac{\partial \xi_i}{\partial \alpha_j}$. The symbol δ_{ij}, as usual, denotes the Kronecker delta. The expression for G_{ij} has been symmetrized by replacing G_{ij} by $\frac{1}{2}(G_{ij} + G_{ji})$.

It is clear from (48.2) and (48.5) that the elongations of linear elements and shears are characterized by the coefficients g_{ii} and G_{ij}. Thus, consider a linear element ds_i directed along one of the coordinate lines α_i. From (48.2), its length is

$$ds_i = \sqrt{g_{ii}}\, d\alpha_i,$$

while the length of the same element after deformation is

$$ds_i' = \sqrt{G_{ii}}\, d\alpha_i.$$

Accordingly, the extension e_{ii} of this element is

$$\begin{aligned} e_{ii} &= \frac{\sqrt{G_{ii}}\, d\alpha_i - \sqrt{g_{ii}}\, d\alpha_i}{\sqrt{g_{ii}}\, d\alpha_i} \\ &= \sqrt{1 + \frac{G_{ii} - g_{ii}}{g_{ii}}} - 1 \doteq \frac{1}{2}\frac{G_{ii} - g_{ii}}{g_{ii}}, \end{aligned}$$

if we neglect the nonlinear terms in the ξ_i and their derivatives. Noting the definitions (48.6), we have

$$(48.7) \qquad \begin{aligned} e_{ii} &= \frac{1}{2}\frac{G_{ii} - g_{ii}}{g_{ii}} = \frac{\partial \xi_i}{\partial \alpha_i} + \frac{1}{2g_{ii}}\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k \\ &= \frac{\partial}{\partial \alpha_i}\frac{u_i}{\sqrt{g_{ii}}} + \frac{1}{2g_{ii}}\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\frac{u_k}{\sqrt{g_{kk}}}. \end{aligned}$$

The cosine of the angle θ_{ij} between the directions of linear elements in the deformed state that were originally directed parallel to the coordinate lines α_i and α_j is given by[1]

$$(48.8) \qquad \cos\theta_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}}.$$

[1] See Prob. 3 at the end of this section.

Just as in Sec. 4, we define the angle α_{ij} by the formula

$$\theta_{ij} = \frac{\pi}{2} - \alpha_{ij};$$

then

$$\cos \theta_{ij} = \sin \alpha_{ij} \doteq \alpha_{ij}.$$

The shear components of the strain tensor are defined by the relation $\alpha_{ij} = 2e_{ij}$. Substituting in (48.8), we get[1]

$$e_{ij} = \frac{1}{2} \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \doteq \frac{1}{2} \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}}.$$

Finally, substituting in the foregoing formula the definitions (48.6), we obtain the shear components of the strain tensor in the form

$$(48.9) \quad \begin{aligned} e_{ij} &= \frac{1}{2\sqrt{g_{ii}g_{jj}}} \left(g_{ii} \frac{\partial \xi_i}{\partial \alpha_j} + g_{jj} \frac{\partial \xi_j}{\partial \alpha_i} \right) \\ &= \frac{1}{2\sqrt{g_{ii}g_{jj}}} \left[g_{ii} \frac{\partial}{\partial \alpha_j} \left(\frac{u_i}{\sqrt{g_{ii}}} \right) + g_{jj} \frac{\partial}{\partial \alpha_i} \left(\frac{u_j}{\sqrt{g_{jj}}} \right) \right], \qquad \text{if } i \neq j. \end{aligned}$$

The components τ_{ij} of the stress tensor in curvilinear coordinates are defined in precisely the same way as they were in the cartesian system. Thus, the component of stress normal to the element of area perpendicular to the coordinate line α_i is denoted by τ_{ii}, and the component of shear associated with the coordinate lines α_i and α_j is written as τ_{ij}.

In this notation,[2] Hooke's law for a homogeneous isotropic medium assumes the form

$$(48.10) \quad \begin{cases} \tau_{ii} = \lambda\vartheta + 2\mu e_{ii} \text{ or } \tau_{ii} = \dfrac{E\sigma}{(1+\sigma)(1-2\sigma)}\vartheta + \dfrac{E}{1+\sigma} e_{ii} \\ \tau_{ij} = 2\mu e_{ij} \qquad \text{or} \qquad \tau_{ij} = \dfrac{E}{1+\sigma} e_{ij}, \qquad \text{if } i \neq j, \end{cases}$$

where the invariant $\vartheta \equiv e_{11} + e_{22} + e_{33}$. Solving the system (48.10) for

[1] Note that

$$\begin{aligned} G_{ii}G_{jj} &= \left(g_{ii} + \sum_{k=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_k} \xi_k + 2g_{ii} \frac{\partial \xi_i}{\partial \alpha_i} \right) \left(g_{jj} + \sum_{k=1}^{3} \frac{\partial g_{jj}}{\partial \alpha_k} \xi_k + 2g_{jj} \frac{\partial \xi_j}{\partial \alpha_j} \right) \\ &= g_{ii}g_{jj} + \sum_{k=1}^{3} \left(\frac{\partial g_{jj}}{\partial \alpha_k} g_{ii} + \frac{\partial g_{ii}}{\partial \alpha_k} g_{jj} \right) \xi_k + 2g_{ii}g_{jj} \left(\frac{\partial \xi_i}{\partial \alpha_i} + \frac{\partial \xi_j}{\partial \alpha_j} \right) \end{aligned}$$

+ terms involving products of ξ_k and its derivatives, terms that were neglected previously

[2] Cf. Secs 22 and 23

the components of strain yields

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta, \tag{48.11}$$

where the invariant $\Theta = \tau_{11} + \tau_{22} + \tau_{33}$.

A somewhat lengthy calculation, although in essence similar to that outlined in Sec. 15, leads to the following equations of equilibrium in curvilinear coordinates:

$$\frac{\partial(g\tau_{ii})}{\partial\alpha_i} - \frac{1}{2}\sum_{j=1}^{3}\frac{g\tau_{jj}}{g_{jj}}\frac{\partial g_{jj}}{\partial\alpha_i} + \sum_{j=1}^{3}\frac{\partial}{\partial\alpha_j}\left(\frac{gg_{ii}\tau_{ij}}{\sqrt{g_{ii}g_{jj}}}\right) + F_i g\sqrt{g_{ii}} = 0, \qquad (i = 1, 2, 3), \tag{48.12}$$

where $g \equiv \sqrt{g_{11}g_{22}g_{33}}$, and the F_i are the components, in the directions of the coordinate axes, of the body force $\mathbf{F}$.

A complete set of the field equations of linear theory of elasticity, valid in all coordinate systems, is recorded here for the benefit of readers familiar with tensor calculus. In these formulas a comma followed by the subscripts $i, j, \ldots$ denotes the covariant derivatives with respect to the variables $x_i, x_j, \ldots$, and a repeated index is summed from 1 to 3. The g_{ij} and g^{ij} are, respectively, the components of covariant and contravariant metric tensors. The meaning of all other symbols is identical with that used previously.

a. Hooke's Law

$$\tau_{ij} = \lambda\vartheta g_{ij} + 2\mu e_{ij}, \qquad \vartheta = g^{ij}e_{ij},$$
$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

b. Equilibrium Equations

$$g^{jk}\tau_{ij,k} + F_i = 0, \qquad \text{in } \tau$$
$$\tau_{ij}\nu^j = T_i, \qquad \text{on } \Sigma.$$

c. Navier's Equations

$$(\lambda + \mu)\vartheta_{,i} + \mu\nabla^2 u_i + F_i = 0, \qquad \text{in } \tau,$$

where $\nabla^2 u_i = g^{jk}u_{i,jk}$.

d. Compatibility Equations

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$$

We write out the expressions for the strain components (48.7) and (48.9) and the equations of equilibrium (48.12) for three important special cases of curvilinear coordinates.

a. Plane Polar Coordinates. In this case, the index i assumes the values 1, 2, and according to the usual notation

$$\alpha_1 = r, \qquad \alpha_2 = \theta.$$

The coordinate surfaces in this case are circular cylinders perpendicular to the xy-plane (r = const) and radial planes through the origin (θ = const). The element of arc is given by

$$ds^2 = dr^2 + r^2\,d\theta^2,$$

so that

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = r^2.$$

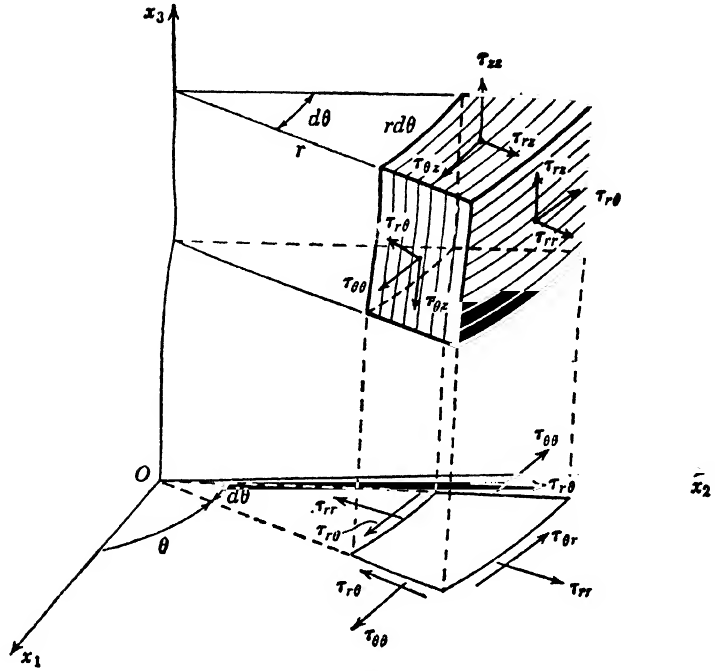

FIG 37

Noting the formulas (48.7) and (48.9), we see that the strain components in this case are

$$
\left\{
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\
e_{r\theta} &= \frac{1}{2r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta + r\frac{\partial u_\theta}{\partial r}\right),
\end{aligned}
\right.
\tag{48.13}
$$

while the equations of equilibrium (48.12) become

$$
\left\{
\begin{aligned}
&\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r = 0, \\
&\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0.
\end{aligned}
\right.
\tag{48.14}
$$

b. Cylindrical Coordinates. The variables involved here are

$$\alpha_1 = r, \qquad \alpha_2 = \theta, \qquad \alpha_3 = z,$$

and the element of arc in cylindrical coordinates is given by

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2,$$

so that

$$g_{11} = 1, \qquad g_{22} = r^2, \qquad g_{33} = 1.$$

The surfaces r = const and θ = const are circular cylinders and radial planes as in case a above, while the surfaces z = const are planes parallel

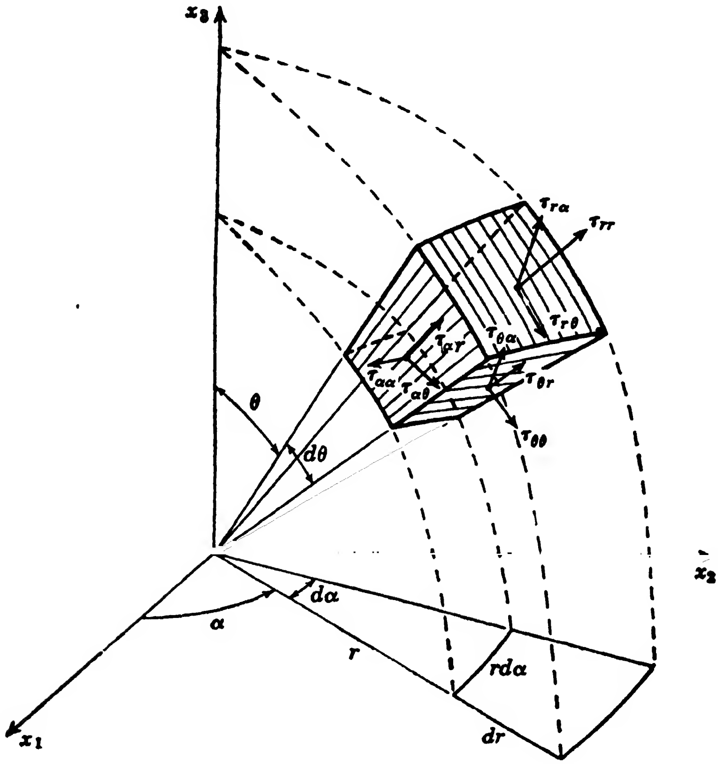

FIG. 38

to the xy-plane (Fig. 37). Substituting the values of the metric coefficients in (48.7), (48.9), and (48.12) gives the expressions for the strain components

$$
(48.15)\qquad
\left\{
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\
e_{zz} &= \frac{\partial u_z}{\partial z}, \\
e_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\
e_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \\
e_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right),
\end{aligned}
\right.
$$

and the equations of equilibrium

$$
(48.16)\qquad \begin{cases}
\dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{\partial \tau_{rz}}{\partial z} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r = 0, \\
\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + \dfrac{\partial \tau_{\theta z}}{\partial z} + \dfrac{2}{r}\tau_{r\theta} + F_\theta = 0, \\
\dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \tau_{zz}}{\partial z} + \dfrac{1}{r}\tau_{rz} + F_z = 0.
\end{cases}
$$

c. Spherical Coordinates (Fig. 38). For this coordinate system, we have $\alpha_1 = r$, $\alpha_2 = \alpha$, $\alpha_3 = \theta$. The coordinate surfaces are the spheres $r =$ const, the radial planes perpendicular to the xy-plane, $\alpha =$ const, and the right-circular cones with vertices at the origin, $\theta =$ const. Since the element of arc is given by

$$ds^2 = dr^2 + r^2 \sin^2 \theta\, d\alpha^2 + r^2\, d\theta^2,$$

we have

$$g_{11} = 1, \qquad g_{22} = r^2 \sin^2 \theta, \qquad g_{33} = r^2.$$

The strain components, in this case, are

$$
(48.17)\qquad \begin{cases}
e_{rr} = \dfrac{\partial u_r}{\partial r}, \\
e_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r}, \\
e_{\alpha\alpha} = \dfrac{1}{r \sin \theta}\dfrac{\partial u_\alpha}{\partial \alpha} + \dfrac{u_r}{r} + u_\theta \dfrac{\cot \theta}{r}, \\
e_{r\alpha} = \dfrac{1}{2}\left(\dfrac{1}{r \sin \theta}\dfrac{\partial u_r}{\partial \alpha} - \dfrac{u_\alpha}{r} + \dfrac{\partial u_\alpha}{\partial r}\right), \\
e_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}\right), \\
e_{\alpha\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_\alpha}{\partial \theta} - \dfrac{u_\alpha \cot \theta}{r} + \dfrac{1}{r \sin \theta}\dfrac{\partial u_\theta}{\partial \alpha}\right),
\end{cases}
$$

and the equations of equilibrium are

$$
(48.18)\qquad \begin{cases}
\dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r \sin \theta}\dfrac{\partial \tau_{r\alpha}}{\partial \alpha} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{2\tau_{rr} - \tau_{\alpha\alpha} - \tau_{\theta\theta} + \tau_{r\theta}\cot \theta}{r} + F_r = 0, \\
\dfrac{\partial \tau_{r\alpha}}{\partial r} + \dfrac{1}{r \sin \theta}\dfrac{\partial \tau_{\alpha\alpha}}{\partial \alpha} + \dfrac{1}{r}\dfrac{\partial \tau_{\alpha\theta}}{\partial \theta} + \dfrac{3\tau_{r\alpha} + 2\tau_{\alpha\theta}\cot \theta}{r} + F_\alpha = 0, \\
\dfrac{\partial \tau_{r\alpha}}{\partial r} + \dfrac{1}{r \sin \theta}\dfrac{\partial \tau_{\alpha\theta}}{\partial \alpha} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + \dfrac{3\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\alpha\alpha})\cot \theta}{r} + F_\theta = 0.
\end{cases}
$$

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PROBLEMS

1. Show that the metric coefficients g_{ij} can be calculated by observing that

$$dx_k = \sum_{i=1}^{3} \frac{\partial x_k}{\partial \alpha_i}\, d\alpha_i, \qquad (dx_k)^2 = \sum_{i,j=1}^{3} \frac{\partial x_k}{\partial \alpha_i}\frac{\partial x_k}{\partial \alpha_j}\, d\alpha_i\, d\alpha_j.$$

It follows that

$$ds^2 = \sum_{k=1}^{3} (dx_k)^2 = \sum_{k,i,j=1}^{3} \frac{\partial x_k}{\partial \alpha_i}\frac{\partial x_k}{\partial \alpha_j}\, d\alpha_i\, d\alpha_j,$$

or

$$ds^2 = \sum_{i,j=1}^{3} g_{ij}\, d\alpha_i\, d\alpha_j,$$

where

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial \alpha_i}\frac{\partial x_k}{\partial \alpha_j}.$$

2. Calculate the metric coefficients g_{ij} for plane polar coordinates $\alpha_1 \equiv r$, $\alpha_2 \equiv \theta$ from the relations

$$x_1 = \alpha_1 \cos \alpha_2, \qquad x_2 = \alpha_1 \sin \alpha_2$$

and

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial \alpha_i}\frac{\partial x_k}{\partial \alpha_j}.$$

3. Consider a curvilinear triangle in the undeformed state, with sides directed parallel to the coordinate lines α_1 and α_2. The increments in the coordinates α_i along the sides can be written as $(d\alpha_1, 0, 0)$ and $(0, d\alpha_2, 0)$, while the "hypotenuse" corresponds to coordinate changes $(-d\alpha_1, d\alpha_2, 0)$. Show from (48.5) that, after defor-

mation, the sides have lengths $\sqrt{G_{11}}\,d\alpha_1$, $\sqrt{G_{22}}\,d\alpha_2$ and include an angle θ_{12}, while the length of the hypotenuse is $\sqrt{G_{11}\,d\alpha_1^2 + G_{22}\,d\alpha_2^2 - 2G_{12}\,d\alpha_1\,d\alpha_2}$. Use the law of cosines to show that

$$\cos\theta_{12} = \frac{G_{12}}{\sqrt{G_{11}G_{22}}},$$

49. Torsion of Shafts of Varying Circular Cross Section. In discussing the torsion by terminal couples of a circular shaft of varying diameter, it is convenient to make use of the cylindrical coordinates r, θ, z introduced in the preceding section. We shall direct the axis of the shaft along the z-axis, and, in order to avoid using subscripts, we shall denote the displacements u_r and u_θ, in the radial and tangential directions, by u and v, respectively. The displacement u_z in the direction of the axis of the shaft will be called w.

It will be recalled that, in the case of a uniform circular shaft twisted by terminal couples, the displacement of points in any cross section is in the tangential direction[1] and that the displacement in the direction of the axis of the shaft vanishes. We shall attempt to solve the torsion problem for a shaft of varying diameter by assuming that, in this case, we also have

$$u = w = 0,$$

and then prove that the solution based on this assumption fulfills all conditions of the problem and hence is the desired one.

On account of the circular symmetry, the tangential displacement v cannot depend on the angle θ and thus will be a function of the variables r and z.

Since the displacements u and w vanish, the formulas (48.15) (with $u_r \equiv u$, $u_\theta \equiv v$, $u_z \equiv w$) give

$$(49.1) \qquad e_{rr} = e_{\theta\theta} = e_{zz} = e_{rz} = 0, \qquad e_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right), \qquad e_{\theta z} = \frac{1}{2}\frac{\partial v}{\partial z},$$

and it follows from (48.10) that the corresponding stresses are:

$$(49.2) \qquad \tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{rz} = 0, \qquad \tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right), \qquad \tau_{\theta z} = \mu\frac{\partial v}{\partial z}.$$

Inserting these expressions in the three equilibrium equations (48.16) shows that two of them are satisfied identically, and the remaining one requires that

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

This equation can be rewritten in the form

$$\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right] + \frac{\partial}{\partial z}\left[r^3\frac{\partial}{\partial z}\left(\frac{v}{r}\right)\right] = 0,$$

[1] See Fig. 20.

and it follows that there exists a function $F(r, z)$ such that

$$\frac{\partial F}{\partial r} = r^3 \frac{\partial}{\partial z}\left(\frac{v}{r}\right), \qquad \frac{\partial F}{\partial z} = -r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right), \tag{49.3}$$

so that

$$\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = \frac{1}{r^3}\frac{\partial F}{\partial r}, \qquad \text{and} \qquad -\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = \frac{1}{r^3}\frac{\partial F}{\partial z}.$$

Differentiating the first of these equations with respect to r, the second with respect to z, and adding gives the equation on the function $F(r, z)$ in the form

$$\frac{\partial^2 F}{\partial r^2} - \frac{3}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0. \tag{49.4}$$

The stress components will now be expressed in terms of the function $F(r, z)$. It follows from formulas (49.3) that

$$\frac{1}{r^2}\frac{\partial F}{\partial r} = r\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = \frac{\partial v}{\partial z},$$

$$-\frac{1}{r^2}\frac{\partial F}{\partial z} = r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = \frac{\partial v}{\partial r} - \frac{v}{r},$$

and comparison of these expressions with the last two of the formulas (49.2) shows that the nonvanishing components of stress are given in terms of the function F by the formulas

$$\tau_{\theta z} = \frac{\mu}{r^2}\frac{\partial F}{\partial r}, \qquad \tau_{r\theta} = -\frac{\mu}{r^2}\frac{\partial F}{\partial z}. \tag{49.5}$$

Since the lateral surface of the shaft is free from external loads, it follows that the resultant shearing stress must be directed along the tangent to the boundary of the axial section. Accordingly, the component of the resultant stress in the direction ν normal to this boundary must vanish, and we have the boundary condition[1]

$$\tau_{\theta z}\cos(z, \nu) + \tau_{r\theta}\cos(r, \nu) = 0.$$

But $\cos(z, \nu) = -\dfrac{dr}{ds}$, $\cos(r, \nu) = \dfrac{dz}{ds}$, where ds is the element of arc along the boundary of the axial section (Fig. 39), and we have

$$-\tau_{\theta z}\frac{dr}{ds} + \tau_{r\theta}\frac{dz}{ds} = 0 \qquad \text{on the boundary.}$$

Substituting in this expression from (49.5), we get

$$\frac{\partial F}{\partial r}\frac{dr}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0,$$

[1] If $\tau_{\theta z}\cos(z, \nu) + \tau_{r\theta}\cos(r, \nu) = T(s)$, the calculations yield $dF/ds = -T(s)$. The theory outlined here is due to J. H. Michell, *Proceedings of the London Mathematical Society*, vol. 31 (1900), p. 140.

or

$$\frac{dF}{ds} = 0 \qquad \text{on the boundary.}$$

Thus, the condition that the lateral surface be free from external loads demands that the function $F(r, z)$ assume a constant value on the boundary of the axial section, determined by θ = const.

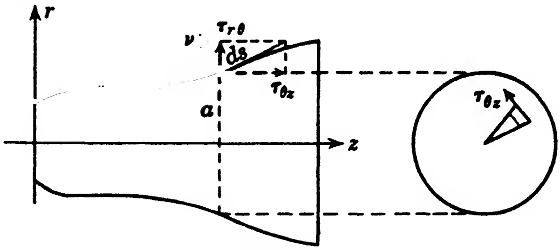

FIG. 39

The twisting moment on any cross section whose radius is a is easily computed. Thus,

$$(49.6) \quad M = \int_0^{2\pi}\int_0^a \tau_{\theta z} r^2\, dr\, d\theta = 2\pi \int_0^a r^2 \tau_{\theta z}\, dr = 2\pi\mu \int_0^a \frac{\partial F}{\partial r}\, dr$$
$$= 2\pi\mu[F(a, z) - F(0, z)].$$

The solution of Eq. (49.4) is quite simple for the case of a conical shaft, shown in Fig. 40. It is easily checked that the function

$$(49.7) \qquad F(r, z) = c\left\{\frac{z}{(r^2 + z^2)^{1/2}} - \frac{1}{3}\left[\frac{z}{(r^2 + z^2)^{1/2}}\right]^3\right\},$$

where c is a constant, satisfies Eq. (49.4). Moreover, the expression $z/(r^2 + z^2)^{1/2}$ is constant on the lateral surface, since it is equal to the cosine of one-half the vertical angle of the cone, and hence the function $F(r, z)$ in (49.7) assumes a constant value on the lateral surface of the cone, that is, for $z = r \cot \alpha$.

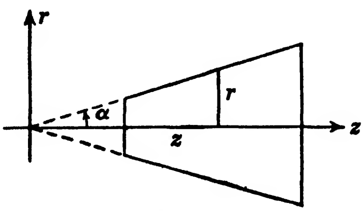

FIG 40

The magnitude of the shearing stresses $\tau_{\theta z}$ and $\tau_{r\theta}$ is given at once by the formulas (49.5), and a simple calculation shows that

$$\tau_{\theta z} = -\frac{\mu c r z}{(r^2 + z^2)^{5/2}}, \qquad \tau_{r\theta} = -\frac{\mu c r^2}{(r^2 + z^2)^{5/2}}.$$

The value of the constant c can be determined from (49.6) when the twisting couple in the terminal section is known.

Indeed, from (49.6),

$$M = 2\pi\mu[F(a, z) - F(0, z)]$$
$$= 2\pi c\mu(\cos \alpha - \tfrac{1}{3} \cos^3 \alpha - \tfrac{2}{3}),$$

since $\cos\alpha = z/(r^2 + z^2)^{1/2}$, and hence

$$c = \frac{M}{2\pi\mu(\cos\alpha - \frac{1}{3}\cos^3\alpha - \frac{2}{3})}$$

We can also show that the maximum value of the shearing stress τ_s occurs at the narrow end of the shaft. For the principal stresses τ_i are,

$$\tau_1 = (\tau_{\theta z}^2 + \tau_{r\theta}^2)^{1/2}, \qquad \tau_2 = -(\tau_{\theta z}^2 + \tau_{r\theta}^2)^{1/2}, \qquad \tau_3 = 0,$$

and

$$(\tau_s)_{\max} = \tfrac{1}{2}(\tau_1 - \tau_2) = (\tau_{\theta z}^2 + \tau_{r\theta}^2)^{1/2} = \tau_1.$$

Reference to formulas for $\tau_{\theta z}$ and $\tau_{r\theta}$ shows that τ_1 has the maximum value at the narrow end.

Since

$$e_{\theta z} = \frac{1}{2\mu}\tau_{\theta z} = \frac{1}{2}\frac{\partial v}{\partial z},$$

$$e_{r\theta} = \frac{1}{2\mu}\tau_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right),$$

we readily find by simple integration that

$$v = \frac{M}{6\pi\mu k}\frac{r}{(z^2 + r^2)^{3/2}} + \omega r,$$

where $k = \cos\alpha - \frac{1}{3}\cos^3\alpha - \frac{2}{3}$ and ω = const representing a rigid rotation of the cone about the z-axis. The constant ω can be determined if we suppose, for example, that the circumference of the shaft is fixed at the wide end.

The applications of the theory sketched above are not numerous.[1] The deformation of the segment of a torus by shearing stresses distributed over its plane ends, so that they produce torsion and give a resultant force in the direction of the axis of the torus, was investigated by a method of successive approximations by Göhner.[2] An exact solution of this problem, in bipolar coordinates, was obtained by Freiberger.[3]

[1] For reviews of the literature see:

T. Poschl, *Zeitschrift für angewandte Mathematik und Mechanik.*, vol. 2 (1921), p. 137;

T. J. Higgins, *Experimental Stress Analysis*, vol. 3 (1945), p. 94.

[2] O. Göhner, *Ingenieur Archiv*, vol. 1 (1930), p. 619. A detailed account of Göhner's work is contained in Timoshenko and Goodier's Theory of Elasticity, pp. 391–395.

A review of the history of this problem was given by R. V. Southwell, *Proceedings of the Royal Society* (*London*) (*A*), vol. 180 (1942), pp. 367–396. In this paper Southwell uses the semi-inverse method to discuss the torsion of a shaft of varying circular cross section, the torsion and flexure of an incomplete tore, the shearing stresses in a toroidal book, and symmetrical strains in a solid of revolution.

[3] W. Freiberger, *Australian Journal of Scientific Research*, ser. A, vol. 2 (1949), pp. 354–375. See also related papers on torsion and stretching of spiral rods by H. Okubo, *Quarterly of Applied Mathematics*, vol. 9 (1951), pp. 263–272, vol. 11 (1954), pp. 499–501; *Journal of Applied Mechanics* Paper 53-APM-2, pp. 1–6.

Torsion of shafts of variable cross section was recently studied by Poritsky, Shapiro, and Wilhoit.[1]

The Saint-Venant torsion problem for a circular cylinder with symmetrically located spherical cavity was treated by Ling,[2] and axially symmetric shafts with cracks were considered briefly by Weinstein and Payne.[3]

We shall make use of some particular solutions of Eq. (49.1) in the next section, which is concerned with a study of local effects near the ends of a twisted circular cylinder in which the distribution of stress at one end differs from that demanded by Saint-Venant's theory.

50. Local Effects. It has already been noted that Saint-Venant's theory of torsion, discussed in Secs. 33 and 34, imposes a requirement that the distribution of stress over the ends of the cylinder be the same as in every other cross section of the cylinder. The theory developed in those sections yields stresses that at the end sections are statically equivalent to the applied twisting couples. The distribution of these end stresses cannot be arbitrarily specified, since the end couples must be applied in a way demanded by the solution of the torsion problem for the particular section under discussion. If the distribution of stresses over the ends of the cylinder differs from that demanded by the theory, there will be some local irregularities in the neighborhood of the ends and it is to be expected from Saint-Venant's principle that the effect of local perturbations will not be felt far from the ends. We proceed to investigate the character of local disturbances in a long circular cylinder of radius a that is twisted by some prescribed distribution of stresses $\tau_{\theta z}$ over the end $z = 0$.

We note first that a particular integral $F = Ar^4$ of Eq. (49.4), as is clear from (49.5), yields

$$\tau_{\theta z} = 4\mu Ar, \qquad \tau_{r\theta} = 0,$$

which become identical with the stress system (33.2) in a circular beam twisted by the end couples, if we set[4] $4A = \alpha$.

A set of particular integrals of Eq. (49.4) can be obtained by assuming solutions in the form $e^{-kz}R(r)$, where $k > 0$ and $R(r)$ is a function of r alone, and it follows from (49.4) that the function $R(r)$ must satisfy the equation

[1] H. Poritsky, *American Mathematical Society Proceedings of Third Symposium in Applied Mathematics*, vol. 3 (1950), pp. 163–186.

G. S. Shapiro, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 17 (1953), pp. 249–252.

J. C. Wilhoit, Jr., *Quarterly of Applied Mathematics*, vol. 11 (1954), pp. 499–501.

[2] C. B. Ling, *Quarterly of Applied Mathematics*, vol. 10 (1952), pp. 149–156.

[3] A. Weinstein, *Quarterly of Applied Mathematics*, vol. 10 (1952), pp. 77–81.

L. E. Payne, *Journal of the Society for Industrial and Applied Mathematics*, vol. 1 (1953), pp. 53–71.

[4] Note that $\tau_{\theta z} = (\tau_{zx}^2 + \tau_{zy}^2)^{1/2}$.

$$\frac{d^2R}{dr^2} - \frac{3}{r}\frac{dR}{dr} + k^2R = 0.$$

This is a well-known differential equation, and it is easy to show[1] that it is satisfied by the function $R = r^2J_2(kr)$, where $J_2(kr)$ is the Bessel function of the first kind and of second order. Since the differential equation (49.4) is linear, a linear combination of solutions of the type $A_nr^2e^{-k_nz}J_2(k_nr)$ will satisfy the equation, and we take the function $F(r, z)$ in the form

$$F(r, z) = \tfrac{1}{4}\alpha r^4 + \sum_{n=1}^{\infty} A_nr^2e^{-k_nz}J_2(k_nr), \tag{50.1}$$

where α and A_i are constants. If the constants k_i are chosen to be the successive roots of the equation $J_2(ka) = 0$, then on the boundary of the axial section we have $F(a, z) = \tfrac{1}{4}\alpha a^4 = \text{const}$, which is the required boundary condition. It is obvious from (49.5) that the distribution of stress in the cylinder corresponding to the choice $A_n = 0$ $(n = 1, 2, \ldots)$ is precisely that required by Saint-Venant's theory.

The expression for the tangential stress $\tau_{\theta z}$ is given by the first of formulas (49.5), and we obtain formally

$$\tau_{\theta z} = \mu\left\{\alpha r + \sum_{n=1}^{\infty} A_ne^{-k_nz}\left[\frac{2}{r}J_2(k_nr) + J_2'(k_nr)k_n\right]\right\}, \tag{50.2}$$

where the prime denotes the derivative with respect to the argument k_nr. But[2]

$$k\left[\frac{2}{kr}J_2(kr) + J_2'(kr)\right] = kJ_1(kr), \tag{50.3}$$

where $J_1(kr)$ stands for the Bessel function of order 1, which is known to satisfy the equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{1}{r^2}\right)J_1(kr) = 0. \tag{50.4}$$

Substituting (50.3) in (50.2) and setting $z = 0$ gives the expression for the distribution of stresses $\tau_{\theta z}$ over the end $z = 0$ of the cylinder,

$$(\tau_{\theta z})_{z=0} = \mu\left[\alpha r + \sum_{n=1}^{\infty} A_nk_nJ_1(k_nr)\right]. \tag{50.5}$$

[1] See, for example, I. S. and E. S. Sokolnikoff, Higher Mathematics for Engineers and Physicists, 2d ed., p. 339.

[2] See G. N. Watson's Theory of Bessel Functions or J. M. MacRobert's Treatise on Bessel Functions.

We proceed to calculate the torque M acting on the end $z = 0$ of the cylinder. Now

$$(50.6) \qquad M = \int_0^a 2\pi r^2 (\tau_{\theta z})_{z=0}\, dr = 2\pi\mu \left[\int_0^a \alpha r^3\, dr + \sum_{n=1}^{\infty} A_n k_n \int_0^a r^2 J_1(k_n r)\, dr \right],$$

and it is easy to show that $\int_0^a r^2 J_1(k_n r)\, dr = 0$. We note from (50.4) that

$$J_1(kr) = -\frac{1}{k^2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) J_1(kr),$$

so that

$$\begin{aligned}
\int_0^a r^2 J_1(k_n r)\, dr &= -\frac{1}{k_n^2}\int_0^a \left(r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} - 1\right) J_1(k_n r)\, dr \\
&= -\frac{1}{k_n^2}\int_0^a \frac{d}{dr}\left[r^2 \frac{dJ_1(k_n r)}{dr} - rJ_1(k_n r)\right] dr \\
&= -\frac{1}{k_n^2}[a^2 J_1'(k_n a)k_n - aJ_1(k_n a)] \\
&= -\frac{a^2}{k_n}\left[J_1'(k_n a) - \frac{1}{k_n a} J_1(k_n a)\right].
\end{aligned}$$

But it is known that

$$J_2(kr) = -J_1'(kr) + \frac{1}{kr} J_1(kr),$$

and if the numbers k_n are the roots of the equation $J_2(ka) = 0$, then $J_1'(k_n a) - (1/k_n a)J_1(k_n a) = 0$. Thus, the integrals in (50.6) involving Bessel's functions vanish, and we get

$$M = \frac{\pi a^4}{2} \mu\alpha,$$

which is the same expression for the moment M as previously obtained in Sec. 33.

Since a suitably restricted function of r defined in the interval $(0, a)$ can be expanded in a series of the form[1] $\sum_{n=0}^{\infty} a_n J_1(k_n r)$, we see from (50.5) that we can obtain the solution of the torsion problem that corresponds to distribution of stress $(\tau_{\theta z})_0$ over the end $z = 0$, where $(\tau_{\theta z})_0$ is a prescribed function of r. It is obvious from (50.2) that the effect of the terms involving the factors $e^{-k_n z}$ diminishes with an increase in z, and hence the distribution of stresses in a long cylinder, over the far end, is sensibly equal to a couple of moment M.

[1] See Hankel-Schläfli expansion on p. 577 of G. N. Watson's Bessel's Functions, 2d ed. (1948).

The results of this section are essentially due to Dougall. They have been extended by Synge, who proposed a more general approach to the Saint-Venant torsion and flexure problems.[1]

Instead of prescribing the distribution of stress over one end of the cylinder, one may impose a requirement that one of the sections of the twisted cylinder remain plane.[2]

51. Torsion of Anisotropic Beams. We saw that the deformation of long isotropic cylinders twisted by end couples is the same in every cross section. The corresponding deformation of anisotropic cylinders is more complicated. The anisotropy of the medium ordinarily gives rise to bending moments which deform the planes containing the axis of the cylinder. If, however, the medium is such that the planes normal to the axis of the cylinder coincide with the planes of elastic symmetry, then the twisting couples produce no bending. This fact was first established by Voigt[3] and, for the special case of an orthotropic medium, by Saint-Venant.[4] As a consequence of this, the torsion problem for such cylinders can be reduced to the solution of the torsion problem for certain isotropic cylinders. We proceed to show how this is done when the material is orthotropic. The corresponding solution for the case when the medium

[1] J. Dougall, *Transactions of the Royal Society of Edinburgh*, vol. 49 (1913), pp 895–978.

J. L. Synge, *Quarterly of Applied Mathematics*, vol. 2 (1945), pp. 307–317.

[2] A solution by energy methods of such a torsion problem for a beam of elliptical section was found by A. Föppl (1920) and is given in A. and L. Föppl, Drang and Zwang, vol. 2, Sec. 77. The beam of rectangular section was considered by S. Timoshenko, *Proceedings of the London Mathematical Society*, vol. 20 (1922), p. 389 and by J. Nowinski, *Arch. Mech. Stos.*, vol. 5 (1953), pp. 47–66 (in Polish). B. P. Netrebko, *Vestnik, Moscow University*, No. 12 (1954), pp. 15–26 (in Russian), used energy methods to investigate the torsion of a rectangular parallelepiped by arbitrarily specified distributions of shearing stresses on the bases. Energy methods are also used by N. V. Zvolinskij in "Angenäherte Lösung der Torsionsaufgabe für einen elastischen zylindrischen Stab mit einem nicht verwolbten Querschnitt," *Bulletin de l'academie des sciences de l'URSS, Classe des sciences mathématiques et naturelles*, No. 8 (1939), pp. 91–100 (in Russian). The problem of flexure of such a beam has been treated by R. Sonntag in "Über Biegung bei verhinderter Querschnittskrümmung," *Ingenieur Archiv*, vol. 4 (1944), pp. 415–420.

The effect of local stresses corresponding to different modes of applying torsional couples to a circular cylinder has been discussed by Wolf and by Deimel [K. Wolf, *Sitzungsberichte der Akademie der Wissenschaften in Wien*, vol. 125 (1916), p. 1149; R. F. Deimel, "The Torsion of a Circular Cylinder," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 21 (1935), pp. 637–642].

[3] W. Voigt, Lehrbuch der Kristallphysik, p. 648. In Chap. VII, Secs. 315–324, of this work Voigt discusses the torsion problem for cylinders with the most general kind of anisotropy. See also S. G. Lekhnitsky, Theory of Elasticity of an Anisotropic Body (1950), pp. 141–172 (in Russian).

[4] B. Saint-Venant, *Mémoires présentés par divers savants à l'académie des sciences, Sciences mathématiques et physiques*, vol. 14 (1856).

has a family of planes of elastic symmetry normal to the axis of the cylinder is essentially the same.[1]

Let the beam have three mutually orthogonal planes of elastic symmetry, and assume that the axis of the beam is perpendicular to one of these planes.

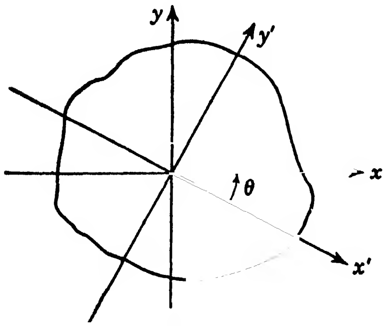

FIG. 41

As in the isotropic case, we let the axis of the beam coincide with the z-axis and choose the x- and y-axes in one end of the beam. The longitudinal planes of elastic symmetry are denoted by $x'z$ and $y'z$ and the shear moduli associated with the axes z and x' and z and y' by μ_1 and μ_2, respectively (Fig. 41).

The components of shear $\tau_{x'z}$ and $\tau_{y'z}$ are connected with the shearing strains $e_{x'z}$ and $e_{y'z}$ by the formulas

(51.1) $$\tau_{x'z} = 2\mu_1 e_{x'z}, \qquad \tau_{y'z} = 2\mu_2 e_{y'z},$$

where

(51.2) $$e_{x'z} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x'}\right), \qquad e_{y'z} = \frac{1}{2}\left(\frac{\partial w}{\partial y'} + \frac{\partial v}{\partial z}\right),$$

and u, v, and w are the components of displacement in the directions of the x'-, y'-, and z-axes, respectively.

We assume, as in the isotropic case,[2] that the displacements u, v and w are given by the formulas

(51.3) $$u = -\alpha z y', \qquad v = \alpha z x', \qquad w = \alpha\varphi(x', y'),$$

where α is the angle of twist per unit length of the bar and $\varphi(x', y')$ is the torsion function associated with this problem.

If we let θ denote the angle between the axes x and x', then the expressions for the nonvanishing components of stress τ_{xz} and τ_{yz} are related to the components $\tau_{x'z}$ and $\tau_{y'z}$ by the formulas[3]

$$\begin{aligned} \tau_{xz} &= \tau_{y'z}\sin\theta + \tau_{x'z}\cos\theta, \\ \tau_{yz} &= \tau_{y'z}\cos\theta - \tau_{x'z}\sin\theta, \end{aligned}$$

[1] See Voigt's Lehrbuch der Kristallphysik cited above. This book contains a number of interesting solutions of special problems. Saint-Venant's memoir, quoted in the preceding footnote, contains explicit solutions and detailed calculations for orthotropic rods of elliptical, rectangular, and several other cross sections. See also I. W. Geckeler, Handbuch der Physik, vol. 6, Elastizitätstheorie anisotroper Körper. A comprehensive modern account of the theory of elasticity of anisotropic media is presented in a book by S. G. Lekhnitzky, Theory of Elasticity of an Anisotropic Body, Moscow (1950) (in Russian).

[2] See Sec. 34.

[3] Note formulas (16.5).

which, on account of the relations (51.1) and (51.3), become

$$(51.4)\qquad \begin{cases} \tau_{xz} = \alpha\mu_2 \sin\theta\left(\dfrac{\partial\varphi}{\partial y'} + x'\right) + \alpha\mu_1 \cos\theta\left(\dfrac{\partial\varphi}{\partial x'} - y'\right), \\ \tau_{yz} = \alpha\mu_2 \cos\theta\left(\dfrac{\partial\varphi}{\partial y'} + x'\right) - \alpha\mu_1 \sin\theta\left(\dfrac{\partial\varphi}{\partial x'} - y'\right). \end{cases}$$

The partial derivatives of the torsion function $\varphi(x', y')$ appearing in the right-hand members of these expressions can be calculated in terms of $\dfrac{\partial\varphi}{\partial x}$ and $\dfrac{\partial\varphi}{\partial y}$, since

$$x = x'\cos\theta + y'\sin\theta,$$
$$y = -x'\sin\theta + y'\cos\theta.$$

We have

$$(51.5)\qquad \begin{cases} \dfrac{\partial\varphi}{\partial x'} = \dfrac{\partial\varphi}{\partial x}\cos\theta - \dfrac{\partial\varphi}{\partial y}\sin\theta, \\ \dfrac{\partial\varphi}{\partial y'} = \dfrac{\partial\varphi}{\partial x}\sin\theta + \dfrac{\partial\varphi}{\partial y}\cos\theta. \end{cases}$$

Inserting the values from (51.5) in (51.4) and introducing the abbreviations

$$A = \mu_2\sin^2\theta + \mu_1\cos^2\theta,$$
$$B = (\mu_2 - \mu_1)\sin\theta\cos\theta,$$
$$C = \mu_2\cos^2\theta + \mu_1\sin^2\theta,$$

we get

$$(51.6)\qquad \begin{cases} \tau_{xz} = \alpha\left(A\dfrac{\partial\varphi}{\partial x} + B\dfrac{\partial\varphi}{\partial y} + Bx - Ay\right), \\ \tau_{yz} = \alpha\left(B\dfrac{\partial\varphi}{\partial x} + C\dfrac{\partial\varphi}{\partial y} + Cx - By\right). \end{cases}$$

Since $\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0$ and τ_{xz} and τ_{yz} are independent of z, the first two of the equilibrium equations (15.3) are identically satisfied and the third one gives the equation

$$(51.7)\qquad A\frac{\partial^2\varphi}{\partial x^2} + 2B\frac{\partial^2\varphi}{\partial x\,\partial y} + C\frac{\partial^2\varphi}{\partial y^2} = 0.$$

Thus, in this case, the torsion function φ no longer satisfies Laplace's equation.

Let the boundary C of the cross section have the equation $f(x, y) = 0$; then the components $\cos(x, \nu)$ and $\cos(y, \nu)$ of the normal $\mathbf{\nu}$ to the boundary C are proportional to $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, respectively, and we can write the boundary condition

$$\tau_{xz}\cos(x, \nu) + \tau_{yz}\cos(y, \nu) = 0 \qquad \text{on } C$$

in the form

$$\left(A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y}\right)\frac{\partial f}{\partial x} + \left(B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y}\right)\frac{\partial f}{\partial y} = (Ay - Bx)\frac{\partial f}{\partial x} + (By - Cx)\frac{\partial f}{\partial y} \quad \text{on } C. \tag{51.8}$$

Equation (51.7) and the boundary condition (51.8) can be simplified by introducing new independent variables ξ and η, defined by the formulas

$$\xi = \frac{\sqrt{\mu_1\mu_2}}{A}x, \qquad \eta = y - \frac{B}{A}x. \tag{51.9}$$

A simple calculation shows that Eq. (51.7) becomes

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} = 0, \tag{51.10}$$

while the equation of the boundary $f(x, y) = 0$ is changed into

$$F(\xi, \eta) = 0. \tag{51.11}$$

Making the corresponding change of variables in the boundary condition (51.8) gives

$$\frac{\sqrt{\mu_1\mu_2}}{A}\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial F}{\partial\xi} + \frac{\partial\varphi}{\partial\eta}\frac{\partial F}{\partial\eta}\right) = \eta\frac{\partial F}{\partial\xi} - \xi\frac{\partial F}{\partial\eta} \quad \text{on } C', \tag{51.12}$$

where C' is the transformed boundary defined by (51.11).

Finally, if we set

$$\varphi'(\xi, \eta) = \frac{\sqrt{\mu_1\mu_2}}{A}\varphi(\xi, \eta),$$

then

$$\frac{\partial^2\varphi'}{\partial\xi^2} + \frac{\partial^2\varphi'}{\partial\eta^2} = 0, \tag{51.13}$$

and (51.12) becomes

$$\frac{\partial\varphi'}{\partial\xi}\frac{\partial F}{\partial\xi} + \frac{\partial\varphi'}{\partial\eta}\frac{\partial F}{\partial\eta} = \eta\frac{\partial F}{\partial\xi} - \xi\frac{\partial F}{\partial\eta} \quad \text{on } C',$$

or

$$\frac{\partial\varphi'}{\partial\nu} = \eta\cos(\xi, \nu) - \xi\cos(\eta, \nu) \quad \text{on } C', \tag{51.14}$$

where ν is the normal to the boundary C'.

The boundary condition (51.14) is precisely of the same form as that appearing in a study of torsion of isotropic cylinders; hence the solution of the torsion problem for a cylinder of nonisotropic material (having three orthogonal planes of elastic symmetry) whose cross section is C is reduced to the solution of the torsion problem for an isotropic bar whose cross section has a different boundary C', defined by Eq. (51.11).

It is not difficult to calculate the torsional rigidity of a nonisotropic cylinder in terms of the torsional rigidity of the corresponding isotropic cylinder. Substituting from (51.6) in the expression for the couple M, we obtain

$$M = \iint_R (\tau_{yz}x - \tau_{xz}y)\,dx\,dy$$

$$= \frac{\alpha A}{\sqrt{\mu_1\mu_2}} \iint_{R'} \left[\sqrt{\mu_1\mu_2}\left(\xi\frac{\partial\varphi}{\partial\eta} - \eta\frac{\partial\varphi}{\partial\xi}\right) + A(\xi^2 + \eta^2)\right] d\xi\,d\eta,$$

where the integration now extends over the region R' bounded by the curve C'. Recalling that $\varphi' \equiv (\sqrt{\mu_1\mu_2}/A)\,\varphi$, we have

$$M = \frac{\alpha A^2}{\sqrt{\mu_1\mu_2}} \iint_{R'} \left(\xi\frac{\partial\varphi'}{\partial\eta} - \eta\frac{\partial\varphi'}{\partial\xi} + \xi^2 + \eta^2\right) d\xi\,d\eta,$$

and since

$$M = \alpha D,$$

where D is the torsional rigidity, we see that the torsional rigidity of a nonisotropic cylinder can be deduced from the torsional rigidity of the isotropic cylinder obtained from the nonisotropic one by a homogeneous deformation (51.9).

We conclude this formulation of the torsion problem for a nonisotropic prism by remarking that the transformation (51.9) changes the boundary of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

into another ellipse, and since the solution of the torsion problem for an isotropic cylinder is known, we can write down at once the solution of the corresponding problem for a nonisotropic elliptical cylinder.

The transformation (51.9), in general, carries a rectangle into a parallelogram, and hence the solution of the torsion problem for a nonisotropic rectangular beam is not covered by the discussion contained in Sec. 38, unless the x'-axis coincides with the x-axis. If these axes coincide, then $B = 0$, and the rectangle will be transformed into another rectangle of different length. The solution corresponding to this case is written out in Love's Treatise, on page 325.

REFERENCES FOR COLLATERAL READING

G. W. Trayer and H. W. March: *National Advisory Committee for Aeronautics Report* 334, pp. 37–44.

A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Sec. 226.

52. Flexure of Beams by Terminal Loads. Let a cantilever beam of uniform cross section have one end ($z = 0$) fixed and the other end ($z = l$) loaded by some distribution of forces that is statically equivalent to a single force (W_x, W_y, 0) lying in the plane $z = l$ and acting at the *load point* (x_0, y_0, l). The z-axis is taken along the central line of the beam, while the x- and y-axes are any orthogonal axes intersecting at the *centroid* of the end $z = 0$ (Fig. 42). The lateral surface of the beam is free from external forces, and the body forces are assumed to vanish.

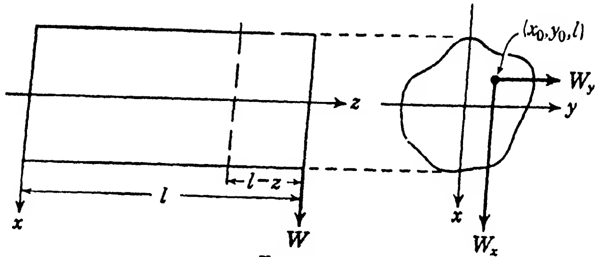

FIG. 42

We shall follow the semi-inverse method of Saint-Venant; that is, we put

$$\tau_{xx} = \tau_{xy} = \tau_{yy} = 0. \tag{52.1}$$

The functions τ_{zx}, τ_{zy}, and τ_{zz} will be so chosen that the equations of equilibrium and compatibility, as well as the boundary conditions, are satisfied.

In writing an expression for τ_{zz}, we shall be guided by an expression for the bending moment M_y that would be produced by the load W_x acting alone. In any cross section z units distant from the fixed end, one would have

$$M_y = W_x(l - z), \tag{52.2}$$

so that the stress distribution (due to W_x alone) in this section would have to be statically equivalent to the moment M_y and to the resultant force W_x. Now in the discussion of the problem of bending of beams by couples applied at the ends,[1] it was found that the stress τ_{zz}, distributed according to the linear relation

$$\tau_{zz} = -\frac{M_y}{I_y} x, \qquad \text{(bending by couples)}, \tag{52.3}$$

with

$$I_y = \iint_R x^2 \, dx \, dy,$$

[1] See Sec. 32. Note that the coordinate axes there were taken to be the principal axes of the cross section, while the choice of axes here is not restricted.

is statically equivalent to a couple of moment M_y. Equations (52.2) and (52.3) suggest that we try to satisfy the conditions of the present problem by assuming

$$\tau_{zz} = -E(l - z)(K_x x + K_y y), \tag{52.4}$$

where the constants K_x, K_y are to be determined from the conditions

$$\iint_R \tau_{zx}\,dx\,dy = W_x, \qquad \iint_R \tau_{zy}\,dx\,dy = W_y. \tag{52.5}$$

Substituting from (52.1) and (52.4) in the equations of equilibrium (29.1), we get

$$\begin{cases} \dfrac{\partial \tau_{zx}}{\partial z} = 0, \qquad \dfrac{\partial \tau_{zy}}{\partial z} = 0, \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zy}}{\partial y} + E(K_x x + K_y y) = 0. \end{cases} \tag{52.6}$$

It follows from the first two of Eqs. (52.6) that the shear components τ_{zx} and τ_{zy} have the same value in all cross sections of the beam, while the third equation can be rearranged to read

$$\frac{\partial}{\partial x}\left(\tau_{zx} + \frac{1}{2} E K_x x^2\right) + \frac{\partial}{\partial y}\left(\tau_{zy} + \frac{1}{2} E K_y y^2\right) = 0.$$

As this equation is of the form

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial F}{\partial x}\right) = 0,$$

it is evident that there exists a function $F(x, y)$ such that

$$\begin{cases} \tau_{zx} = \dfrac{\partial F}{\partial y} - \dfrac{1}{2} E K_x x^2, \\ \tau_{zy} = -\dfrac{\partial F}{\partial x} - \dfrac{1}{2} E K_y y^2. \end{cases} \tag{52.7}$$

The conditions to be satisfied by the function $F(x, y)$ can be determined from the Beltrami-Michell compatibility equations (24.15), which reduce in this case to

$$\nabla^2 \tau_{zx} + \frac{E K_x}{1 + \sigma} = 0, \qquad \nabla^2 \tau_{zy} + \frac{E K_y}{1 + \sigma} = 0.$$

Substituting from (52.7) in these equations, we see that the latter will be fulfilled if

$$\frac{\partial}{\partial y}(\nabla^2 F) = 2\mu\sigma K_x, \qquad \frac{\partial}{\partial x}(\nabla^2 F) = -2\mu\sigma K_y,$$

from which it follows that

$$\nabla^2 F(x, y) = -2\mu\sigma K_y x + 2\mu\sigma K_x y - 2\mu\alpha. \tag{52.8}$$

The physical significance of the constant of integration $-2\mu\alpha$ will be discovered presently. It is not difficult to obtain a particular integral of (52.8) in the form of a polynomial, and it is readily verified that the solution of (52.8) is

(52.9) $$F(x, y) = f(x, y) - \tfrac{1}{3}\mu\sigma(K_y x^3 - K_x y^3) - \tfrac{1}{2}\mu\alpha(x^2 + y^2),$$

where $f(x, y)$ is a harmonic function.

It will prove advantageous to write the stresses τ_{zx}, τ_{zy} not in terms of $f(x, y)$ but in terms of its harmonic conjugate $g(x, y)$, where

(52.10) $$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \qquad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}.$$

Equations (52.7) can then be written as

(52.11) $$\begin{cases} \tau_{zx} = -\dfrac{\partial g}{\partial x} - \mu\alpha y + \mu\sigma K_x y^2 - \dfrac{1}{2}EK_x x^2, \\ \tau_{zy} = -\dfrac{\partial g}{\partial y} + \mu\alpha x + \mu\sigma K_y x^2 - \dfrac{1}{2}EK_y y^2. \end{cases}$$

The constant of integration $-2\mu\alpha$ in (52.8) can easily be interpreted physically. Each element of area of a cross section is rotated in its own plane through an angle [see (7.5)]

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

The *local twist* at a point (x, y) of a cross section is defined as

$$\begin{aligned} \frac{\partial \omega}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial z\,\partial x} - \frac{\partial^2 u}{\partial z\,\partial y}\right) &= \frac{\partial e_{zy}}{\partial x} - \frac{\partial e_{zx}}{\partial y} \\ &= \frac{1}{2\mu}\left(\frac{\partial \tau_{zy}}{\partial x} - \frac{\partial \tau_{zx}}{\partial y}\right). \end{aligned}$$

Substituting the values of the shear stresses from (52.11), one gets

$$\frac{\partial \omega}{\partial z} = \alpha + \sigma(K_y x - K_x y).$$

The mean value of the local twist over the section (or, equally well, the value of the local twist at the centroid of the section) is just the constant α. Thus, we see that the terms in (52.11) that involve α represent a twist of the beam, and, indeed, the terms $-\mu\alpha y$ and $\mu\alpha x$ in these expressions also appear in the solution of the torsion problem (see Sec. 34). In the latter problem, one has

$$\tau_{zx} = \mu\alpha\left(\frac{\partial \varphi}{\partial x} - y\right), \qquad \tau_{zy} = \mu\alpha\left(\frac{\partial \varphi}{\partial y} + x\right), \qquad \text{(pure torsion)}.$$

We are thus led to introduce the torsion function $\varphi(x, y)$ into the flexure problem by writing

(52.12) $$g(x, y) = -\mu\alpha\varphi(x, y) - \mu[K_x\varphi_1(x, y) + K_y\varphi_2(x, y)],$$

where $\varphi(x, y)$, $\varphi_1(x, y)$, and $\varphi_2(x, y)$ are harmonic functions. We can now write

$$(52.13)\qquad \begin{cases} \tau_{zx} = \mu\alpha\left(\dfrac{\partial \varphi}{\partial x} - y\right) + \mu K_z\left[\dfrac{\partial \varphi_1}{\partial x} - x^2 - \sigma(x^2 - y^2)\right] + \mu K_y\dfrac{\partial \varphi_2}{\partial x}, \\ \tau_{zy} = \mu\alpha\left(\dfrac{\partial \varphi}{\partial y} + x\right) + \mu K_y\left[\dfrac{\partial \varphi_2}{\partial y} - y^2 - \sigma(y^2 - x^2)\right] + \mu K_z\dfrac{\partial \varphi_1}{\partial y}. \end{cases}$$

The boundary conditions on the functions φ_1 and φ_2 may be derived from the relation

$$(52.14)\qquad \tau_{zx}\cos(x, \nu) + \tau_{zy}\cos(y, \nu) = 0,$$

which expresses the vanishing of external force on the lateral surface of the cylinder, and, from the boundary condition on the torsion function φ [see (34.6)],

$$(52.15)\qquad \frac{d\varphi}{d\nu} = y\cos(x, \nu) - x\cos(y, \nu).$$

Inserting Eqs. (52.13) in (52.14) and taking account of (52.15) yields

$$K_x\frac{d\varphi_1}{d\nu} + K_y\frac{d\varphi_2}{d\nu} = K_x[(1 + \sigma)x^2 - \sigma y^2]\cos(x, \nu) + K_y[(1 + \sigma)y^2 - \sigma x^2]\cos(y, \nu) \quad \text{on } C,$$

and this will be satisfied if the functions φ_1 and φ_2 are subject to the conditions

$$(52.16)\qquad \begin{cases} \dfrac{d\varphi_1}{d\nu} = [(1 + \sigma)x^2 - \sigma y^2]\cos(x, \nu) & \text{on } C, \\ \dfrac{d\varphi_2}{d\nu} = [(1 + \sigma)y^2 - \sigma x^2]\cos(y, \nu) & \text{on } C. \end{cases}$$

The flexure problem has thus been reduced to the task of finding three functions, harmonic within the region R of the cross section, whose normal derivatives are prescribed on the boundary C; that is, we have been led to the problem of Neumann. In order to see that the condition of the existence of a solution of this problem is fulfilled, we observe that

$$\int_C \frac{d\varphi_1}{d\nu}\,ds = \int_C [(1 + \sigma)x^2 - \sigma y^2]\,dy = 2(1 + \sigma)\iint_R x\,dx\,dy = 0,$$

$$\int_C \frac{d\varphi_2}{d\nu}\,ds = -\int_C [(1 + \sigma)y^2 - \sigma x^2]\,dx = 2(1 + \sigma)\iint_R y\,dx\,dy = 0,$$

since the origin is at the centroid of the section.

In considering the torsion of a beam by couples, it was seen that the solution could be made to depend upon either a problem of Neumann, that is, the problem of finding a function $\varphi(x, y)$, harmonic in R and such that

$$\frac{d\varphi}{d\nu} = y \cos(x, \nu) - x \cos(y, \nu) \qquad \text{on } C,$$

or upon a problem of Dirichlet, with

$$\psi = \tfrac{1}{2}(x^2 + y^2) + \text{const} \qquad \text{on } C.$$

The torsion functions φ and ψ are harmonic conjugates, so that $\varphi + i\psi$ is an analytic function of $x + iy$. The flexure problem may also be reduced to a problem of Dirichlet by introducing the harmonic functions ψ_1, ψ_2, conjugate to φ_1 and φ_2, respectively. Then

$$\frac{\partial \varphi_i}{\partial x} = \frac{\partial \psi_i}{\partial y}, \qquad \frac{\partial \varphi_i}{\partial y} = -\frac{\partial \psi_i}{\partial x}, \qquad \frac{d\varphi_i}{d\nu} = \frac{d\psi_i}{ds}, \qquad (i = 1, 2),$$

and the boundary conditions (52.16) can be written as

$$\frac{d\psi_1}{ds} = [(1 + \sigma)x^2 - \sigma y^2]\frac{dy}{ds}, \qquad \frac{d\psi_2}{ds} = -[(1 + \sigma)y^2 - \sigma x^2]\frac{dx}{ds},$$

or

$$(52.17) \qquad \begin{cases} \psi_1 = -\tfrac{1}{3}\sigma y^3 + (1 + \sigma)\displaystyle\int_{(x_0,y_0)}^{(x,y)} x^2\, dy + \text{const} & \text{on } C, \\ \psi_2 = \tfrac{1}{3}\sigma x^3 - (1 + \sigma)\displaystyle\int_{(x_0,y_0)}^{(x,y)} y^2\, dx + \text{const} & \text{on } C, \end{cases}$$

where the line integrals are to be evaluated along the contour C.

We turn now to the determination of the constants K_x and K_y. Since the resultant of the stresses τ_{zx} acting over any cross section must equal the component W_x of the applied load, we have

$$W_x = \iint_R \tau_{zx}\, dx\, dy,$$

or, substituting from (52.13),

$$(52.18) \qquad \begin{aligned} W_x = \mu\alpha \iint_R \frac{\partial \varphi}{\partial x}\, dx\, dy &+ \mu K_x \iint_R \frac{\partial \varphi_1}{\partial x}\, dx\, dy \\ &+ \mu K_y \iint_R \frac{\partial \varphi_2}{\partial x}\, dx\, dy \\ &+ \mu K_x[-(1 + \sigma)I_y + \sigma I_x], \end{aligned}$$

where

$$I_x = \iint_R y^2\, dx\, dy, \qquad I_y = \iint_R x^2\, dx\, dy.$$

Now if Φ is any harmonic function, then[1]

$$\iint\limits_R \frac{\partial \Phi}{\partial x}\, dx\, dy = \int_C x \frac{d\Phi}{d\nu}\, ds.$$

With the aid of this identity and the boundary conditions (52.15) and (52.16), Eq. (52.18) becomes

$$W_x = \mu\alpha \int_C (xy\, dy + x^2\, dx) + \mu K_x \int_C [(1 + \sigma)x^3 - \sigma x y^2]\, dy + \mu K_y \int_C [-(1 + \sigma)xy^2 + \sigma x^3]\, dx + \mu K_x[-(1 + \sigma)I_y + \sigma I_x]$$

Upon applying Green's Theorem and recalling that

$$E = 2\mu(1 + \sigma),$$

the last equation can be written as

$$W_x = E(K_x I_y + K_y I_{xy}), \tag{52.19}$$

where

$$I_{xy} = \iint\limits_R xy\, dx\, dy$$

is the product of inertia of the section. Similarly, from

$$W_y = \iint\limits_R \tau_{zy}\, dx\, dy,$$

it follows that

$$W_y = E(K_y I_x + K_x I_{xy}). \tag{52.20}$$

Equations (52.19) and (52.20) can be solved for K_x, K_y to give

$$\begin{cases} EK_x = \dfrac{I_x W_x - I_{xy} W_y}{I_x I_y - I_{xy}^2}, \\[2ex] EK_y = \dfrac{I_y W_y - I_{xy} W_x}{I_x I_y - I_{xy}^2}, \end{cases} \tag{52.21}$$

since the denominator in these expressions never vanishes.

[1] For

$$\iint\limits_R \frac{\partial \Phi}{\partial x}\, dx\, dy = \iint\limits_R \left[\frac{\partial}{\partial x}\left(x \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(x \frac{\partial \Phi}{\partial y}\right)\right] dx\, dy$$
$$= \int_C \left(-x \frac{\partial \Phi}{\partial y}\, dx + x \frac{\partial \Phi}{\partial x}\, dy\right) = \int_C x \frac{d\Phi}{d\nu}\, ds.$$

Similarly, it follows that

$$\iint\limits_R \frac{\partial \Phi}{\partial y}\, dx\, dy = \int_C y \frac{d\Phi}{d\nu}\, ds.$$

Using the values of K_x and K_y determined by (52.21) in the formula (52.4), we easily check that

$$M_x \equiv \iint_R y\tau_{zz}\,dx\,dy = -(l-z)W_y,$$
$$M_y \equiv \iint_R -x\tau_{zz}\,dx\,dy = (l-z)W_x,$$

which are precisely the bending moments produced in the section z = const by the forces W_x and W_y.

The stress distribution over any cross section is easily seen to be statically equivalent to the load $(W_x, W_y, 0)$;

$$\iint_R \tau_{xz}\,dx\,dy = W_x, \qquad \iint_R \tau_{zy}\,dx\,dy = W_y, \qquad \iint_R \tau_{zz}\,dx\,dy = 0.$$

The first two equations are satisfied by virtue of our choice of the constants K_x and K_y, while the third follows from our assumption that the z-axis passes through centroids of cross sections. The constant α in formulas (52.13), for shear stresses, is determined by the condition that the twisting moment M_z be such that

$$\iint_R (x\tau_{yz} - y\tau_{xz})\,dx\,dy = x_0W_y - y_0W_x. \tag{52.22}$$

In (52.22), (x_0, y_0) are the coordinates of the load point relative to any set of axes intersecting at the centroid of the section.

We see that the solution of the general Saint-Venant flexure problem is reduced to the determination of harmonic functions φ, φ_1, and φ_2 that satisfy the boundary conditions (52.15) and (52.16). The boundary conditions (52.16) are somewhat unwieldy, and we shall show in Sec. 53 how the formulation of the problem can be simplified by introducing the idea of center of flexure.[1]

53. Center of Flexure. The formulas (52.13) for shear stresses suggest a resolution of the general flexure problem into the following simpler problems:

[1] The treatment of the flexure problem given here is influenced by L. S. Leibenson, *Central Aero-hydrodynamical Institute Technical Notes* 45, Moscow (1933), and A. C. Stevenson, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 237 (1938–1939), pp. 161–229. These authors have departed from Saint-Venant's formulation by supposing that the load acts not at the centroid of the end section but at an arbitrary load point (x_0, y_0, l). Also, they abandoned Saint-Venant's assumption that the x- and y-axes are the principal axes of inertia of the cross section. Freedom in the choice of axes is of importance for asymmetric cross sections because, for such sections, the principal axes seldom provide the most convenient mathematical description of the boundary. Leibenson obtained formulas, equivalent to those given here, by a transformation of coordinates in the classical Saint-Venant's solution.

1. A flexure problem in which the mean local twist α is set equal to zero. The position of the load point $(\bar{x}, \bar{y}, l)$, corresponding to this stress distribution, is then determined by the condition

$$\iint_R (x\tau_{zy} - y\tau_{zx})\, dx\, dy = \bar{x}W_y - \bar{y}W_x,$$

which must hold for an arbitrary choice of W_x and W_y. The load point $(\bar{x}, \bar{y}, l)$, corresponding to $\alpha = 0$ is called the *center of flexure* and is denoted by (x_{cf}, y_{cf}, l).

2. A torsion problem with the twist α due to a couple of moment

$$W_y(x_0 - x_{cf}) - W_x(y_0 - y_{cf}),$$

and with shear stresses determined by (52.13), with K_x and K_y set equal to zero.

We can thus think of the load $\mathbf{W}$ at the point (x_0, y_0, l) as being replaced by an equal load at the center of flexure and by a couple producing the twist α. The solution of the general flexure problem is then got by superposing the solutions of these two simpler problems. The decomposition of the general flexure problem into problems 1 and 2 amounts to resolving the twisting moment

$$M_z = x_0W_y - y_0W_x,$$

determined by (52.22), into two parts:

$$\iint_R (x\tau_{zy} - y\tau_{zx})\, dx\, dy = x_{cf}W_y - y_{cf}W_x, \tag{53.1}$$

where τ_{zy} and τ_{zx} are given by formulas (52.13) with $\alpha = 0$, and

$$\iint_R (x\tau_{zy} - y\tau_{zx})\, dx\, dy = (x_0 - x_{cf})W_y - (y_0 - y_{cf})W_x, \tag{53.2}$$

where τ_{zy} and τ_{zx} are given by (52.13) with $K_x = K_y = 0$.

The position of the center of flexure is determined from the formula (53.1), and it is really found that

$$\begin{cases} x_{cf} = J(I_yS_2 - I_{xy}S_1), \\ y_{cf} = J(I_{xy}S_2 - I_xS_1), \end{cases} \tag{53.3}$$

where

$$S_1 = \iint_R \left[x\frac{\partial\varphi_1}{\partial y} - y\frac{\partial\varphi_1}{\partial x} + (1+\sigma)x^2y - \sigma y^3\right] dx\, dy,$$

$$S_2 = \iint_R \left[x\frac{\partial\varphi_2}{\partial y} - y\frac{\partial\varphi_2}{\partial x} - (1+\sigma)xy^2 + \sigma x^3\right] dx\, dy,$$

$$J^{-1} = 2(1+\sigma)(I_xI_y - I_{xy}^2).$$

If the cross section R is symmetric about the x-axis, then it is evident from the symmetry of the differential equation and the symmetry of the boundary conditions that $\varphi_1(x, y)$ is an even function in y; hence, $x \frac{\partial \varphi_1}{\partial y}$ and $y \frac{\partial \varphi_1}{\partial x}$ are odd in y. In this case the foregoing formulas reduce to

$$S_1 = 0, \qquad J^{-1} = 2(1 + \sigma) I_x I_y,$$

and

$$(53.4) \qquad \begin{cases} x_{cf} = \dfrac{1}{2(1+\sigma) I_x} \displaystyle\iint_R \left[x \frac{\partial \varphi_2}{\partial y} - y \frac{\partial \varphi_2}{\partial x} - (1 + \sigma) x y^2 + \sigma x^3 \right] dx\, dy, \\ y_{cf} = 0, \end{cases}$$

which state that the center of flexure lies on the axis of symmetry of the section. Accordingly, if the cross section has two perpendicular axes of symmetry, then the center of flexure coincides with the centroid of the section.

In general, the center of flexure does not lie on either of the principal axes and may even be outside the cross section of the beam.[1]

The solution of the simple flexure problem is given by the harmonic functions φ_1 and φ_2, which satisfy the conditions (52.16) on the boundary. Simpler boundary conditions can be realized by subdividing the problem once more. We define the harmonic functions φ_{11}, φ_{12}, φ_{21}, φ_{22} by the relations

$$(53.5) \qquad \begin{cases} \varphi_1 = (1 + \sigma)\varphi_{11} - \sigma \varphi_{12}, \\ \varphi_2 = (1 + \sigma)\varphi_{22} + \sigma \varphi_{21}. \end{cases}$$

Equations (52.16) now become

$$(53.6) \qquad \begin{cases} (1 + \sigma) \dfrac{d\varphi_{11}}{d\nu} - \sigma \dfrac{d\varphi_{12}}{d\nu} = (1 + \sigma) x^2 \cos(x, \nu) - \sigma y^2 \cos(x, \nu), \\ (1 + \sigma) \dfrac{d\varphi_{22}}{d\nu} + \sigma \dfrac{d\varphi_{21}}{d\nu} = (1 + \sigma) y^2 \cos(y, \nu) - \sigma x^2 \cos(y, \nu). \end{cases}$$

[1] There is some confusion in the literature concerning the relation of the flexural center to the *center of twist*, the latter being defined as the point at rest in every cross section of the beam fixed at one end and twisted at the other by a couple. The center of flexure is sometimes vaguely defined as the point in the end section of a cantilever beam such that the load applied at that point produces "torsionless bending." There are different definitions of torsionless bending [E. Trefftz, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 15 (1935), pp. 220–225; J. N. Goodier, *Journal of the Aeronautical Science*, vol 11 (1944), pp. 272–280], and the confusion in the identification of the two centers generally stems from the failure to define torsionless bending and to specify the mode of fixing the beam. It is possible to define the center of flexure (also called the *center of shear*) and the center of twist so that both centers coincide. See A. Weinstein, *Quarterly of Applied Mathematics*, vol. 5 (1947), pp. 97–99. The centers of flexure for several beams with polygonal cross sections have been calculated by N. Kh. Arutiunyan and N. O. Gulkanyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 18 (1954), pp. 597–618.

We are at liberty to prescribe arbitrary boundary conditions on the individual functions φ_{ij}, subject only to the restriction that the relations (53.6) be satisfied on C. Boundary conditions that are simple in form and independent of the elastic constants of the material will be realized if it is required that the functions φ_{ij} satisfy conditions

$$(53.7)\qquad \begin{cases} \dfrac{d\varphi_{11}}{d\nu} = x^2 \cos(x, \nu) = \dfrac{d}{d\nu}\left(\dfrac{1}{3}x^3\right), \\ \dfrac{d\varphi_{22}}{d\nu} = y^2 \cos(y, \nu) = \dfrac{d}{d\nu}\left(\dfrac{1}{3}y^3\right), \\ \dfrac{d\varphi_{12}}{d\nu} = y^2 \cos(x, \nu) = \dfrac{d}{ds}\left(\dfrac{1}{3}y^3\right), \\ \dfrac{d\varphi_{21}}{d\nu} = -x^2 \cos(y, \nu) = \dfrac{d}{ds}\left(\dfrac{1}{3}x^3\right). \end{cases}$$

We introduce the conjugate harmonic functions ψ_{12}, ψ_{21} with

$$\frac{\partial\varphi_{12}}{\partial x} = \frac{\partial\psi_{12}}{\partial y}, \qquad \frac{\partial\varphi_{12}}{\partial y} = -\frac{\partial\psi_{12}}{\partial x},$$
$$\frac{\partial\varphi_{21}}{\partial x} = \frac{\partial\psi_{21}}{\partial y}, \qquad \frac{\partial\varphi_{21}}{\partial y} = -\frac{\partial\psi_{21}}{\partial x},$$

and in terms of these functions the last two boundary conditions can be written as

$$\frac{d\psi_{12}}{ds} = \frac{d}{ds}\left(\frac{1}{3}y^3\right), \qquad \frac{d\psi_{21}}{ds} = \frac{d}{ds}\left(\frac{1}{3}x^3\right),$$

or

$$(53.8)\qquad \psi_{12} = \tfrac{1}{3}y^3 + \text{const}, \qquad \psi_{21} = \tfrac{1}{3}x^3 + \text{const} \qquad \text{on } C.$$

The solution of the simple flexure problem in which the applied load $(W_x, W_y, 0)$ acts at the center of flexure (with $\alpha = 0$) is thus given by the stresses

$$\tau_{xx} = \tau_{xy} = \tau_{yy} = 0,$$
$$\tau_{zz} = -E(l - z)(K_x x + K_y y),$$
$$\tau_{xz} = \mu K_x\left[(1 + \sigma)\left(\frac{\partial\varphi_{11}}{\partial x} - x^2\right) - \sigma\left(\frac{\partial\psi_{12}}{\partial y} - y^2\right)\right] + \mu K_y\left[(1 + \sigma)\frac{\partial\varphi_{22}}{\partial x} + \sigma\frac{\partial\psi_{21}}{\partial y}\right],$$
$$\tau_{zy} = \mu K_y\left[(1 + \sigma)\left(\frac{\partial\varphi_{22}}{\partial y} - y^2\right) - \sigma\left(\frac{\partial\psi_{21}}{\partial x} - x^2\right)\right] + \mu K_x\left[(1 + \sigma)\frac{\partial\varphi_{11}}{\partial y} + \sigma\frac{\partial\psi_{12}}{\partial x}\right],$$

where

$$EK_x = \frac{I_x W_x - I_{xy} W_y}{I_x I_y - I_{xy}^2}, \qquad EK_y = \frac{I_y W_y - I_{xy} W_x}{I_x I_y - I_{xy}^2},$$

and where the harmonic functions φ_{11}, φ_{22}, ψ_{12}, ψ_{21} satisfy the boundary conditions

$$\frac{d\varphi_{11}}{d\nu} = x^2 \cos(x, \nu), \qquad \frac{d\varphi_{22}}{d\nu} = y^2 \cos(y, \nu),$$
$$\psi_{21} = \tfrac{1}{3}x^3 + \text{const}, \qquad \psi_{12} = \tfrac{1}{3}y^3 + \text{const} \qquad \text{on } C.$$

The connection between the functions employed in this section and the classical Saint-Venant flexure function is discussed in the next section.

54. Bending by a Load along a Principal Axis. The general problem considered in the last two sections will now be specialized to an important particular case, namely, that in which the axes are taken to be the principal axes of the section and the load $(W_x, 0, 0)$ is directed parallel to one of these axes. In this case, $I_{xy} = 0$, and Eqs. (52.21) yield

$$K_x = \frac{W_x}{EI_y} = \frac{W_x}{2\mu(1+\sigma)I_y}, \qquad K_y = 0,$$

while Eqs. (52.13) become

$$\tau_{zx} = \mu\alpha\left(\frac{\partial\varphi}{\partial x} - y\right) + \frac{W_x}{2(1+\sigma)I_y}\left[\frac{\partial\varphi_1}{\partial x} - (1+\sigma)x^2 + \sigma y^2\right],$$
$$\tau_{zy} = \mu\alpha\left(\frac{\partial\varphi}{\partial y} + x\right) + \frac{W_x}{2(1+\sigma)I_y}\frac{\partial\varphi_1}{\partial y}.$$

The flexure function $\varphi_1(x, y)$ is not of the same form as the classical Saint-Venant flexure function $\Phi(x, y)$ used by most writers; the two functions (together with their harmonic conjugates ψ_1 and Ψ) are related, in fact, by the expression

$$\varphi_1 + i\psi_1 = -(\Phi + i\Psi) + \tfrac{1}{3}(1 + \tfrac{1}{2}\sigma)(x+iy)^3,$$

or

$$(54.1) \qquad \begin{cases} \varphi_1 = -\Phi + \tfrac{1}{3}(1+\tfrac{1}{2}\sigma)(x^3 - 3xy^2), \\ \psi_1 = -\Psi + \tfrac{1}{3}(1+\tfrac{1}{2}\sigma)(3x^2y - y^3). \end{cases}$$

In terms of the harmonic function $\Phi(x, y)$, the stresses can be written as

$$(54.2) \qquad \begin{cases} \tau_{xx} = \tau_{xy} = \tau_{yy} = 0, \\ \tau_{zz} = -\dfrac{W_x}{I_y}(l-z)x, \\ \tau_{zx} = \mu\alpha\left(\dfrac{\partial\varphi}{\partial x} - y\right) - \dfrac{W_x}{2(1+\sigma)I_y}\left[\dfrac{\partial\Phi}{\partial x} + \dfrac{1}{2}\sigma x^2 + \left(1 - \dfrac{1}{2}\sigma\right)y^2\right], \\ \tau_{zy} = \mu\alpha\left(\dfrac{\partial\varphi}{\partial y} + x\right) - \dfrac{W_x}{2(1+\sigma)I_y}\left[\dfrac{\partial\Phi}{\partial y} + (2+\sigma)xy\right]. \end{cases}$$

From (54.1) and from the boundary condition (52.16) on the function φ_1, it follows that the harmonic function Φ must satisfy the condition

$$(54.3)\quad \frac{d\Phi}{d\nu} = -\left[\frac{1}{2}\sigma x^2 + \left(1 - \frac{1}{2}\sigma\right) y^2\right] \cos (x, \nu) - (2 + \sigma)xy \cos (y, \nu) \qquad \text{on } C.$$

This special case of the general flexure problem has been formulated in terms of the torsion function φ and the flexure function Φ as a problem of Neumann. It may be rephrased as a problem of Dirichlet by writing the stresses in terms of the conjugate harmonic functions ψ and Ψ. The appropriate boundary condition on Ψ is seen from (52.17) and (54.1) to be

$$(54.4)\quad \Psi = -\tfrac{1}{3}(1 - \tfrac{1}{2}\sigma)y^3 + (1 + \tfrac{1}{2}\sigma)x^2y - (1 + \sigma) \int_{(x_0,y_0)}^{(x,y)} x^2\, dy + \text{const}$$
$$= -\tfrac{1}{3}(1 - \tfrac{1}{2}\sigma)y^3 - \tfrac{1}{2}\sigma x^2y + 2(1 + \sigma) \int_{(x_0,y_0)}^{(x,y)} xy\, dx + \text{const} \qquad \text{on } C,$$

where the last step makes use of integration by parts, and where the line integral is to be evaluated along the contour C.

The y-coordinate of the center of flexure is found from (53.3) and (54.1) to be given by

$$(54.5)\quad y_{cf} = \frac{1}{2(1 + \sigma)I_y} \iint\limits_R \left(x \frac{\partial \Phi}{\partial y} - y \frac{\partial \Phi}{\partial x}\right) dx\, dy + \frac{1}{2(1 + \sigma)I_y} \iint\limits_R \left[\left(2 + \frac{1}{2}\sigma\right) x^2y - \left(1 - \frac{1}{2}\sigma\right) y^3\right] dx\, dy$$
$$= \frac{1}{2(1 + \sigma)I_y} \iint\limits_R \left(y \frac{\partial \varphi_1}{\partial x} - x \frac{\partial \varphi_1}{\partial y}\right) dx\, dy - \frac{1}{2(1 + \sigma)I_y} \iint\limits_R [(1 + \sigma)x^2y - \sigma y^3]\, dx\, dy.$$

Since we have set $W_y = 0$, the x-coordinate of the flexural center cannot be determined from (53.1); that is, the mean twist over every section will vanish, provided the load W_x is applied at any point along the line $y = y_{cf}$.

55. The Displacement in a Bent Beam. In this section, expressions for the displacement components u, v, w will be given in terms of the torsion function φ and the flexure functions φ_1, φ_2. Some conclusions about the state of deformation can then be drawn from these expressions without explicitly determining the functions φ, φ_1, φ_2. The procedure is to substitute the expressions for the stresses found in Sec. 52 in formulas (29.2) and to carry out the integrations in a manner analogous to that used in Secs. 31 and 32. Since the calculation presents no points of interest, we shall merely list the final results and it is a simple matter to verify that the formulas for the components of displacement lead to the expressions for the stresses found in the preceding section.

In the case of the general flexure problem, discussed in Sec. 52, the expressions for the components of displacement are:

$$
(55.1)\quad \begin{cases}
u = -\alpha yz + K_x[\tfrac{1}{2}\sigma(l - z)(x^2 - y^2) - \tfrac{1}{6}z^3 + \tfrac{1}{2}lz^2] \\
\qquad\qquad + K_y\sigma(l - z)xy - cy + bz + a', \\
v = \alpha xz + K_y[\tfrac{1}{2}\sigma(l - z)(y^2 - x^2) - \tfrac{1}{6}z^3 + \tfrac{1}{2}lz^2] \\
\qquad\qquad + K_x\sigma(l - z)xy + cx - az + b', \\
w = \alpha\varphi(x, y) - bx + ay + c' \\
\qquad + K_x[\varphi_1(x, y) - (lz - \tfrac{1}{2}z^2)x - \tfrac{1}{6}(2 + \sigma)x^3 + \tfrac{1}{2}\sigma xy^2] \\
\qquad\quad + K_y[\varphi_2(x, y) - (lz - \tfrac{1}{2}z^2)y - \tfrac{1}{6}(2 + \sigma)y^3 + \tfrac{1}{2}\sigma x^2y].
\end{cases}
$$

The linear terms that arise in deriving Eqs. (55.1) represent a rigid body displacement and can be made to vanish by imposing suitable conditions of fixity.

When the flexure problem is specialized to the case of bending by a load $(W_x, 0, 0)$ along a principal axis (Sec. 54), then Eqs. (55.1) take the form

$$
(55.2)\quad \begin{cases}
u = -\alpha yz + \dfrac{W_x}{EI_y}\left[\dfrac{1}{2}\sigma(l - z)(x^2 - y^2) - \dfrac{1}{6}z^3 + \dfrac{1}{2}lz^2\right], \\
v = \alpha xz + \dfrac{W_x}{EI_y}\sigma(l - z)xy, \\
w = \alpha\varphi(x, y) - \dfrac{W_x}{EI_y}\left[\Phi(x, y) + xy^2 + \left(lz - \dfrac{1}{2}z^2\right)x\right],
\end{cases}
$$

where the function Φ is defined in Sec. 54.

The linear terms in (55.2) were made to vanish by fixing the end of the beam. Thus if the origin (0, 0, 0) is fixed and $\Phi(x, y)$ is chosen so that $\Phi(0, 0) = 0$, then $a' = b' = c' = 0$. If, in addition, an element of the z-axis is fixed at the origin, then $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ at (0, 0, 0), and if an element of the plane $x = 0$ is fixed there, then $\dfrac{\partial u}{\partial z} = 0$. These give $a = b = c = 0$.

Some interesting conclusions regarding the state of deformation can be drawn directly from Eqs. (55.1). We note first that points $(0, 0, z)$ lying on the central line of the undeformed beam are carried into points (x', y', z'), with

$$
(55.3)\quad \begin{cases}
x' = u = K_x(-\tfrac{1}{6}z^3 + \tfrac{1}{2}lz^2), \\
y' = v = K_y(-\tfrac{1}{6}z^3 + \tfrac{1}{2}lz^2);
\end{cases}
$$

that is, the deformed central line of the beam lies in the *plane of bending*

$$
(55.4)\quad y = \frac{K_y}{K_x}x = \frac{I_yW_y - I_{xy}W_x}{I_xW_x - I_{xy}W_y}x.
$$

The greatest deflection of the central line of the beam occurs at the loaded end $z = l$, where

$$
u = \frac{1}{3}K_xl^3 = \frac{1}{3}\frac{I_xW_x - I_{xy}W_y}{E(I_xI_y - I_{xy}^2)}l^3,
$$

$$
v = \frac{1}{3}K_yl^3 = \frac{1}{3}\frac{I_yW_y - I_{xy}W_x}{E(I_xI_y - I_{xy}^2)}l^3.
$$

If the axes are principal axes of a cross section, then

$$u = \frac{1}{3}\frac{W_x}{EI_y}l^3, \qquad v = \frac{1}{3}\frac{W_y}{EI_x}l^3,$$

while, for bending by a load W_x along a principal axis, the end deflection is

$$u = \frac{1}{3}\frac{W_x}{EI_y}l^3$$

The *plane of the load* (the plane containing the z-axis and the line in the direction of the load) does not, in general, coincide with the plane of bending, since the equation of the former is

$$y = \frac{W_y}{W_x}x.$$

The *neutral plane* is defined as that plane whose filaments are not altered in length; that is, it is characterized by the equation $e_{zz} = 0$. Since

$$e_{zz} = \frac{\partial w}{\partial z} = -K_x(l - z)x - K_y(l - z)y,$$

we have as the equation of the neutral plane

$$y = -\frac{K_x}{K_y}x. \tag{55.5}$$

The planes defined by (55.4) and (55.5) are orthogonal, and hence the neutral plane is perpendicular to the plane of bending.

In the case of bending by a load $(W_x, 0, 0)$ along a principal axis (Sec. 54), the xz-plane contains the deformed central line, while the yz-plane is the neutral plane.

Consider now the curvature of the deformed central line of the beam. Taking coordinates $r = \sqrt{x'^2 + y'^2}$ and z in the plane of bending, we have from (55.3)

$$r = \sqrt{K_x^2 + K_y^2}\,(-\tfrac{1}{6}z^3 + \tfrac{1}{2}lz^2).$$

If the displacements and their derivatives are small, one can write

$$r = \sqrt{K_x^2 + K_y^2}\,(-\tfrac{1}{6}z'^3 + \tfrac{1}{2}lz'^2),$$

from which it follows that the curvature of the central line is given approximately by

$$\frac{1}{R} = \frac{d^2r}{dz'^2} = \sqrt{K_x^2 + K_y^2}\,(l - z').$$

That the curvature is proportional to the bending moments M_x, M_y is easily seen by referring to Sec. 52, where it was found that

$$M_x = \iint\limits_R y\tau_{zz}\, dx\, dy = -(l - z)W_y,$$

$$M_y = \iint\limits_R - x\tau_{zz}\, dx\, dy = (l - z)W_x,$$

and hence

$$M_x = \frac{-W_y}{R\sqrt{K_x^2 + K_y^2}}, \qquad M_y = \frac{W_x}{R\sqrt{K_x^2 + K_y^2}}.$$

For the case of bending by a load W_x along a principal axis (Sec. 54), these relations become

$$M_x = 0, \qquad M_y = \frac{EI_y}{R}.$$

Thus, the Bernoulli-Euler law is also valid in the case of bending of beams by transverse end loads.

The changes in the cross section of the beam are determined from a study of the terms in u, v, and w that are independent of the twist α, and one can carry out an analysis similar to that given in Sec. 32. The neutral plane is deformed into a saddle-shaped surface, of which the central line is one of the principal lines of curvature. The cross sections $z = c$ of the beam do not remain plane even when the term $\alpha\varphi(x, y)$, which is due to the twisting of the beam by the load, disappears. This can be seen by examining the equation

$$\begin{aligned}(55.6)\quad z' = c + w = c &+ \alpha\varphi(x, y) \\ &+ K_x[\varphi_1(x, y) - (lc - \tfrac{1}{2}c^2)x - \tfrac{1}{6}(2 + \sigma)x^3 + \tfrac{1}{2}\sigma x y^2] \\ &+ K_y[\varphi_2(x, y) - (lc - \tfrac{1}{2}c^2)y - \tfrac{1}{6}(2 + \sigma)y^3 + \tfrac{1}{2}\sigma x^2 y].\end{aligned}$$

For the special case considered in Sec. 54, this takes the form

$$(55.7)\quad z' = c + \alpha\varphi(x, y) - \frac{W_x}{EI_y}\left[\Phi(x, y) + xy^2 + \left(lc - \frac{1}{2}c^2\right)x\right].$$

The nature of the distortion of cross sections and the distribution of stresses can be discussed with more profit after the solutions of the flexure problem for specific cross sections have been deduced. It is not difficult, however, to write down the differential equation of the lines of shearing stress. The directions of these lines are given by the equation

$$\frac{dy}{dx} = \frac{\tau_{zy}}{\tau_{zx}},$$

so that, disregarding the terms in (52.13) that depend on α, we have the differential equation

$$\begin{aligned}(55.8)\quad &\left\{K_y\left[\frac{\partial\varphi_2}{\partial y} - y^2 - \sigma(y^2 - x^2)\right] + K_x\frac{\partial\varphi_1}{\partial y}\right\}dx \\ &\qquad - \left\{K_x\left[\frac{\partial\varphi_1}{\partial x} - x^2 - \sigma(x^2 - y^2)\right] + K_y\frac{\partial\varphi_2}{\partial x}\right\}dy = 0.\end{aligned}$$

For the special case of bending by a load along a principal axis, this becomes

$$\left[2\frac{\partial\Phi}{\partial y} + (4 + 2\sigma)xy\right]dx - \left[2\frac{\partial\Phi}{\partial x} + \sigma x^2 + (2 - \sigma)y^2\right]dy = 0. \tag{55.9}$$

This equation will be used to determine the distribution of lines of shearing stress in a bent circular beam.

56. Flexure of Circular and Elliptical Beams. Let the equation of the boundary of cross section of a beam of length l be

$$x^2 + y^2 = a^2,$$

and let the terminal load W be applied at the centroid of the end section and directed along the x-axis. The form of the boundary suggests the use of polar coordinates (r, θ). In terms of these coordinates, the equation of the boundary assumes the simple form $r = a$, and the boundary condition (54.4) becomes

$$\begin{aligned}\Psi = &-\tfrac{1}{3}(1 - \tfrac{1}{2}\sigma)a^3 \sin^3\theta - \tfrac{1}{2}\sigma a^3 \cos^2\theta \sin\theta \\ &- 2(1 + \sigma)a^3 \textstyle\int \sin^2\theta \cos\theta \, d\theta,\end{aligned}$$

or

$$\Psi = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)a^3 \sin\theta + \tfrac{1}{4}a^3 \sin 3\theta, \qquad \text{on } r = a.$$

Since the function Ψ is harmonic in the interior of the circle $r = a$, the appropriate particular solutions of the equation $\nabla^2\Psi = 0$ in polar coordinates are of the form $r^n \sin n\theta$. Hence we must have

$$\Psi = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)a^2 r \sin\theta + \tfrac{1}{4}r^3 \sin 3\theta,$$

while the conjugate flexure function is

$$\Phi = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)a^2 r \cos\theta + \tfrac{1}{4}r^3 \cos 3\theta.$$

Recalling that $x = r\cos\theta$, $y = r\sin\theta$, we get

$$\Phi(x, y) = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)a^2 x + \tfrac{1}{4}(x^3 - 3xy^2). \tag{56.1}$$

From the symmetry of the cross section, it is seen that the center of flexure coincides with the centroid of the end section, and as the load point has also been taken at the centroid, it follows that in this example $\alpha = 0$. The stress components are found from (54.2) to be

$$\left\{\begin{aligned}\tau_{xz} &= \frac{(3 + 2\sigma)W}{2\pi a^4(1 + \sigma)}\left(a^2 - x^2 - \frac{1 - 2\sigma}{3 + 2\sigma}y^2\right), \\ \tau_{yz} &= -\frac{(1 + 2\sigma)W}{\pi a^4(1 + \sigma)}xy, \\ \tau_{zz} &= -\frac{4W}{\pi a^4}(l - z)x.\end{aligned}\right. \tag{56.2}$$

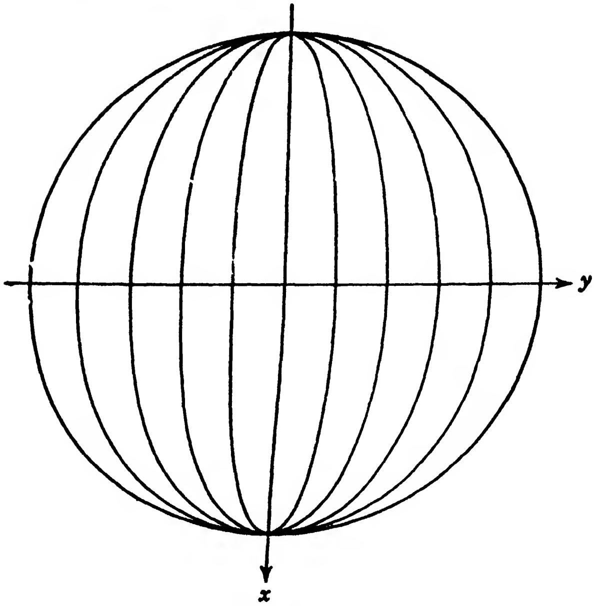

FIG. 43

Along the diameter $x = 0$,

$$(56.3) \qquad \tau_{zy} = 0, \qquad \tau_{zx} = \frac{(3 + 2\sigma)W}{2\pi a^4(1 + \sigma)}\left(a^2 - \frac{1 - 2\sigma}{3 + 2\sigma}y^2\right),$$

and it is evident that τ_{zx} takes its maximum value at the center of the circle, where

$$(\tau_{zx})_{\max} = \frac{3 + 2\sigma}{2(1 + \sigma)}\frac{W}{\pi a^2}.$$

The shearing stress at the ends of this diameter is

$$(\tau_{zx})_{y=\pm a} = \frac{1 + 2\sigma}{1 + \sigma}\frac{W}{\pi a^2}$$

The distribution of the lines of shearing stress can be determined with the aid of Eq. (55.9) or directly from the defining relation

$$\frac{dy}{dx} = \frac{\tau_{zy}}{\tau_{zx}}.$$

The differential equation of the lines of shearing stress is easily found to be

$$2(1 + 2\sigma)xy\,dx - (3 + 2\sigma)\left(-a^2 + x^2 + \frac{1 - 2\sigma}{3 + 2\sigma}y^2\right)dy = 0,$$

the solution of which is given by

$$x^2 + y^2 = a^2 + cy^{\frac{3+2\sigma}{1+2\sigma}},$$

where c is an arbitrary constant. Several of these lines of shearing stress are indicated in Fig. 43 for $\sigma = 0.3$.

The distortion of cross sections is given by Eq. (55.7), which becomes in this case

$$z' - c = -\frac{W}{\pi a^4 E}[-(3 + 2\sigma)a^2 + 2c(2l - c)]x - \frac{W}{\pi a^4 E}(x^2 + y^2)x.$$

The linear term corresponds to a rigid rotation of the section $z = c$ about the y-axis, whereas the nonlinear terms represent the distortion of the section $z = c$ out of a plane. The contour lines of the section are given by

$$-\frac{W}{\pi a^4 E}(x^2 + y^2)x = \text{const.}$$

Some of the contour lines are shown in Fig. 44.

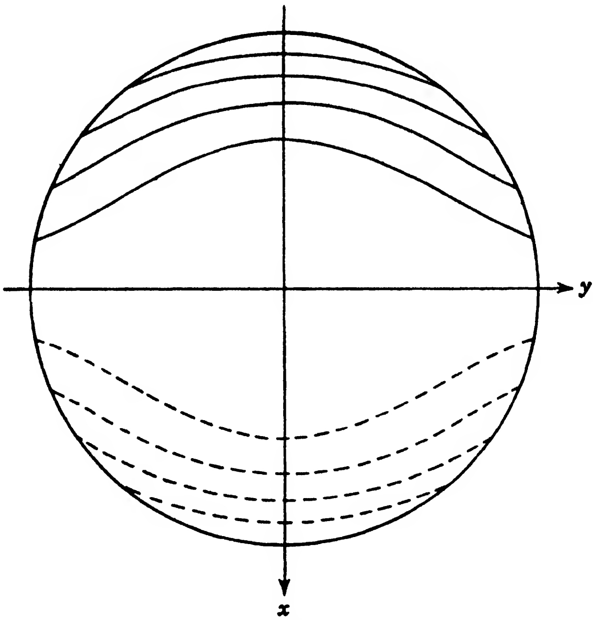

FIG. 44

An analysis similar to that used in the preceding section can be applied to determine the flexure function for a beam whose cross section is given by the equation (see also Sec. 60),

(56.4) $$f(x, y) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

We assume, as above, that the load acts in the direction of the x-axis and is applied at the centroid of the end section.

Now the direction cosines $\cos(x, \nu)$ and $\cos(y, \nu)$ of the normal to the boundary of the ellipse are proportional to $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, respectively. Hence

$$\frac{\cos(x, \nu)}{\cos(y, \nu)} = \frac{x/a^2}{y/b^2}.$$

On the boundary C of the section, the flexure function Φ satisfies the condition (54.3); hence

$$\frac{\partial \Phi}{\partial x} b^2 x + \frac{\partial \Phi}{\partial y} a^2 y = -\left[\frac{1}{2}\sigma x^2 + \left(1 - \frac{1}{2}\sigma\right) y^2\right] b^2 x - (2+\sigma) a^2 x y^2 \qquad \text{on } C. \tag{56.5}$$

Since the right-hand member of (56.5) is a homogeneous polynomial in x and y, it is natural to seek a solution in the form of the sum of integral harmonics. Assuming

$$\Phi + i\Psi = c_1(x + iy) + c_2(x + iy)^3,$$

that is,

$$\Phi = c_1 x + c_2(x^3 - 3xy^2), \tag{56.6}$$

and substituting in the boundary condition (56.5) gives

$$[c_1 + 3c_2(x^2 - y^2)]b^2 - 6c_2 a^2 y^2 = -[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2]b^2 - (2 + \sigma)a^2 y^2.$$

But on the boundary of the section,

$$x^2 = a^2 - \frac{a^2 y^2}{b^2},$$

and the preceding equation demands that

$$c_1 = \frac{-a^2}{3a^2 + b^2}[2(1 + \sigma)a^2 + b^2],$$
$$c_2 = \frac{(2 + \tfrac{1}{2}\sigma)a^2 + (1 - \tfrac{1}{2}\sigma)b^2}{9a^2 + 3b^2}.$$

The expression (56.6) for the flexure function now becomes

$$\Phi = -\frac{a^2[2(1 + \sigma)a^2 + b^2]}{3a^2 + b^2} x + \frac{2a^2 + b^2 + \tfrac{1}{2}\sigma(a^2 - b^2)}{3(3a^2 + b^2)}(x^3 - 3xy^2). \tag{56.7}$$

Calculating stresses with the aid of formulas (54.2), we find that

$$\begin{cases} \tau_{zx} = \dfrac{2W}{\pi a^3 b} \dfrac{2(1 + \sigma)a^2 + b^2}{(1 + \sigma)(3a^2 + b^2)} \left[a^2 - x^2 - \dfrac{(1 - 2\sigma)a^2}{2(1 + \sigma)a^2 + b^2} y^2\right], \\ \tau_{zy} = -\dfrac{4W}{\pi a^3 b} \dfrac{(1 + \sigma)a^2 + \sigma b^2}{(1 + \sigma)(3a^2 + b^2)} xy. \end{cases} \tag{56.8}$$

It is obvious from these formulas that the y-component, τ_{zy}, of the shearing stress vanishes on the horizontal axis ($x = 0$) of the cross section and that

$$(\tau_{zx})_{x=0} = \frac{2W}{\pi a^3 b}\frac{2(1+\sigma)a^2+b^2}{(1+\sigma)(3a^2+b^2)}\left[a^2 - \frac{(1-2\sigma)a^2}{2(1+\sigma)a^2+b^2}y^2\right].$$

Hence τ_{zx} takes its maximum value at the center of the ellipse where

$$(\tau_{zx})_{\max} = \frac{2W}{A}\frac{2(1+\sigma)a^2+b^2}{(1+\sigma)(3a^2+b^2)},$$

and $A = \pi ab$ is the area of the cross section. Evidently τ_{zy} reaches its maximum on the boundary, and if we put $x = a\cos\theta$, $y = b\sin\theta$, then $(xy)_{\max} = (\frac{1}{2}ab \sin 2\theta)_{\max} = \frac{1}{2}ab$, and it is seen that

$$(\tau_{zy})_{\max} = \frac{2W}{A}\frac{b}{a}\frac{(1+\sigma)a^2+\sigma b^2}{(1+\sigma)(3a^2+b^2)}.$$

If $b \ll a$, then the shape of the beam approaches that of a thin rectangular plank loaded parallel to its longer side. In this case, neglecting terms of order b^2/a^2, we get

$$(\tau_{zx})_{\max} \doteq \frac{4W}{3A}, \qquad (\tau_{zy})_{\max} \doteq \frac{4W}{3A}\frac{b}{2a}, \qquad b \ll a,$$

so that $\tau_{zy} \ll \tau_{zx}$. On the other hand, if $a \ll b$, then the load acts along the shorter axis, and

$$(\tau_{zx})_{\max} \doteq \frac{2}{1+\sigma}\frac{W}{A}, \qquad (\tau_{zy})_{\max} \doteq \frac{2}{1+\sigma}\frac{W}{A}\frac{\sigma b}{a}, \qquad a \ll b.$$

57. Bending of Rectangular Beams. The problems in the preceding section illustrate the solution of the boundary-value problems of flexure by forming those combinations of particular solutions of the differential equation

$$\nabla^2\Phi = 0 \qquad \text{or} \qquad \nabla^2\Psi = 0$$

that satisfy the boundary conditions on the function Φ or Ψ. The flexure problem of circular beams was treated by inspection of the boundary values of Ψ in polar coordinates and by utilizing the particular solutions of the form $r^n \sin n\theta$. Beams of elliptical cross section were handled by observing that the boundary condition on $\frac{d\Phi}{d\nu}$ involved only homogeneous polynomials in x and y, and this fact suggested that a solution for the complex flexure function $\Phi + i\Psi$ be sought as a sum of terms of the form $c_n(x + iy)^n$. In this section, the solution of the flexure problem for a beam of rectangular cross section is given as an infinite series of particular solutions $A_n \sinh \alpha x \cos \beta y$, the coefficients A_n being so chosen as to ensure the satisfaction of the boundary conditions. The

next two sections will illustrate the use of analytic functions in solving the flexure problems.

Let the equation of the boundary of the cross section of the beam be

$$(x^2 - a^2)(y^2 - b^2) = 0,$$

and let the terminal load be directed along the positive x-axis and applied at the origin.

A reference to the boundary conditions (54.3) shows that, on the sides $x = \pm a$, we must have

$$\frac{\partial \Phi}{\partial x} = -\frac{1}{2}\sigma a^2 - \left(1 - \frac{1}{2}\sigma\right) y^2, \qquad -b < y < b,$$

while, on the sides $y = \pm b$, we must satisfy the condition

$$\frac{\partial \Phi}{\partial y} = \mp (2 + \sigma) bx, \qquad -a < x < a.$$

In order to simplify the boundary conditions, we define a harmonic function $f(x, y)$ by the relation

$$f(x, y) = \Phi(x, y) - \tfrac{1}{6}(2 + \sigma)(x^3 - 3xy^2);$$

then the boundary conditions to be satisfied by the function $f(x, y)$ are

$$\frac{\partial f}{\partial x} = -(1 + \sigma)a^2 + \sigma y^2, \qquad \text{on } x = \pm a,$$

$$\frac{\partial f}{\partial y} = 0, \qquad \text{on } y = \pm b.$$

It follows from the discussion in Sec. 38 that one can build up the desired solution by forming an infinite series of particular solutions

$$f(x, y) = Ax + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}.$$

The boundary condition on $y = \pm b$ is satisfied by each term of the series, while the satisfaction of the boundary condition on $x = \pm a$ is readily effected by noting the expansion

$$y^2 = \frac{b^2}{3} + \frac{4b^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi y}{b}, \qquad -b \le y \le b.$$

The condition on the boundary $x = \pm a$ now takes the form

$$A + \sum_{n=1}^{\infty} \frac{n\pi}{b} A_n \cosh \frac{n\pi a}{b} \cos \frac{n\pi y}{b}$$
$$= -(1 + \sigma)a^2 + \sigma \left[\frac{b^2}{3} + \frac{4b^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi y}{b}\right],$$

and equating the coefficients of cos $(n\pi y/b)$ leads to the result

$$f(x, y) = \left[-(1 + \sigma)a^2 + \frac{1}{3}\sigma b^2 \right] x + \frac{4\sigma b^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{\sinh \frac{n\pi x}{b}}{\cosh \frac{n\pi a}{b}} \cos \frac{n\pi y}{b}.$$

The flexure function $\Phi(x, y)$ can now be found from the relation

$$\Phi(x, y) = f(x, y) + \tfrac{1}{6}(2 + \sigma)(x^3 - 3xy^2).$$

We shall dispense with the calculation of shear stresses. An elaborate discussion of the bending problem for a rectangular beam is given by Timoshenko,[1] who, however, approaches the problem in an entirely different way, by using an analogy between a certain stress function and the deflection of a stretched membrane under nonuniform pressure. We shall discuss this analogy in Sec. 60.

58. Conformal Mapping and the General Problem of Flexure; the Cardioid Section. The examples considered in the preceding sections give illustrations of the specialized problem of flexure by a load $(W_x, 0, 0)$ directed along a principal axis (which was also one of the two axes of symmetry) of the cross section of the beam. The analysis was also simplified by taking the centroid of the section as the point of application of the load. We consider now, as an illustration of the general problem of flexure, the problem of bending of a beam whose cross section is bounded by a cardioid and thus has only one axis of symmetry. The load $(W_x, W_y, 0)$ will be considered to act at some point (x_0, y_0, l) of the end section, and the origin of coordinates will be taken in the fixed end at the centroid of the cardioid. Upon this problem we shall bring to bear the powerful weapon of analytic function theory, which was used earlier in the case of torsion of a beam.[2]

In Sec. 53, it was seen that the general problem of flexure by a load $(W_x, W_y, 0)$ acting at any point (x_0, y_0, l) can be resolved into (1) a simpler flexure problem with α, the mean local twist, set equal to zero, and with the load applied at the center of flexure (x_{cf}, y_{cf}, l), and (2) a torsion problem with a twist α due to a couple of moment $W_y(x_0 - x_{cf}) - W_x(y_0 - y_{cf})$. The problem of torsion of a cylinder was reduced in Sec. 35 to the boundary-value problem of finding the analytic function $\varphi(x, y) + i\psi(x, y)$ with

$$\psi = \tfrac{1}{2}(x^2 + y^2) \qquad \text{on the boundary } C.$$

[1] S. Timoshenko and J. N. Goodier, Theory of Elasticity, Sec. 109.

[2] For solutions of this problem by other methods, see W. M. Shepherd, *Proceedings of the Royal Society (London)*, (*A*), vol. 154 (1936), p. 500; A. C. Stevenson, "Flexure with Shear and Associated Torsion in Prisms of Uni-axial and Asymmetric Cross-sections," *Philosophical Transactions of the Royal Society (London)* (*A*), vol. 237 (1939), pp. 161–229; R. M. Morris, "Some General Solutions of St. Venant's Flexure and Torsion Problem I," *Proceedings of the London Mathematical Society*, ser. 2, vol. 46 (1940), pp. 81–98.

The simpler flexure problem (1) was seen in Sec. 53 to be equivalent to four boundary-value problems involving the determination of four analytic functions $\varphi_{11} + i\psi_{11}$, $\varphi_{22} + i\psi_{22}$, $\varphi_{12} + i\psi_{12}$, and $\varphi_{21} + i\psi_{21}$, such that

$$\left.\begin{aligned} \frac{d\varphi_{11}}{d\nu} &= x^2 \cos(x, \nu), & \frac{d\varphi_{22}}{d\nu} &= y^2 \cos(y, \nu), \\ \psi_{21} &= \tfrac{1}{3}x^3 + \text{const}, & \psi_{12} &= \tfrac{1}{3}y^3 + \text{const} \end{aligned}\right\} \quad \text{on } C.$$

The boundary conditions on the normal derivatives of φ_{11} and φ_{22} can be replaced by conditions on the boundary values of the conjugate functions ψ_{11} and ψ_{22} by noting that

$$\frac{d\varphi_{11}}{d\nu} = \frac{\partial\varphi_{11}}{\partial x}\frac{dx}{d\nu} + \frac{\partial\varphi_{11}}{\partial y}\frac{dy}{d\nu} = \frac{\partial\psi_{11}}{\partial y}\frac{dy}{ds} + \frac{\partial\psi_{11}}{\partial x}\frac{dx}{ds} = \frac{d\psi_{11}}{ds}.$$

The condition on ψ_{11} now becomes

$$\frac{d\psi_{11}}{ds} = x^2 \cos(x, \nu) = x^2 \cos(y, s) \qquad \text{on } C,$$

or

$$\psi_{11} = \int x^2 \frac{dy}{ds}\, ds = \int x^2\, dy \qquad \text{on } C.$$

Similarly, we have

$$\psi_{22} = -\int y^2\, dx \qquad \text{on } C.$$

The general problem of flexure is thus made to depend on the solution of five boundary-value problems of Dirichlet for the conjugate torsion function ψ and the four flexure functions ψ_{11}, ψ_{22}, ψ_{21}, and ψ_{12}.

We have already seen, in Sec. 44, how to solve the boundary-value problem

$$\nabla^2\psi = 0 \qquad \text{in } R, \qquad \psi = \tfrac{1}{2}(x^2 + y^2) \qquad \text{on } C,$$

by mapping the region R on the interior of the unit circle $|\zeta| \leq 1$ and applying the formula of Schwarz [Eq. (42.4)]. The same procedure can be used, of course, to write down the solution, in the form of an integral, for any problem of Dirichlet for any region that can be mapped conformally on the interior of a unit circle. In particular, the Schwarz integral affords solutions for the boundary-value problems of flexure.

It is not difficult to express the boundary conditions imposed on the functions ψ_{ij} in Sec. 53 in terms of the complex variables $z = x + iy$ and $\bar{z} = x - iy$ or, when the mapping function $z = \omega(\zeta)$ is known, in terms of ζ and $\bar{\zeta}$. The application of the Schwarz formula (42.6) would then yield the complex flexure functions $\varphi_{ij} + i\psi_{ij}$ in a form analogous to (44.5). Because of the complicated form of the boundary conditions which the ψ_{ij} satisfy, the resulting general formulas are of doubtful value in specific

applications, and we dispense with recording them here.[1] Instead we illustrate in detail how the complex torsion and flexure functions can be determined in a specific problem by considering a beam whose cross section is a cardioid. The technique illustrated here can be used[2] to solve flexure problems for beams with cross sections considered in Sec. 45.

Consider a beam whose cross section is shown in Fig. 45, where the origin of the cartesian axes is at the centroid of the section. The polar and rectangular coordinates are related by

$$x = \frac{5c}{3} + r\cos t, \qquad y = r\sin t,$$

when the origin of polar coordinates (r, t) is taken at the cusp of the cardioid.

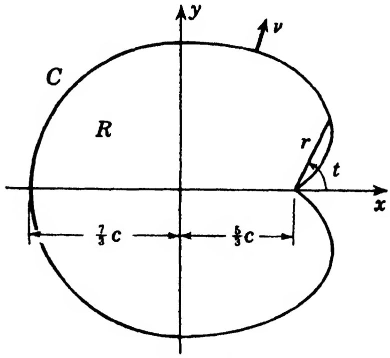

Fig. 45

The polar equation of the boundary C of the section is

$$r = 2c(1 - \cos t),$$

and we can write

$$(58.1) \qquad \begin{cases} x = \dfrac{5c}{3} + 2c\cos t(1 - \cos t) = \dfrac{5c}{3} - c + 2c\cos t - c\cos 2t, \\ y = 2c\sin t(1 - \cos t) = 2c\sin t - c\sin 2t. \end{cases}$$

Then

$$x + iy = \frac{5c}{3} - c(1 - 2e^{it} + e^{2it}) = \frac{5c}{3} - c(1 - e^{it})^2,$$

and it is seen that the analytic function

$$(58.2) \qquad z = \omega(\zeta) \equiv \frac{5c}{3} - c(1 - \zeta)^2,$$

with

$$z = x + iy = \frac{5c}{3} + re^{it}, \qquad \zeta = \xi + i\eta = \rho e^{i\theta},$$

maps the interior of the cardioid on the unit circle $|\zeta| \leq 1$, with $\theta = t$. The inverse transformation is

$$(58.3) \qquad \zeta = 1 + i\left(\frac{r}{c}\right)^{\frac{1}{2}} e^{it/2}, \qquad 0 \leq t \leq 2\pi.$$

[1] See S. Gosh, *Bulletin of the Calcutta Mathematical Society*, vol. 39 (1947), pp. 1–14.

[2] It should be noted, however, that a suitable choice of curvilinear coordinates may yield simpler solutions. Thus the flexure problem for a cylinder whose cross section is formed by the arcs of two intersecting circles of different radii, which includes the segment of the circle as a special case, is more easily solved in bipolar coordinates. See Ya. S. Uflyand, *Doklady Akademii Nauk SSSR*, New Series, vol. 69 (1949), pp. 751–754.

The mapping function can be written as

$$\omega(\sigma) = \frac{5c}{3} - c(1-\sigma)^2 = \frac{c}{3}(2 + 6\sigma - 3\sigma^2).$$

and we have

$$\omega(\sigma)\bar{\omega}\left(\frac{1}{\sigma}\right) = \frac{c^2}{9\sigma^2} f_1(\sigma),$$

where

$$f_1(\sigma) = -6 - 6\sigma + 49\sigma^2 - 6\sigma^3 - 6\sigma^4.$$

The complex torsion function is found from (44.5) to be given by [cf. (44.16)]

$$\varphi + i\psi = \frac{c^2}{18\pi}\int_\gamma \frac{f_1(\sigma)}{\sigma^2(\sigma - \zeta)}\,d\sigma = \frac{ic^2}{9}(R_1 + R_2),$$

where R_1 and R_2 are the residues of the integrand at $\sigma = 0$ and $\sigma = \zeta$, respectively, and where γ denotes the contour $|\zeta| = 1$. We have

$$R_1 = \frac{d}{d\sigma}\left[\frac{f_1(\sigma)}{\sigma - \zeta}\right]_{\sigma=0} = \frac{6}{\zeta} + \frac{6}{\zeta^2},$$

$$R_2 = \frac{f_1(\zeta)}{\zeta^2} = -\frac{6}{\zeta^2} - \frac{6}{\zeta} + 49 - 6\zeta - 6\zeta^2,$$

and

$$\varphi + i\psi = \frac{ic^2}{9}(49 - 6\zeta - 6\zeta^2)$$

$$= \frac{ic^2}{9}\left[37 - 18i\left(\frac{r}{c}\right)^{\frac{1}{2}} e^{it/2} + 6\frac{r}{c}e^{it}\right],$$

or

$$\varphi = 2r^{\frac{1}{2}}c^{\frac{3}{2}}\cos\frac{t}{2} - \frac{2}{3}rc\sin t,$$

$$\psi = 2r^{\frac{1}{2}}c^{\frac{3}{2}}\sin\frac{t}{2} + \frac{2}{3}rc\cos t.$$

Equation (44.8), for the moment of inertia I_0, takes the form

$$I_0 = \frac{1}{4i}\frac{2c^4}{27}\int_\gamma \frac{f_2(\sigma)}{\sigma^4}\,d\sigma,$$

where

$$f_2(\sigma) = 18 - 36\sigma - 177\sigma^2 + 495\sigma^3 - 220\sigma^4 - 128\sigma^5 + 36\sigma^6 + 12\sigma^7,$$

and hence

$$I_0 = \frac{\pi c^4}{27} R_3,$$

where

$$R_3 = \frac{1}{3!}\frac{d^3}{d\sigma^3}[f_2(\sigma)]_{\sigma=0} = 495$$

is the residue of the integrand at $\sigma = 0$. Thus,

$$I_0 = \frac{55\pi c^4}{3} \tag{58.4}$$

is the polar moment of inertia of the cross section.[1]

Similarly, Eq. (44.7) yields

$$D_0 = -\frac{1}{4}\frac{i4c^4}{9}\int_\gamma \frac{f_3(\sigma)}{\sigma^5}\,d\sigma,$$

with

$$f_3(\sigma) = 2 + 3\sigma + \sigma^2 - 3\sigma^3 - 6\sigma^4 - 3\sigma^5 + \sigma^6 + 3\sigma^7 + 2\sigma^8.$$

Then

$$D_0 = \frac{2\pi c^4}{9} R_4 = \frac{2\pi c^4}{9}\frac{1}{4!}\frac{d^4}{d\sigma^4}[f_3(\sigma)]_{\sigma=0} = -\frac{4\pi c^4}{3}.$$

The torsional rigidity D is

$$D = \mu(I_0 + D_0) = 17\mu\pi c^4.$$

The shearing stresses may be found either from Eq. (44.10) or from the relation

$$\tau_{zx} = \mu\alpha\frac{\partial\Psi}{\partial y}, \qquad \tau_{zy} = -\mu\alpha\frac{\partial\Psi}{\partial x},$$
$$\Psi = \psi - \tfrac{1}{2}(x^2 + y^2).$$

We turn now to the determination of the harmonic flexure function ψ_{21}, which takes the values

$$\psi_{21} = \tfrac{1}{3}x^3 + \text{const} \qquad \text{on } C.$$

From

$$\cos t = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right), \qquad \sigma = e^{i\theta} = e^{it}$$

and Eq. (58.1), we get

$$x = \frac{c}{3}[5 + 6(1 - \cos\theta)\cos\theta]$$
$$= \frac{c(-3 + 6\sigma + 4\sigma^2 + 6\sigma^3 - 3\sigma^4)}{6\sigma^2}$$

and

$$x^3 = \frac{c^3 f_4(\sigma)}{216\sigma^6},$$

with

$$f_4(\sigma) = -27 + 162\sigma - 216\sigma^2 - 54\sigma^3 - 441\sigma^4 + 828\sigma^5 + 496\sigma^6$$
$$+ 828\sigma^7 - 441\sigma^8 - 54\sigma^9 - 216\sigma^{10} + 162\sigma^{11} - 27\sigma^{12}.$$

[1] This result could have been obtained more simply by calculus. The detailed calculations included here are intended to illustrate the step-by-step procedures and to provide a review of the residue method of evaluating simple integrals. The reader versed in such matters is advised to omit the rest of this section

The boundary condition becomes

$$\psi_{21} = \frac{c^3 f_4(\sigma)}{648\sigma^6} \qquad \text{on } \gamma,$$

and since ψ_{21} is the real part of $(1/i)(\varphi_{21} + i\psi_{21})$, we have by the Schwarz integral (42.4)

$$\frac{1}{i}(\varphi_{21} + i\psi_{21}) = \frac{1}{\pi i}\frac{c^3}{648}\int_\gamma \frac{f_4(\sigma)}{\sigma^6(\sigma - \zeta)}\,d\sigma,$$

or

$$\varphi_{21} + i\psi_{21} = \frac{ic^3}{324}(R_5 + R_6),$$

where R_5 and R_6 are the residues of the integrand at $\sigma = 0$ and $\sigma = \zeta$, respectively. We have

$$\begin{aligned} R_5 &= \frac{1}{5!}\frac{d^5}{d\sigma^5}\left[\frac{f_4(\sigma)}{\sigma - \zeta}\right]_{\sigma=0} \\ &= -\frac{828}{\zeta} + \frac{441}{\zeta^2} + \frac{54}{\zeta^3} + \frac{216}{\zeta^4} - \frac{162}{\zeta^5} + \frac{27}{\zeta^6}, \end{aligned}$$

and

$$R_6 = \frac{f_4(\zeta)}{\zeta^6}.$$

Then

$$\varphi_{21} + i\psi_{21} = \frac{ic^3}{324}(248 + 828\zeta - 441\zeta^2 - 54\zeta^3 - 216\zeta^4 + 162\zeta^5 - 27\zeta^6)$$

With the help of the inverse transformation (58.3), we find

$$(58.5)\quad \left\{\begin{aligned} \psi_{21} &= \frac{1}{324}\Big(500c^3 + 432r^{1/2}c^{5/2}\sin\frac{t}{2} + 684rc^2\cos t \\ &\qquad + 162r^{3/2}c^{3/2}\sin\frac{3t}{2} + 189r^2c\cos 2t + 27r^3\cos 3t\Big), \\ \varphi_{21} &= \frac{1}{36}\Big(48r^{1/2}c^{5/2}\cos\frac{t}{2} - 76rc^2\sin t + 18r^{3/2}c^{3/2}\cos\frac{3t}{2} \\ &\qquad - 21r^2c\sin 2t - 3r^3\sin 3t\Big) \end{aligned}\right.$$

Similarly, the function ψ_{12} can be determined by noting that (58.1) and the relations

$$\cos t = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right), \qquad \sin t = \frac{1}{2i}\left(\sigma - \frac{1}{\sigma}\right),$$

yield

$$y = \frac{c(1 - 2\sigma + 2\sigma^3 - \sigma^4)}{2i\sigma^2},$$

$$\begin{aligned} y^3 &= \frac{ic^3}{8\sigma^6}(1 - 6\sigma + 12\sigma^2 - 2\sigma^3 - 27\sigma^4 + 36\sigma^5 - 36\sigma^7 + 27\sigma^8 + 2\sigma^9 \\ &\qquad - 12\sigma^{10} + 6\sigma^{11} - \sigma^{12}) \\ &= \frac{ic^3}{8\sigma^6}f_5(\sigma). \end{aligned}$$

The condition on the function ψ_{12} is

$$\psi_{12} = \frac{1}{3}\frac{ic^3}{8}\frac{f_5(\sigma)}{\sigma^6} \qquad \text{on } \gamma.$$

The Schwarz integral (42.4) yields the complex flexure function

$$\varphi_{12} + i\psi_{12} = \frac{ic^3}{24\pi}\int_\gamma \frac{f_5(\sigma)}{\sigma^6(\sigma - \zeta)}\,d\sigma = -\frac{c^3}{12}(R_7 + R_8),$$

where

$$R_7 = \frac{1}{5!}\frac{d^5}{d\sigma^5}\left[\frac{f_5(\sigma)}{\sigma - \zeta}\right]_{\sigma=0}, \qquad R_8 = \frac{f_5(\zeta)}{\zeta^6}.$$

Hence

$$\varphi_{12} + i\psi_{12} = \frac{c^3}{12}(36\zeta - 27\zeta^2 - 2\zeta^3 + 12\zeta^4 - 6\zeta^5 + \zeta^6),$$

from which it follows that

$$(58.6)\quad \begin{cases} \psi_{12} = \dfrac{1}{12}\left(6rc^2 \sin t - 6r^{3/2}c^{3/2}\cos\dfrac{3t}{2} - 3r^2c\sin 2t - r^3\sin 3t\right), \\ \varphi_{12} = \dfrac{1}{12}\left(14c^3 + 6rc^2\cos t + 6r^{3/2}c^{3/2}\sin\dfrac{3t}{2} - 3r^2c\cos 2t - r^3\cos 3t\right). \end{cases}$$

The boundary values of the function ψ_{11} are themselves given in terms of a line integral, which must first be evaluated. From (58.1) we get

$$x^2 = \frac{c^2}{36\sigma^4}(9 - 36\sigma + 12\sigma^2 + 12\sigma^3 + 106\sigma^4 + 12\sigma^5 + 12\sigma^6 - 36\sigma^7 + 9\sigma^8),$$

$$dy = \frac{ic}{\sigma^3}(1 - \sigma - \sigma^3 + \sigma^4)\,d\sigma,$$

and

$$\int x^2\,dy = \psi_{11}\Big|_\gamma = \frac{ic^3}{72}\frac{f_6(\sigma)}{\sigma^6},$$

with

$$f_6(\sigma) = -3 + 18\sigma - 24\sigma^2 + 6\sigma^3 - 139\sigma^4 + 284\sigma^5 - 284\sigma^7 + 139\sigma^8 - 6\sigma^9 + 24\sigma^{10} - 18\sigma^{11} + 3\sigma^{12}.$$

From (42.4) we get

$$\varphi_{11} + i\psi_{11} = \frac{ic^3}{72\pi}\int_\gamma \frac{f_6(\sigma)}{\sigma^6(\sigma - \zeta)}\,d\sigma = \frac{-c^3}{36}(R_9 + R_{10}),$$

where

$$R_9 = \frac{1}{5!}\frac{d^5}{d\sigma^5}\left[\frac{f_6(\sigma)}{\sigma - \zeta}\right]_{\sigma=0}, \qquad R_{10} = \frac{f_6(\zeta)}{\zeta^6},$$

and hence

$$\varphi_{11} + i\psi_{11} = \frac{c^3}{36}(284\zeta - 139\zeta^2 + 6\zeta^3 - 24\zeta^4 + 18\zeta^5 - 3\zeta^6),$$

and

$$(58.7)\quad \varphi_{11} = \frac{1}{36}\left(142c^3 + 130rc^2\cos t + 30r^{3/2}c^{3/2}\sin\frac{3t}{2} + 21r^2c\cos 2t + 3r^3\cos 3t\right).$$

The last flexure function ψ_{22} is found in a similar way. The boundary values are calculated by observing that

$$y^2 = 4c^2(1-\mu)^2(1-\mu^2), \qquad \mu = \cos t,$$
$$dx = 2c(1-2\mu)\,d\mu,$$

and

$$\int y^2\,dx = \frac{4c^3}{3}(6\mu - 12\mu^2 + 8\mu^3 + 3\mu^4 - 6\mu^5 + 2\mu^6).$$

But $\mu = \frac{1}{2}(\sigma + 1/\sigma)$, and hence

$$-\int y^2\,dx = \psi_{22}\Big|_\gamma = -\frac{c^3}{24}\frac{f_7(\sigma)}{\sigma^6},$$

where

$$f_7(\sigma) = 1 - 6\sigma + 12\sigma^2 + 2\sigma^3 - 57\sigma^4 + 132\sigma^5 - 136\sigma^6 + 132\sigma^7 - 57\sigma^8 + 2\sigma^9 + 12\sigma^{10} - 6\sigma^{11} + \sigma^{12}.$$

The Schwarz formula (42.4) yields

$$\varphi_{22} + i\psi_{22} = -\frac{c^3}{24\pi}\int_\gamma \frac{f_7(\sigma)}{\sigma^6(\sigma - \zeta)}\,d\sigma = -\frac{ic^3}{12}(R_{11} + R_{12}),$$

with

$$R_{11} = \frac{1}{5!}\frac{d^5}{d\sigma^5}\left[\frac{f_7(\sigma)}{\sigma - \zeta}\right]_{\sigma=0}, \qquad R_{12} = \frac{f_7(\zeta)}{\zeta^6},$$

or

$$\varphi_{22} + i\psi_{22} = \frac{ic^3}{12}(136 - 132\zeta + 57\zeta^2 - 2\zeta^3 - 12\zeta^4 + 6\zeta^5 - \zeta^6),$$

and

$$\varphi_{22} = \frac{1}{12}\left(48r^{1/2}c^{5/2}\cos\frac{t}{2} + 24rc^2\sin t - 10r^{3/2}c^{3/2}\cos\frac{3t}{2} - 3r^2c\sin 2t - r^3\sin 3t\right). \tag{58.8}$$

Before the stresses can be found, the constants K_x and K_y must first be evaluated. From (52.21) we get, since $I_{xy} = 0$,

$$K_x = \frac{W_x}{EI_y}, \qquad K_y = \frac{W_y}{EI_x}.$$

Now

$$I_x = \iint_R y^2\,dx\,dy = -\frac{1}{3}\int_C y^3\,dx,$$

and from (58.1) we have

$$y = 2c\sin t(1 - \cos t), \qquad dx = -2c\sin t(1 - 2\cos t)\,dt.$$

Integration of $y^3\,dx$ from $t = 0$ to $t = 2\pi$ yields

$$I_x = \frac{21\pi c^4}{2}.$$

The moment of inertia I_y is found with the aid of Eq. (58.4) to be

$$I_y = I_0 - I_z = \frac{55\pi c^4}{3} - \frac{21\pi c^4}{2} = \frac{47\pi c^4}{6}.$$

The stresses can now be found from formulas in Sec. 53.

The coordinates of the center of flexure will now be determined. The first term in the expression for x_{cf} in (53.4) becomes, with the help of Green's Theorem and Eqs. (53.5),

$$\iint_R x \frac{\partial \varphi_2}{\partial y}\, dx\, dy = \int_C -x\varphi_2\, dx$$
$$= \int_C -x[(1 + \sigma)\varphi_{22} + \sigma\varphi_{21}]\, dx.$$

This integral can be evaluated by noting that Eqs. (58.1) yield

$$x = \frac{c}{3}(5 + 6 \cos t - 6 \cos^2 t), \qquad dx = -2c(1 - 2 \cos t) \sin t\, dt,$$

while the polar equation of the boundary C is $r = 2c(1 - \cos t)$, or

$$r^{1/2} = 2c^{1/2} \sin \frac{t}{2}, \qquad 0 \leq t \leq 2\pi.$$

Substitution of these expressions in Eqs. (58.5) and (58.8) yields the boundary values of the functions φ_{21} and φ_{22} in terms of the variable t. The integration indicated above is now carried out from $t = 0$ to $t = 2\pi$, with the result

$$\iint_R x \frac{\partial \varphi_2}{\partial y}\, dx\, dy = (1 + \sigma)\pi c^5 + \frac{\sigma 2\pi c^5}{9}$$
$$= \frac{(9 + 11\sigma)\pi c^5}{9}.$$

Similarly,

$$\iint_R y \frac{\partial \varphi_2}{\partial x}\, dx\, dy = \int_C y[(1 + \sigma)\varphi_{22} + \sigma\varphi_{21}]\, dy$$
$$= (1 + \sigma)2\pi c^5 + \frac{\sigma\pi c^5}{3} = \frac{(6 + 7\sigma)\pi c^5}{3},$$

$$\iint_R xy^2\, dx\, dy = -\tfrac{1}{3} \int_C xy^3\, dx = \pi c^5,$$

$$\iint_R x^3\, dx\, dy = \frac{1}{4} \int_C x^4\, dy = \frac{-5\pi c^5}{9},$$

and finally, from Eqs. (53.4),

$$x_{cf} = -\frac{2(3 + 4\sigma)}{63(1 + \sigma)}\, c, \qquad y_{cf} = 0.$$

Before concluding this section, it should be remarked that some of the foregoing results were obtained by W. M. Shepherd in 1936, and somewhat earlier by N. M. Mushtari. However, Mushtari's work was published in two journals that are not readily obtainable,[1] and not until after the appearance of Shepherd's paper did Mushtari publish a summary of his earlier papers. Mushtari considers the problems of torsion and flexure of beams whose boundaries of cross sections have the forms

$$r = a + b(1 + \cos\theta)$$

and

$$r^2 = a^2 + b^2 \cos 2\theta.$$

His method of solution consists essentially in assuming the complex torsion and flexure functions to have certain forms that involve integral and fractional powers of the complex variable z.

59. Bending of Circular Pipe. As an illustration of a simple application of the theory of analytic functions in determining the flexure function Φ (Sec. 54), consider a beam whose cross section is bounded by two concentric circles, that is, a pipe with inner radius r_i and outer radius r_0.

The complex flexure function $F(z) = \Phi + i\Psi$, being analytic and single-valued in the circular ring $r_i \leq r \leq r_0$, admits of an expansion in a Laurent series, so that

$$\Phi + i\Psi = \sum_{n=-\infty}^{\infty} (a_n + ib_n)z^n.$$

Setting $z = re^{i\theta}$, we obtain

$$\begin{aligned}\Phi + i\Psi &= \sum_{-\infty}^{\infty} r^n(a_n + ib_n)e^{in\theta}\\ &= \sum_{-\infty}^{\infty} r^n(a_n + ib_n)(\cos n\theta + i\sin n\theta),\end{aligned}$$

or, separating real and imaginary parts,

$$(59.1)\qquad \left\{\begin{aligned}\Phi &= \sum_{-\infty}^{\infty} r^n(a_n\cos n\theta - b_n\sin n\theta),\\ \Psi &= \sum_{-\infty}^{\infty} r^n(a_n\sin n\theta + b_n\cos n\theta).\end{aligned}\right.$$

[1] N. M. Mushtari, *Transactions of the Kazan Aviational Institute*, No. 1 (1933), pp. 17–32, and *Transactions of the KIIKS*, No. 1 (1935), pp. 53–67. These references are given by Mushtari in a paper published in *Applied Mathematics and Mechanics*, New Series, vol. 1 (1938), pp. 427–440. See also D. Z. Avazashvili, "On the Application of Functions of a Complex Variable to the Torsion and Flexure Problems," *Applied Mathematics and Mechanics*, vol. 4, No. 1 (1940). pp. 129–134 (in Russian).

The constants a_n, b_n may be determined either from the boundary condition on the normal derivative of the function Φ [Eq. (54.3)] or from the boundary values of the function Ψ [Eq. (54.4)]. That is, we may solve either a problem of Neumann for the flexure function Φ or a Dirichlet problem for the conjugate function Ψ. The latter course will be followed, since the boundary condition on Ψ has already been given for a circular boundary [see Eq. (56.1)].

Rewriting Eq. (56.1) for the boundaries with radii r_i and r_0 and using Eqs. (59.1), we get

$$\sum_{-\infty}^{\infty} r_i^n(a_n \sin n\theta + b_n \cos n\theta) = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)r_i^3 \sin \theta + \tfrac{1}{4}r_i^3 \sin 3\theta,$$

$$\sum_{-\infty}^{\infty} r_0^n(a_n \sin n\theta + b_n \cos n\theta) = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)r_0^3 \sin \theta + \tfrac{1}{4}r_0^3 \sin 3\theta.$$

Comparing the coefficients of $\sin n\theta$ and $\cos n\theta$ gives a system of equations for the determination of the constants a_n and b_n. We have

$$\begin{aligned} -r_i^{-1}a_{-1} + r_i a_1 &= -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)r_i^3, & -r_i^{-3}a_{-3} + r_i^3 a_3 &= \tfrac{1}{4}r_i^3,\\ -r_0^{-1}a_{-1} + r_0 a_1 &= -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)r_0^3, & -r_0^{-3}a_{-3} + r_0^3 a_3 &= \tfrac{1}{4}r_0^3,\\ b_n &= 0 \quad \text{if } n = \pm 1, \pm 2, \pm 3, \ldots,\\ a_n &= 0 \quad \text{if } n = \pm 2, \pm 4, \pm 5, \pm 6, \ldots. \end{aligned}$$

Solving these equations, we get

$$\begin{aligned} a_1 &= -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)(r_i^2 + r_0^2), & a_{-1} &= -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)r_i^2 r_0^2,\\ a_3 &= \tfrac{1}{4}, & a_{-3} &= 0, \end{aligned}$$

while the coefficients a_0 and b_0 are undetermined, since the boundary condition on Ψ involves an arbitrary constant.

Substituting these values in (59.1), we find that

$$\Phi = -\left(\frac{3}{4} + \frac{1}{2}\sigma\right)\left[(r_i^2 + r_0^2)r + \frac{r_i^2 r_0^2}{r}\right]\cos\theta + \frac{1}{4}r^3\cos 3\theta + \text{const},$$

$$\Psi = -\left(\frac{3}{4} + \frac{1}{2}\sigma\right)\left[(r_i^2 + r_0^2)r - \frac{r_i^2 r_0^2}{r}\right]\sin\theta + \frac{1}{4}r^3\sin 3\theta + \text{const}.$$

The expressions for the stresses can be easily calculated with the aid of Eqs. (54.2).

If r_i is set equal to zero, we get the flexure function for the solid circular beam discussed in Sec. 56.

PROBLEM

Calculate the stresses in a circular pipe of thickness t, fixed at one end and subjected to bending by an end load W, and show that the following approximate formulas are valid:

$$\tau_{zz} = \frac{-W(l-z)x}{\pi r_0^3 t},$$
$$(\tau_{zz})_{\max} = \frac{W}{\pi r_0 t},$$
$$(\tau_{zy})_{\max} = \frac{W}{2\pi r_0 t}.$$

60. Stress Functions and Analogies; Beams of Elliptical and Equilateral Triangular Sections. We recall that in Sec. 52 the equilibrium equations (52.6) led to the definition of the stress function $F(x, y)$, in terms of which the stresses τ_{zx} and τ_{zy} were determined by (52.7):

$$\tau_{zx} = \frac{\partial F}{\partial y} - \frac{1}{2} EK_x x^2, \qquad \tau_{zy} = -\frac{\partial F}{\partial x} - \frac{1}{2} EK_y y^2.$$

The function $F(x, y)$ was seen to be determined by the differential equation (52.8)

$$\nabla^2 F(x, y) = -2\mu\sigma K_y x + 2\mu\sigma K_x y - 2\mu\alpha,$$

and by the boundary condition (52.14)

$$\tau_{zx} \cos(x, \nu) + \tau_{zy} \cos(y, \nu) = 0.$$

In the course of the solution of the general flexure problem in Sec. 53, it was found convenient to phrase it not as a boundary-value problem for the determination of the function $F(x, y)$ but rather in terms of the torsion function φ and the flexure functions φ_1, φ_2 or φ_{11}, φ_{12}, φ_{21}, φ_{22}. In this section, the flexure problem will be stated in terms of a new stress function $T(x, y)$, which will be seen to be of value in certain problems.

We introduce the stress function $T(x, y)$ by defining

$$T(x, y) \equiv F(x, y) - \int R(x)\, dx - \int S(y)\, dy. \tag{60.1}$$

The functions $R(x)$ and $S(y)$ may be so chosen as to yield either a simple boundary condition or a simple differential equation for $T(x, y)$. The stresses can be written in terms of $T(x, y)$, with the aid of (52.7), as

$$\begin{cases} \tau_{zx} = \dfrac{\partial T}{\partial y} + S(y) - \dfrac{1}{2} EK_x x^2, \\ \tau_{zy} = -\dfrac{\partial T}{\partial x} - R(x) - \dfrac{1}{2} EK_y y^2, \end{cases} \tag{60.2}$$

while (52.8) yields the following differential equation for $T(x, y)$:

$$\nabla^2 T(x, y) = -2\mu\sigma K_y x - \frac{dR(x)}{dx} + 2\mu\sigma K_x y - \frac{dS(y)}{dy} - 2\mu\alpha. \tag{60.3}$$

The insertion of Eqs. (60.2) in (52.14) yields the boundary condition

$$\frac{\partial T}{\partial y}\frac{dy}{ds} + \frac{\partial T}{\partial x}\frac{dx}{ds} \equiv \frac{dT}{ds} = \left[\frac{1}{2} EK_x x^2 - S(y)\right]\frac{dy}{ds} - \left[\frac{1}{2} EK_y y^2 + R(x)\right]\frac{dx}{ds}, \tag{60.4}$$

where we make use of the relations

$$\cos(x, \nu) = \frac{dy}{ds}, \qquad \cos(y, \nu) = -\frac{dx}{ds}.$$

The functions $R(x)$, $S(y)$ may be prescribed arbitrarily. We choose them now to be any functions satisfying the relations

$$R(x) = -\tfrac{1}{2}EK_y y^2, \qquad S(y) = \tfrac{1}{2}EK_x x^2, \qquad \text{on } C; \tag{60.5}$$

then the condition on $T(x, y)$ becomes

$$\frac{dT}{ds} = 0 \qquad \text{on } C.$$

Thus, the function $T(x, y)$ is constant along the contour C, and since the choice of this constant cannot affect the stresses, we shall take it equal to zero. With this choice of the functions $R(x)$ and $S(y)$, the stress function $T(x, y)$ is determined by the differential equation (60.3) and by the condition

$$T = 0 \qquad \text{on } C. \tag{60.6}$$

It is to be noted that the function $R(x)$ [or $S(y)$] may take any value along a portion of the boundary where $\dfrac{dx}{ds}\left(\text{or } \dfrac{dy}{ds}\right)$ vanishes.

It should be recalled that the constant of integration α was seen in Sec. 52 to be the mean value of the local twist $\dfrac{\partial\omega}{\partial z}$ over the section (or the value of the local twist at the centroid). As noted in that section, the constant α is to be chosen equal to zero if the load is applied at the center of flexure of the end section.

The stress function $T(x, y)$ can be given an interesting physical interpretation. Comparison of the differential equations (60.3) and (46.1) shows that $T(x, y)$ may be thought of as the deflection of an elastic membrane stretched, with tension t, over an opening of contour C in a rigid plane plate and distorted by a nonuniform load $p(x, y)$, where

$$\frac{-p(x, y)}{t} = -2\mu\sigma K_y x - \frac{dR(x)}{dx} + 2\mu\sigma K_x y - \frac{dS(y)}{dy} - 2\mu\alpha.$$

When the general flexure problem considered above is specialized to the case of bending by a load $(W_x, 0, 0)$ along a principal axis (Sec. 54), we have

$$K_x = \frac{W_x}{EI_y} = \frac{W_x}{2\mu(1+\sigma)I_y}, \qquad K_y = 0.$$

The stress function $T(x, y)$ is determined by the conditions

$$\nabla^2 T(x, y) = \frac{\sigma}{1+\sigma}\frac{W_x}{I_y}y - \frac{dS(y)}{dy} - 2\mu\alpha, \tag{60.7}$$
$$T = 0 \qquad \text{on } C,$$

where $S(y)$ is any function such that

$$S(y) = \frac{W_x}{2I_y} x^2 \qquad \text{on } C, \tag{60.8}$$

except that $S(y)$ may take any boundary value along a portion of the contour where $\frac{dy}{ds}$ is zero. The stresses are given by

$$\begin{cases} \tau_{zx} = \dfrac{\partial T}{\partial y} + S(y) - \dfrac{W_x}{2I_y} x^2, \\ \tau_{zy} = -\dfrac{\partial T}{\partial x}. \end{cases} \tag{60.9}$$

The position of the center of flexure can be found in terms of the function $T(x, y)$ by applying the definition given in Eq. (53.1) and using Eqs. (60.2). We have

$$\begin{aligned} x_{cf}W_y - y_{cf}W_x &= \iint\limits_R \left\{x\left[-\frac{\partial T}{\partial x} - R(x) - \frac{1}{2} EK_y y^2\right]\right. \\ &\qquad \left. - y\left[\frac{\partial T}{\partial y} + S(y) - \frac{1}{2} EK_x x^2\right]\right\} dx\,dy \\ &= \iint\limits_R \left\{2T(x, y) + \frac{\partial}{\partial x}\left[-xT(x, y) - xyS(y) + \frac{1}{6} EK_x x^3 y\right]\right. \\ &\qquad \left. - \frac{\partial}{\partial y}\left[yT(x, y) + xyR(x) + \frac{1}{6} EK_y xy^3\right]\right\} dx\,dy \\ &= 2\iint\limits_R T(x, y)\,dx\,dy \\ &\qquad + \int_C \left[yT(x, y) + xyR(x) + \frac{1}{6} EK_y xy^3\right] dx \\ &\qquad + \int_C \left[-xT(x, y) - xyS(y) + \frac{1}{6} EK_x x^3 y\right] dy, \end{aligned}$$

where Green's Theorem was used in the last step. A reference to the boundary conditions (60.4) shows that this can be written as

$$\begin{aligned} x_{cf}W_y - y_{cf}W_x &= 2\iint\limits_R T(x, y)\,dx\,dy \\ &\qquad - \tfrac{1}{3}E\left(K_y \int_C xy^3\,dx + K_x \int_C x^3 y\,dy\right), \end{aligned} \tag{60.10}$$

wherein we are to set in the function $T(x, y)$ the constant α equal to zero. The coordinates x_{cf}, y_{cf} of the center of flexure are then found by comparing the coefficients of W_x and W_y. For the special case of bending

by a load W_x along a principal axis, Eq. (60.10) becomes

$$(60.11)\qquad y_{cf} = -\frac{2}{W_x}\iint_R T(x, y)\,dx\,dy + \frac{1}{3I_y}\int_C x^3 y\,dy.$$

As an illustration of the use of the stress function $T(x, y)$ in the solution of the flexure problem, we consider the bending of a beam of elliptical cross section with a contour given by the equation $x^2/a^2 + y^2/b^2 = 1$. We suppose that the load W_x is applied at the centroid of the end section. Since on the boundary one has

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2),$$

one can, evidently, choose

$$S(y) = \frac{W_x}{2I_y}\frac{a^2}{b^2}(b^2 - y^2).$$

From Eq. (60.7) it is seen that the function $T(x, y)$ is subject to the conditions

$$\nabla^2 T(x, y) = \frac{W_x}{I_y}\left(\frac{\sigma}{1+\sigma} + \frac{a^2}{b^2}\right) y \quad \text{in } R,$$

$$T(x, y) = 0 \qquad \text{on} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

The differential equation and boundary condition suggest that we seek a solution of the form

$$T(x, y) = ky\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right),$$

and it is readily found that

$$T(x, y) = \frac{a^2[(1+\sigma)a^2 + \sigma b^2]}{2(1+\sigma)(3a^2+b^2)}\frac{W_x}{I_y}\,y\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right).$$

The stresses τ_{zx}, τ_{zy} can now be found from Eqs. (60.9). This method of solution should be compared with that applied in Sec. 56 to the same problem.

The stress function $T(x, y)$ will now be used to solve a special case[1] of the flexure problem for a beam whose cross section is an equilateral triangle (Fig. 46).

The boundary of the triangular section can be written as

$$(y - a)\left(x + \frac{2a + y}{\sqrt{3}}\right)\left(x - \frac{2a + y}{\sqrt{3}}\right) = 0,$$

[1] The flexure function for a beam with an arbitrary triangular cross section is not known. Some special triangular cross sections have been considered by B. R. Seth, *Proceedings of the London Mathematical Society*, ser. 2, vol. 41 (1936), pp. 323–331.

where the origin has been taken at the centroid of the section. Along the side $y = a$ we have $\frac{dy}{ds} = 0$, and hence no condition is imposed on the boundary values of the function $S(y)$ along this side, whereas we require that

$$S(y) = \frac{W_x}{2I_y} x^2 = \frac{W_x}{2I_y} \frac{(2a + y)^2}{3} \qquad \text{on} \qquad x = \pm \frac{2a + y}{\sqrt{3}}.$$

Therefore we take

$$S(y) \equiv \frac{W_x}{6I_y} (2a + y)^2,$$

and from (60.7) it follows that

$$\nabla^2 T(x, y) = \frac{W_x}{I_y} \left[\frac{2(\sigma - \frac{1}{2})}{3(\sigma + 1)} y - \frac{2}{3} a \right],$$

where we have set $\alpha = 0$ and are, accordingly, solving the problem of pure flexure by a load applied at the center of flexure.

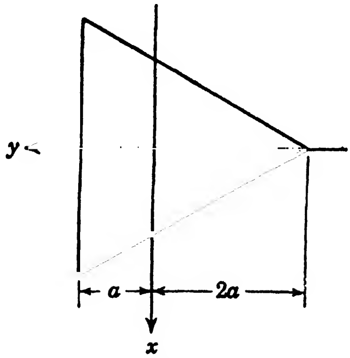

Fig. 46

The differential equation and boundary conditions on $T(x, y)$ can be readily satisfied when Poisson's ratio takes a particular value, namely, $\sigma = \frac{1}{2}$, which corresponds to incompressible materials. In this case, we have

$$\nabla^2 T(x, y) = -\frac{2}{3} \frac{W_x}{I_y} a \qquad \text{in } R,$$

$$T(x, y) = 0 \qquad \text{on} \left\{ \begin{array}{l} y = a \\ x^2 = \frac{1}{3}(2a + y)^2. \end{array} \right.$$

We try

$$T(x, y) = k[x^2 - \tfrac{1}{3}(2a + y)^2](y - a)$$

and find that the stress function is given by

$$\text{(60.12)} \qquad T(x, y) = \frac{W_x}{6I_y} \left[x^2 - \frac{1}{3} (2a + y)^2 \right] (y - a).$$

Equation (60.11) now yields the position of the center of flexure. Straightforward calculations give

$$I_y = \frac{3\sqrt{3}\,a^4}{2}$$

$$\iint_R T(x, y)\,dx\,dy = \frac{3\sqrt{3}\,W_x a^5}{10 I_y} = \frac{aW_x}{5},$$

$$\int_C x^3 y\,dy = 2\int_0^{a\sqrt{3}} x^3(\sqrt{3}\,x - 2a)\sqrt{3}\,dx = \frac{9\sqrt{3}\,a^5}{5},$$

and therefore $y_{cf} = 0$. Since the cross section is symmetrical about the y-axis, we see that the center of flexure is at the origin, and hence at the centroid of the section. Thus, the function $T(x, y)$, given above, furnishes the solution of the flexure problem for an incompressible beam of equilateral triangular section when the load W_x is applied at the centroid.

The flexure function $T(x, y)$ was introduced, for the case of bending by a load along a principal axis, by Timoshenko[1] and was used by him to solve the flexure problem for a number of cross sections.

It will be recalled that, when the functions $R(x)$ and $S(y)$ were introduced, it was remarked that they might be so chosen as to yield either a simple boundary condition or a simple differential equation for the function $T(x, y)$. The first course led to Timoshenko's stress function $T(x, y)$, discussed above, which can be interpreted physically as representing the deflection of an elastic membrane stretched over an opening of boundary C in a plane plate and subjected to a nonuniform load. We follow now the alternative course and choose $R(x)$ and $S(y)$ to make $T(x, y)$ a harmonic function.

Let us define

$$\text{(60.13)} \qquad \begin{cases} R(x) = -\mu\sigma K_y x^2 - \mu\alpha x, \\ S(y) = \mu\sigma K_x y^2 - \mu\alpha y. \end{cases}$$

We shall designate the function $T(x, y)$ defined by Eqs. (60.3) and (60.4) with this choice of $R(x)$ and $S(y)$ by $M(x, y)$. Then these equations become

$$\text{(60.14)} \qquad \nabla^2 M(x, y) = 0$$

and

$$\frac{dM}{ds} = [\mu(1+\sigma)K_x x^2 - \mu\sigma K_x y^2 + \mu\alpha y]\frac{dy}{ds} - [\mu(1+\sigma)K_y y^2 - \mu\sigma K_y x^2 - \mu\alpha x]\frac{dx}{ds} \qquad \text{on } C.$$

[1] S. Timoshenko, *Proceedings of the London Mathematical Society*, ser. 2, vol. 20 (1922), p. 398.

An account of this work will be found in S. Timoshenko and J. N. Goodier, Theory of Elasticity, Secs. 106–113.

This boundary condition can be integrated in part to give

$$
\begin{aligned}
(60.15)\quad M = \tfrac{1}{2}\mu\alpha(x^2 + y^2) &+ \mu K_x\left[-\tfrac{1}{3}\sigma y^3 + (1+\sigma)\int_{(x_0,y_0)}^{(x,y)} x^2\,dy\right] \\
&+ \mu K_y\left[\tfrac{1}{3}\sigma x^3 - (1+\sigma)\int_{(x_0,y_0)}^{(x,y)} y^2\,dx\right] + \text{const} \qquad \text{on } C.
\end{aligned}
$$

The stresses τ_{zx}, τ_{zy} can be written from Eqs. (60.2) and (60.13) as

$$
\begin{aligned}
\tau_{zx} &= \frac{\partial M}{\partial y} + \mu K_x[\sigma y^2 - (1+\sigma)x^2] - \mu\alpha y, \\
\tau_{zy} &= -\frac{\partial M}{\partial x} + \mu K_y[\sigma x^2 - (1+\sigma)y^2] + \mu\alpha x.
\end{aligned}
$$

In the case of bending by a load W_x along a principal axis, the formulas for the stresses become

$$
\begin{aligned}
\tau_{zx} &= \frac{\partial M}{\partial y} + \frac{W_x}{2I_y}\left(\frac{\sigma}{1+\sigma}y^2 - x^2\right) - \mu\alpha y, \\
\tau_{zy} &= -\frac{\partial M}{\partial x} + \mu\alpha x,
\end{aligned}
$$

while $M(x, y)$ is subject to the condition

$$
(60.16)\quad M = \frac{1}{2}\mu\alpha(x^2 + y^2) - \frac{\sigma}{6(1+\sigma)}\frac{W_x}{I_y}y^3 + \frac{W_x}{2I_y}\int_{(x_0,y_0)}^{(x,y)} x^2\,dy \qquad \text{on } C.
$$

One can interpret the determination of the harmonic function $M(x, y)$, subject to the condition (60.15) or (60.16) on C, in terms of a membrane analogy, as was done in connection with the torsion problem in Sec. 46. Thus, the solution of the flexure problem by means of the *membrane function* $M(x, y)$ is mathematically identical with the determination of the deflection of an unloaded elastic membrane stretched across a closed space curve whose projection on the xy-plane is the contour C and whose variable height above the plane is given by the boundary values of $M(x, y)$ [(60.15) or (60.16)]. This analogy, for the case in which the boundary values are given by Eq. (60.16), has been used by Griffith and Taylor,[1] among others, to obtain experimental solutions of the flexure problem for beams whose cross sections are such that the problem does not yield readily to mathematical treatment. Neményi[2] has derived both $M(x, y)$ and another flexure function $F_1(x, y)$ as special cases of a more

[1] A. A. Griffith and G. I. Taylor, *National Advisory Committee on Aeronautics Technical Report*, Great Britain, vol. 3 (1917–1918), p. 950.

See S. Timoshenko and J. N. Goodier, Theory of Elasticity, Sec. 113, for a description of the experimental procedure.

[2] P. Neményi, "Über die Berechnung der Schubspannungen im gebogenen Balken," *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 1 (1921), pp. 89–96.

general flexure function and has discussed these two formulations of the flexure problem in the light of the membrane analogy and of numerical solutions, respectively.

There exists a very close connection between the membrane function $M(x, y)$ and the canonical flexure functions φ_{11}, φ_{12}, φ_{21}, φ_{22}, discussed in Sec. 53. A comparison of the boundary condition (60.15) with the boundary values taken by the torsion function ψ [Eq. (35.4)] and by the flexure functions ψ_1 and ψ_2 [Eqs. (52.17)] shows that

$$M(x, y) = \mu\alpha\psi(x, y) + \mu K_x\psi_1(x, y) + \mu K_y\psi_2(x, y) \qquad \text{on } C. \tag{60.17}$$

Since a harmonic function is uniquely determined by its boundary values, it follows that Eq. (60.17) holds throughout the region R of the cross

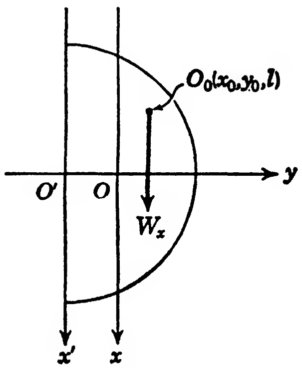

FIG. 47

section. If we define the conjugate membrane function $L(x, y)$ so that $L + iM$ is analytic in R, then we can write

$$L + iM = \mu\alpha(\varphi + i\psi) + \mu K_x(\varphi_1 + i\psi_1) + \mu K_y(\varphi_2 + i\psi_2).$$

Equations (53.5) now furnish the following relation between the complex membrane function $L + iM$, on the one hand, and the torsion function $\varphi + i\psi$ and the canonical flexure functions $\varphi_{11} + i\psi_{11}$, . . . , on the other hand:

$$\begin{aligned} L + iM = \mu\alpha(\varphi + i\psi) &+ \mu K_x[(1 + \sigma)(\varphi_{11} + i\psi_{11}) - \sigma(\varphi_{12} + i\psi_{12})] \\ &+ \mu K_y[(1 + \sigma)(\varphi_{22} + i\psi_{22}) + \sigma(\varphi_{21} + i\psi_{21})]. \end{aligned}$$

61. Flexure of Semicircular Beams. As a further illustration of the usefulness of the function $T(x, y)$, introduced in Sec. 60, we outline a solution of the flexure problem for the semicircular beam, shown in Fig. 47. If the load $(W_x, 0, 0)$ is applied at an arbitrary point $O_0(x_0, y_0, l)$ of the end section and the origin of coordinates O is chosen at the centroid of the fixed end, the function $T(x, y)$ satisfies Eq. (60.7) in the semicircle and vanishes on its boundary. To obtain a more convenient form for the equation of the boundary, we introduce new coordinates x', y' defined

by

$$(61.1)\qquad \begin{cases} x' = x \\ y' = y + \bar{y}, \end{cases}$$

where $\bar{y} = 4a/3\pi$ is the distance of the centroid O from the point O' on the diameter of the semicircle of radius a.

Making this change of variables in Eq. (60.7), we get

$$(61.2)\qquad \nabla^2 T(x', y') = \frac{\sigma}{1+\sigma}\frac{W_x}{I_y}(y' - \bar{y}) + \frac{W_x}{I_y}y' - 2\mu\alpha,$$

where we take $S(y)$ in accordance with (60.8), and note that along the circular part of the boundary $x'^2 = a^2 - y'^2$, and along the diameter $dy'/ds = 0$. The corresponding boundary conditions are:

$$(61.3)\qquad \begin{cases} T = 0 & \text{on } y' = 0, \\ T = 0 & \text{on } y' = \sqrt{a^2 - x'^2}. \end{cases}$$

A particular integral of (61.2) can be taken in the form

$$T_0(x', y') = Ay'(x'^2 + y'^2) + B(x'^2 + y'^2),$$

and, on setting $y' = r\cos\theta$ and $x' = r\sin\theta$, we easily find that

$$(61.4)\qquad T_0 = Ar^3\cos\theta + Br^2,$$

where

$$(61.5)\qquad A = \frac{1}{8}\frac{W_x}{I_y}\frac{1+2\sigma}{1+\sigma}, \qquad B = -\frac{1}{4}\left(\frac{W_x}{I_y}\frac{\sigma}{1+\sigma}\bar{y} + 2\mu\alpha\right)$$

and $I_y = \pi a^4/8$.

Thus the general solution of (61.2) can be written as

$$(61.6)\qquad T(r, \theta) = Ar^3\cos\theta + Br^2 + \sum_{n=0}^{\infty}(A_n r^n \cos n\theta + B_n r^n \sin n\theta).$$

The coefficients A_n, B_n must be chosen so that the conditions (61.3) are fulfilled; that is,

$$(61.7)\qquad \begin{cases} T\left(r, \pm\dfrac{\pi}{2}\right) = 0, \\ T(a, \theta) = 0, \qquad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}. \end{cases}$$

The first of these equations will be satisfied if we take

$$(61.8)\qquad T(r, \theta) = Ar^3\cos\theta + Br^2 + Cr^2\cos 2\theta + \sum_{n=0}^{\infty} A_{2n+1}r^{2n+1}\cos(2n+1)\theta,$$

and choose $B = C$. To satisfy the second of Eqs. (61.7), we select the A_{2n+1} so that

$$Aa^3 \cos \theta + Ba^2(1 + \cos 2\theta) = - \sum_{n=0}^{\infty} A_{2n+1}a^{2n+1} \cos (2n + 1)\theta,$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

On multiplying both members of this equation by $\cos (2m + 1)\theta$ and integrating with respect to θ between the limits $-\pi/2$ and $\pi/2$, we find

$$(61.9) \quad \begin{cases} A_1 = -\dfrac{16a}{3\pi} B - a^2A, \\ A_{2m+1} = \dfrac{a^{-2m-1}16a^2(-1)^m B}{\pi(2m + 1)[(2m + 1)^2 - 4]}, \qquad m = 1, 2, \ldots . \end{cases}$$

Accordingly, the solution of our problem is given by the uniformly and absolutely convergent series

$$(61.10) \quad T(r, \theta) = Ar^3 \cos \theta + Br^2(1 + \cos 2\theta) + \sum_{n=0}^{\infty} A_{2n+1}r^{2n+1} \cos (2n + 1)\theta.$$

The center of flexure obviously lies on the y-axis, and, by using (60.11), it can be shown that[1]

$$y'_{cf} = \frac{8a}{15(1 + \sigma)\pi}\left[3 + \sigma\left(\frac{40}{\pi^2} - 1\right)\right].$$

Leibenson[2] used a similar method to obtain an approximate solution of the flexure problem for a semicircular tube of small thickness.

The flexure problem for a cylindrical beam whose cross section is a segment of the circle was solved in bipolar coordinates with the aid of Fourier integrals by Uflyand.[3]

62. Multiply Connected Domains. Deformation of Nonhomogeneous Beams with Free Sides. Other Developments. Although the mathematical formulation of the flexure problem for beams with multiply connected cross sections is quite straightforward, an explicit determina-

[1] In obtaining this result we have noted that

$$\sum_{m=2}^{\infty} \frac{1}{(2m + 1)^2(2m - 1)^2(2m - 3)} = \frac{1}{8} - \frac{3\pi^2}{128}.$$

[2] L. S. Leibenson, A Course in the Theory of Elasticity (1947), pp. 298–305 (in Russian).

[3] Ya. S. Uflyand, *Doklady Akademiya Nauk SSSR*, vol. 69 (1949), pp. 751–754. See also an interesting monograph by this author entitled Bipolar Coordinates in the Theory of Elasticity (1950) (in Russian), which contains a solution of the flexure problem for a cylinder with the lens-shaped cross section formed by two circular axes. A special case of this, when the section is a segment of the circle, is treated on pp. 50–59.

tion of flexure functions, in specific problems, presents computational difficulties.[1]

The deformation of beams with multiply connected cross sections is a special case of the deformation of compound beams. The deformation of compound beams by end loads was first treated from a general point of view by Muskhelishvili and his treatment was extended by Rukhadze and Vekua.[2]

The mathematical formulation of the problems of simple extension, pure bending, torsion, and flexure of compound cylinders differs from that for homogeneous beams only in the added boundary conditions on the interfaces of the media with different elastic properties. Thus, consider

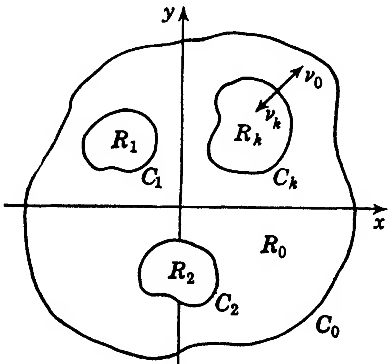

FIG. 48

the cross section bounded by the exterior contour C_0 and several interior contours C_k $(k = 1, 2, \ldots, m)$, shown in Fig. 48. The regions R_k $(k = 1, 2, \ldots, m)$, enclosed by the C_k, are filled with materials whose

[1] The general flexure problem for a hollow beam bounded by two eccentric circles was solved by A. C. Stevenson, *Proceedings of the London Mathematical Society*, vol. 50 (1949), pp. 536–549, and R. Capildeo, *Proceedings of the Cambridge Philosophical Society*, vol. 49, Part II (1953), pp. 308–318. Capildeo also discusses the flexure problem for a beam with cross section bounded by two confocal limaçons.

[2] N. I. Muskhelishvili, "Sur le problème de torsion des poutres élastiques composées, *Comptes rendus hebdomadaires des séances de l'académie des sciences*, Paris, vol. 194 (1932), p. 1435; "On the Problem of Torsion and Flexure of Elastic Beams Composed of Different Materials," *Izvestiya Akademiya Nauk SSSR* (1932), pp. 907–945 (in Russian). See also Chaps. 22–25 in Muskhelishvili's monograph Some Basic Problems of the Mathematical Theory of Elasticity (1953), pp. 561–655.

I. N. Vekua and A. K. Rukhadze, "Torsion Problem for a Circular Cylinder Reinforced by a Longitudinal Circular Rod," *Izvestiya Akademiya Nauk SSSR* (1933), pp. 1297–1308; "On the Problem of Bending of Elastic Beams Composed of Different Materials," *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 1 (1940), pp. 107–114. See also A. I. Uzdalev, "Bending of an Anisotropic Two-layered Cylinder by a Transverse Force," *Inzhenernyi Sbornik*, vol. 15 (1953), pp. 35–42, and I. V. Suharevskiĭ, "On the Problem of Torsion of a Composite Multiconnected Bar," *Inzhenernyi Sbornik*, vol. 19 (1954), pp. 107–124. These papers are in Russian.

elastic properties differ from those of the surrounding medium in the region R_0.

If the components of such a beam are glued or welded so that in the course of deformation there is no separation of material along the contours C_k, the displacements u_i and the internal stresses $\overset{\nu}{T}_i = \tau_{ij}\nu_j$ will be continuous across the contours C_k. Accordingly, the boundary conditions on the C_k can be formulated as follows:

$$(62.1) \quad \begin{cases} (a) \quad \tau_{ij}\nu_j = 0 & \text{on } C_0, \\ (b) \quad (\tau_{ij}\nu_j)_0 = (\tau_{ij}\nu_j)_k & \text{on } C_k \ (k = 1, 2, \ldots, m), \\ (c) \quad (u_i)_0 = (u_i)_k & \text{on } C_k \ (k = 1, 2, \ldots, m). \end{cases}$$

The subscripts 0 and k outside the parentheses in these expressions indicate that the values of affected quantities are computed along the interior contours C_k for the regions R_0 and R_k, respectively. The unit normal vectors $(\nu_i)_0$ and $(\nu_i)_k$ along such contours point into the regions R_0 and R_k as shown in Fig. 48.

The satisfaction of boundary conditions (62.1), in the instance of the Saint-Venant torsion problem, when it is patterned along the lines of Sec. 34, presents no logical difficulties. However, in the problems of extension, pure bending, and flexure by the transverse force, the Saint-Venant assumption that τ_{xx}, τ_{yy}, and τ_{xy} vanish does not lead to continuous displacements along the contours C_k unless the media on either side of the contours have the same Poisson's ratios. The reason why no difficulty of this sort arises in the torsion problem is that the displacements there do not depend on Poisson's ratios.

To remove the discontinuities in displacements, it is necessary to superimpose on displacements resulting from the hypothesis

$$\tau_{xx} = \tau_{yy} = \tau_{xy} = 0$$

the displacements present in certain two-dimensional elastostatic problems. Such problems are discussed in the next chapter. In this manner valid solutions have been deduced for several interesting problems on deformation of compound beams. Besides solutions of the torsion and flexure problems for circular beams reinforced by circular cores (in general, eccentric), solutions are available for the torsion of a composite rectangular beam formed by gluing two rectangular beams along their sides[1] and for the torsion of an elliptical cylinder reinforced by a circular rod whose axis coincides with the axis of the cylinder.[2]

The Saint-Venant torsion and flexure problems for cylinders having a small initial twist in the natural state has been considered in a series of

[1] See papers in the preceding footnote and a paper by L. E. Payne, "Torsion of Composite Sections," *Iowa State College Journal of Science*, vol. 23 (1949), pp. 381–395.

[2] D. I. Sherman, *Inzhenernyi Sbornik*, vol. 10 (1951), pp. 81–108 (in Russian).

papers by Riz, Lourie, Dzhanelidze, Gorgidze, and Rukhadze.[1] All these authors assume that the cylinder with free sides has a small initial twist determined by the angle $\alpha = kz$, where z is measured along the length of the rod and k is a small parameter. This problem is essentially a nonlinear one, since it is necessary to take account of the twist produced by bending. It is generally treated by a method of perturbations on the small parameter k.

With the exception of Sec. 49, rods considered in this chapter have been cylinders with cross sections defined by an equation of the form $f(x, y) = 0$. Consider now the surface defined by

$$f[x(1 - kz), y(1 - kz)] = 0,$$

where the parameter k is such that $kz \ll 1$. This equation defines the surface of a slightly tapered rod. The torsion and bending of such rods with free sides were analyzed by Panov and Rukhadze[2] by methods similar to those used by Riz in solving the corresponding problems for naturally twisted rods.

A problem similar to that just mentioned is also encountered in the study of the deformation of cylinders whose surface is defined by

$$f(x, y + kz^2) = 0,$$

with k, again, a small parameter. This cylinder has a slightly curved axis. The torsion and bending of such cylinders were studied by Riz and Rukhadze.[3]

[1] P. M. Riz, *Doklady Akademii Nauk SSSR*, New Series, vol. 23 (1939), pp. 17–20, 441–444, 765–767.

A. I. Lourie, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, New Series, vol. 2 (1938), pp. 55–68.

A. I. Lourie and G. Dzhanelidze, *Doklady Akademii Nauk SSSR*, New Series, vol. 24 (1939), pp. 24–27, 227–228; vol. 25 (1939), pp. 577–579; vol. 27 (1940), pp. 436–439.

A. Gorgidze and A. Rukhadze, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 5 (1944), pp. 253–262.

A. K. Rukhadze, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 5 (1944), pp. 483–492; *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 11 (1947), pp. 533–542. This paper contains an explicit solution of the flexure problem for an elliptical rod with a small initial twist.

All these papers are in Russian.

[2] D. Y. Panov, "Concerning the Torsion of Nearly Prismatic Rods," *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, New Series, vol. 2 (1938), pp. 159–180; *Doklady Akademii Nauk SSSR*, New Series, vol. 20 (1938), pp. 251–253.

A. K. Rukhadze, "The Problem of Bending Nearly Prismatic Beams," *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 1 (1940), pp. 577–582; *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 6 (1942), pp. 123–138.

All these are in Russian.

[3] P. Riz, *Doklady Akademii Nauk SSSR*, New Series, vol. 24 (1939), pp. 110–113, 229–232 (in Russian).

A. K. Rukhadze, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 2 (1941), pp. 35–42 (in Russian).

Approximate solutions of the torsion, flexure, and pure bending of compound beams with slightly curved axes were obtained by Gorgidze, Minasyan, and Rukhadze[1] and similar solutions for the initially stretched compound beams by Gorgidze and Mecugov.[2] The behavior of twisted compound beams under stretching and pure bending was studied by Rukhadze and Shangriya.[3]

63. Deformation of Cylinders by Lateral Loads. In all the foregoing considerations the lateral surfaces of cylinders were free of external loads, and it remains to investigate the deformation of cylinders by forces distributed over their surfaces. If the cylinder is long and the load does not vary along its axis, the resulting deformation does not depend on the coordinate measured along the length of the cylinder. This case of *plane deformation* is treated in Chap. 5, where some effective methods of solving such problems are provided. If, however, the load varies along the length, the problem becomes a three-dimensional one, and the difficulties of obtaining useful solutions of three-dimensional elastostatic problems are very great. An engineer faced with the necessity of dealing with such problems is obliged to introduce a variety of simplifying assumptions that reduce them to problems in two or, even, in one dimension. In the category of such one-dimensional problems is the problem of the elastica, which is concerned with the determination of deflection of the central line of the beam. The underlying assumption of the theory of the elastica, which forms the core of the technical theory of beams, is that the curvature $1/R$ of the central line is related to the bending moment M by the Bernoulli-Euler law, $M = EI/R$. This relation leads at once to a differential equation for the elastica inasmuch as the moment M, at any point of the central line, can be calculated from the specified distribution of external loads.[4] In actual fact it is possible to load the beam so that the Bernoulli-Euler law is not satisfied even approximately.

[1] A. Ya. Gorgidze, *Trudy Tbilisi Mat. Inst., Akademii Nauk Gruzinskoĭ SSR*, vol. 17 (1949), pp. 95–130 (in Russian).

A. K. Rukhadze, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 14 (1953), pp. 525–532 (in Russian).

R. S. Minasyan, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 15 (1954), pp. 207–214 (in Russian).

[2] A. Ya. Gorgidze, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 14 (1953), pp. 589–594 (in Russian).

V. H. Mecugov, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 14 (1953), p. 459 (in Russian).

[3] A. K. Rukhadze, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 13 (1952), pp. 137–144, 265–272 (in Russian).

A. G. Shangriya, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 13 (1952), pp. 389–396 (in Russian).

[4] A brief discussion of the technical theory of beams and some comments on the validity of the Bernoulli-Euler law in the theory of continuous beams are contained in A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity (1927), Chaps. 16 and 17.

The problem of deformation of a homogeneous cylindrical beam by forces T_i distributed over its surface so that the T_i do not vary along the length of the beam is easily formulated. Indeed, if the x_3-coordinate is taken along the axis of the beam, the problem reduces to the solution of the following system of equations:

a. Equilibrium equations

$$\tau_{ij,j} = 0, \qquad (i = 1, 2, 3),$$

b. Beltrami's compatibility equations

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = 0, \qquad (i, j = 1, 2, 3),$$

c. Boundary conditions[1]

$$\begin{aligned} \tau_{11}\nu_1 + \tau_{12}\nu_2 &= T_1(x_1, x_2), \\ \tau_{21}\nu_1 + \tau_{22}\nu_2 &= T_2(x_1, x_2), \\ \tau_{31}\nu_1 + \tau_{32}\nu_2 &= T_3(x_1, x_2), \qquad \text{on the lateral surface.} \end{aligned}$$

This problem was first considered by Almansi and Michell,[2] who demonstrated in effect that it is possible to reduce it to the determination of two stress functions, one of which is harmonic and the other biharmonic. Some broad classes of boundary-value problems in the biharmonic equation are discussed in Chap. 5, but, as an introduction to them, we consider in the following section the problem of torsion of a long cylinder by tractions suitably distributed over its surface.

64. Torsion of a Cylinder by Forces on the Lateral Surface. Let a cylinder with an arbitrary cross section R be twisted by tractions $\mathbf{T}$ applied to the lateral surface. We suppose that the cylinder is of length l and that one of its ends is fixed in the plane $x_3 = 0$, while the end $x_3 = l$ is free. The surface tractions $\mathbf{T}$, assumed independent of the x_3-coordinate, are directed parallel to the x_1x_2-plane and produce a twisting moment in the cross section $x_3 =$ const. We take the magnitude of this moment to be Ml, so that $\mathbf{M}$ is the torque per unit length of the

[1] If the system of stresses T_i is not self-equilibrating, it is necessary to apply a suitable distribution of forces on one of the ends of the cylinder in order to maintain the cylinder as a whole in equilibrium. This can be done by supposing that the end $x_3 = 0$ is fixed and the other end is free.

[2] E. Almansi, Nota II, *Atti della Accademia nazionale dei Lincei Rendiconti*, Rome, ser. 5, vol. 10 (1901).

J. H. Michell, *Quarterly Journal of Mathematics*, vol. 32 (1901). Almansi also considers the case when the external stresses T_i are polynomials in x_3. A summary of these contributions is contained in Love's Treatise (1927), Secs. 239–241. The corresponding problem for compound beams when Poisson's ratios are identical throughout the cross section, but Young's moduli are different for each component of the beam, was treated by G. M. Hatiashvili, *Soobshcheniya Akademii Nauk Gruzinskoĭ SSR*, vol. 13 (1952), pp. 335–341; vol. 14 (1953), pp. 197–204 (in Russian).

cylinder. Thus,

$$T_3 = 0, \qquad \int_C \mathbf{T}(x_1, x_2)\, ds = 0,$$

$$\int_C [x_1 T_2(x_1, x_2) - x_2 T_1(x_1, x_2)]\, ds = M,$$

where C is the contour bounding R.

Since the end $x_3 = l$ is free, we demand that

$$\tau_{13} = \tau_{23} = \tau_{33} = 0, \qquad \text{for } x_3 = l, \tag{64.1}$$

while at the fixed end $x_3 = 0$ the τ_{ij} must yield:

$$\begin{cases} \iint\limits_R \tau_{13}\, d\sigma = \iint\limits_R \tau_{23}\, d\sigma = \iint\limits_R \tau_{33}\, d\sigma = 0, \\ \iint\limits_R (x_1\tau_{32} - x_2\tau_{31})\, d\sigma = Ml, \\ \iint\limits_R x_1\tau_{33}\, d\sigma = \iint\limits_R x_2\tau_{33}\, d\sigma = 0. \end{cases} \tag{64.2}$$

The conditions in the first line in (64.2) demand that the resultant force vanish on the end $x_3 = 0$, the second line states that the twisting moment in the section $x_3 = 0$ is Ml, and the last line requires that there be no bending by couples.

The conditions on the lateral surface of the cylinder, clearly, are

$$\begin{cases} \tau_{11}\nu_1 + \tau_{12}\nu_2 = T_1(x_1, x_2), \\ \tau_{21}\nu_1 + \tau_{22}\nu_2 = T_2(x_1, x_2), \\ \tau_{31}\nu_1 + \tau_{32}\nu_2 = 0, \qquad \text{on } C. \end{cases} \tag{64.3}$$

The problem thus consists in determining the set of functions τ_{ij} satisfying the equilibrium and Beltrami's equations and the boundary conditions (64.1) to (64.3).

A solution of the problem stated with this degree of generality presents complications because the third of the boundary conditions in (64.1) is difficult to satisfy. If, however, we relax the condition $\tau_{33} = 0$ for $x_3 = l$, by merely requiring that the resultant force in the direction of the x_3-axis vanish, that is,

$$\iint\limits_R \tau_{33}\, d\sigma = 0 \qquad \text{for } x_3 = l, \tag{64.4}$$

and that the distribution of τ_{33} over $x_3 = l$ yield no bending moments,

$$\iint\limits_R x_1\tau_{33}\, d\sigma = \iint\limits_R x_2\tau_{33}\, d\sigma = 0, \tag{64.5}$$

then the problem can be solved quite readily.

One can argue on the basis of Saint-Venant's principle that the solutions of the original and relaxed problems can differ significantly only near the end $x_3 = l$ of the cylinder.[1]

The fact that the forces assigned on the lateral surface produce torsion of the cylinder suggests that the expressions for the τ_{ij} in the relaxed problem have, in part, an appearance similar to the stresses in Saint-Venant's torsion problem. Taking cognizance of the linear variation of the twisting moment along the length of the beam, it is reasonable to consider [cf. Eqs. (34.4)] the shearing stresses in the form

$$\text{(64.6)} \qquad \begin{cases} \tau_{23}^{(1)} = \mu\alpha(l - x_3)(\varphi_{,2} + x_1), \\ \tau_{13}^{(1)} = \mu\alpha(l - x_3)(\varphi_{,1} - x_2). \end{cases}$$

If we now make use of the equilibrium and Beltrami's equations and take

$$\begin{aligned} \tau_{33}^{(1)} &= -2\mu\alpha\varphi, \\ \tau_{11}^{(1)} &= \tau_{22}^{(1)} = \mu\alpha\varphi, \\ \tau_{12}^{(1)} &= \tfrac{1}{2}\mu\alpha(x_1^2 - x_2^2), \end{aligned}$$

we find that φ satisfies Laplace's equation, and the stress system $\tau_{ij}^{(1)}$ is an admissible system. Inserting from (64.6) in the third of the boundary conditions (64.3) yields

$$\frac{d\varphi}{d\nu} = x_2\nu_1 - x_1\nu_2, \qquad \text{on } C,$$

which is precisely the condition (34.6).

The nonvanishing distribution of the normal stress τ_{33} further suggests that we consider the stress system,

$$\begin{aligned} \tau_{11}^{(2)} &= \tau_{22}^{(2)} = \tau_{12}^{(2)} = \tau_{13}^{(2)} = \tau_{23}^{(2)} = 0, \\ \tau_{33}^{(2)} &= A + Bx_1 + Cx_2, \end{aligned}$$

since a system of this sort arises in the problems of stretching and pure bending. Finally, we must consider a system that gives rise to a deformation

$$\text{(64.7)} \qquad u_\alpha = u_\alpha(x_1, x_2), \qquad u_3 = 0 \qquad (\alpha = 1, 2),$$

which is independent of the length of the cylinder. A deformation of this type may be expected to be present in the relaxed problem since applied tractions do not vary along the length of the cylinder.

If we denote the stress system corresponding to the plane deformation (64.7) by $\tau_{ij}^{(3)}$, we can assume a solution of the relaxed problem in the form

$$\text{(64.8)} \qquad \tau_{ij} = \tau_{ij}^{(1)} + \tau_{ij}^{(2)} + \tau_{ij}^{(3)}.$$

[1] Indeed, we have already invoked the Saint-Venant principle in formulating the macroscopic boundary conditions (64.2) on the fixed end $x_3 = 0$, where the actual stress distribution is not known.

We shall see in Sec. 69 that the system of stresses $\tau_{ij}^{(3)}$ corresponding to a plane deformation (64.7) has the form[1]

$$\begin{aligned} \tau_{11}^{(3)} &= -\alpha U_{,22}, \qquad \tau_{22}^{(3)} = -\alpha U_{,11}, \qquad \tau_{12}^{(3)} = \alpha U_{,12}, \\ \tau_{23}^{(3)} &= \tau_{13}^{(3)} = 0, \qquad \tau_{33}^{(3)} = -\alpha\sigma\nabla^2 U, \end{aligned}$$

where $U(x_1, x_2)$ satisfies the biharmonic equation

$$\nabla^4 U = U_{,1111} + 2U_{,1122} + U_{,2222} = 0 \qquad \text{in } R.$$

On combining the systems $\tau_{ij}^{(1)}$, $\tau_{ij}^{(2)}$, $\tau_{ij}^{(3)}$, we get

$$(64.9) \qquad \begin{cases} \dfrac{\tau_{11}}{\alpha} = -U_{,22} + \mu\varphi, \qquad \dfrac{\tau_{22}}{\alpha} = -U_{,11} + \mu\varphi, \\ \dfrac{\tau_{12}}{\alpha} = U_{,12} + \dfrac{\mu}{2}(x_1^2 - x_2^2), \\ \dfrac{\tau_{13}}{\alpha} = \mu(l - x_3)(\varphi_{,1} - x_2), \\ \dfrac{\tau_{23}}{\alpha} = \mu(l - x_3)(\varphi_{,2} + x_1), \\ \dfrac{\tau_{33}}{\alpha} = -2\mu\varphi - \sigma\nabla^2 U + A + Bx_1 + Cx_2. \end{cases}$$

If we now insert from (64.9) in the first two boundary conditions (64.3) and recall the familiar relations $\nu_1 = \dfrac{dx_2}{ds}$, $\nu_2 = -\dfrac{dx_1}{ds}$, we easily find

$$(64.10) \qquad \begin{cases} \dfrac{d}{ds}(U_{,1}) = T_2/\alpha - \mu\varphi\nu_2 - \dfrac{1}{2}\mu(x_1^2 - x_2^2)\nu_1, & \text{on } C, \\ -\dfrac{d}{ds}(U_{,2}) = T_1/\alpha - \mu\varphi\nu_1 - \dfrac{1}{2}\mu(x_1^2 - x_2^2)\nu_2, & \text{on } C. \end{cases}$$

Thus, if the torsion function φ is known for the region R, the values of the derivatives $U_{,\alpha}$ of U can be calculated on the contour C. The problem of determining the biharmonic function U from prescribed values of the partial derivatives of U on the contour C is known as the *fundamental boundary-value problem in the biharmonic equation.* We shall see in the next chapter that there are effective methods for solving it.

It remains to show that the constants α, A, B, and C can be chosen in such a way that the end conditions (64.2), (64.4), and (64.5) are fulfilled. The fact that $\tau_{13} = \tau_{23} = 0$ on $x_3 = l$ is obvious from (64.9). The verification that the resultant forces $\iint \tau_{13}\, d\sigma$ and $\iint \tau_{23}\, d\sigma$ vanish over the end $x_3 = 0$ is, in every detail, identical with that given in Sec. 34. The condition $\iint_R (x_1\tau_{32} - x_2\tau_{31})\, d\sigma = Ml$ yields at once the result that

$$\alpha = \frac{M}{D},$$

[1] See Eqs. (69.4) and (69.5). The constant $-\alpha$ was introduced here for convenience. It can clearly be absorbed in $U(x_1, x_2)$.

where D is the torsional rigidity of the section. For the determination of the remaining constants A, B, and C, we have three equations (64.4) and (64.5). If the x_3-axis is taken through the centroid of the section R, the formulas for A, B, and C become quite simple.

Using the theory outlined in this section, it is not difficult to deduce formulas for stresses in a beam of elliptical cross section twisted by constant tangential tractions.[1] The corresponding problem for the circular section was first considered by Filon[2] and for cylinders of arbitrary cross section by Zvolinsky and Riz.[3] Extensions of this theory to the anisotropic media have been made by Lekhnitzky and Luxenberg.[4]

Solutions of the biharmonic equation suitable for the investigation of axially symmetrically loaded thick-walled circular tubes of finite length have been constructed by Prokopov.[5] An exact solution of the torsion problem for a solid cylindrical shaft consisting of two circular cylinders of different radii twisted by axially symmetric tractions applied to the lateral surface of the shaft was deduced by Abramyan and Dzerbashyan.[6] An approximate treatment of the deformation of cylinders of variable cross sections by forces distributed on the lateral surface was sketched by Shapiro,[7] who considered the equilibrium of cones and paraboloids of revolution.

[1] The biharmonic function U, in this case, is

$$U(x_1, x_2) = \frac{b^2x_1x_2^3 - a^2x_1^3x_2}{12(a^2 + b^2)},$$

where a and b are the semiaxes. The torsion function φ for this section is given by formula (36.6).

[2] L. N. G. Filon, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 198 (1902), p. 147. This problem was reconsidered by A. Timpe, *Mathematische Annalen*, vol. 71 (1912), p. 480.

[3] N. V. Zvolinsky and P. M. Riz, *Izvestiya Akademiya Nauk SSSR*, No. 10 (1939), pp. 21–26.

P. M. Riz, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, New Series, vol. 4 (1940), pp. 121–122.

These authors also consider the case where T varies linearly along the length of the cylinder. The paper by Zvolinsky and Riz contains explicit formulas for the displacements and stresses in a circular cylinder twisted by tangential tractions.

[4] S. G. Lekhnitzky, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 2 (1939), pp. 345–368; vol. 6 (1942), pp. 3–18. See also this author's monograph Theory of Elasticity of an Anisotropic Body (1950) (in Russian).

H. Luxenberg, *Journal of Research of the National Bureau of Standards*, vol. 50 (1953), pp. 263–276.

[5] V. K. Prokopov, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 13 (1949), pp. 135–144 (in Russian).

[6] B. L. Abramyan and M. M. Dzerbashyan, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 15 (1951), pp. 451–472 (in Russian). See also B. A. Kostandyan, *Izvestiya Akademii Nauk Armyanskoi SSR*, Physics and Math Series 7, No. 4 (1954), pp. 23–53 (in Russian).

[7] G. S. Shapiro, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 8 (1944); vol. 17 (1953), pp. 248–252 (in Russian).

CHAPTER 5

TWO-DIMENSIONAL ELASTOSTATIC PROBLEMS

65. Introductory Remarks. This chapter is devoted to a concise presentation of one general method of solution of certain broad classes of two-dimensional boundary-value problems in elasticity. In contradistinction to familiar general methods, which rarely provide more than the proof of the existence of solutions, the method presented here proved effective in deducing explicit solutions of many technically important problems. It also gave a powerful impetus to several new theoretical developments, particularly in the domain of the contact problems in elasticity.

The method is based on a reduction of the boundary-value problems in elasticity to the solutions of certain functional equations in a complex domain, and, in its simpler aspects, its effectiveness has already been demonstrated in the preceding chapter.

Although the systematic use of the complex variable theory in plane elasticity was proposed by Kolossoff[1] as early as 1909, nearly forty years elapsed before the theory, based on Kolossoff's idea, was brought to a successful conclusion. This was accomplished, in the main, by a group of Russian mathematicians inspired by the work of Muskhelishvili.[2]

The two-dimensional problems with which we shall be concerned in this chapter fall into two physically distinct types. One of these arises

[1] G. V. Kolossoff, "On One Application of the Theory of Functions of a Complex Variable to a Plane Problem in the Mathematical Theory of Elasticity," a dissertation at Dorpat (Yurieff) University (1909) (in Russian). See also G. V. Kolossoff, *Zeitschrift für Mathematik und Physik*, vol. 62 (1914), pp. 384–409, and his Russian monograph An Application of the Complex Variable in the Theory of Elasticity (1935).

[2] An accessible account of the earlier work is contained in N. I. Muskhelishvili's paper entitled, "Recherches sur les problèmes aux limites relatifs à l'équation biharmonique et aux équations de l'élasticité a deux dimensions," *Mathematische Annalen*, vol. 107 (1932), pp. 282–312. A comprehensive up-to-date treatment, based, in part, on Muskhelishvili's Theory of Singular Integral Equations, was published in the third edition of Muskhelishvili's remarkable monograph entitled Some Basic Problems of the Mathematical Theory of Elasticity (1949). An English translation of these books was released in December, 1953, by P. Noordhoff, N. V., of Groningen:

N. I. Muskhelishvili, Singular Integral Equations (1953).

N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity (1953).

in the study of deformation of large cylindrical bodies acted on by the external forces so distributed that the component of deformation in the direction of the axis of the cylinder vanishes and the remaining components do not vary along the length of the cylinder. This is the class of problems in *plane deformation*, or *plane strain*. The other type appears in the study of the deformation of thin plates, the state of stress in which is characterized by the vanishing of the stress components in the direction of the thickness of the plate. These are the problems in *plane stress*. It turns out that the mathematical formulation of these physically distinct types of problems is identical and that their solution hinges on the determination of two functions of a complex variable from certain functional equations.

We proceed to derive these equations and solve them for several technically important problems.

The coordinates x_i used throughout this chapter are rectangular cartesian, and we use Latin indices for the range 1, 2, 3 and Greek indices for the range 1, 2. As in the earlier chapters of this book, a repeated index represents the sum for all allowable values of that index. The notation for all symbols is identical with that introduced in the first three chapters, except that we omit writing the superscript ν in the designation of the components $\overset{\nu}{T}_i$ of stress acting on an element of surface with the unit normal $\mathbf{v}$.

66. Plane Deformation. A body is said to be in the state of *plane deformation*, or *plane strain*, parallel to the x_1x_2-plane, if the component u_3 of the displacement vector $\mathbf{u}$ vanishes[1] and the components u_1 and u_2 are functions of the coordinates x_1 and x_2, but not of x_3. Thus, the state of plane deformation is characterized by the formulas,

$$\text{(66.1)} \qquad \begin{cases} u_\alpha = u_\alpha(x_1, x_2), \\ u_3 = 0. \end{cases}$$

It follows from (66.1) and from the definitions (7.5) of the strain and rotation tensors that the nonvanishing components of these tensors are given by the formulas,

$$\text{(66.2)} \qquad \begin{cases} e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \\ \omega_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}), \end{cases}$$

which, clearly, do not depend on x_3.

If we insert from (66.2) in the stress-strain relations,

$$\text{[22.3]} \qquad \tau_{ij} = \lambda\vartheta\delta_{ij} + 2\mu e_{ij},$$

[1] Some writers define the state of plane strain by requiring that u_3 = const and u_1 and u_2 be the functions of x_1 and x_2.

we get

$$\tau_{\alpha\beta} = \lambda\vartheta\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha}), \tag{66.3}$$

$$\tau_{33} = \lambda\vartheta, \qquad \tau_{13} = \tau_{23} = 0, \tag{66.4}$$

where the dilatation $\vartheta = u_{\alpha,\alpha}$.

It is easy to show that τ_{33} is proportional to the sum $\tau_{11} + \tau_{22}$. Indeed, using (66.3), we get

$$\begin{aligned}\tau_{11} + \tau_{22} &= (\lambda\vartheta + 2\mu u_{1,1}) + (\lambda\vartheta + 2\mu u_{2,2}) \\ &= 2(\lambda + \mu)\vartheta,\end{aligned}$$

and, noting (66.4), we have

$$\begin{aligned}\tau_{33} &= \frac{\lambda}{2(\lambda + \mu)}(\tau_{11} + \tau_{22}) \\ &= \sigma(\tau_{11} + \tau_{22}),\end{aligned} \tag{66.5}$$

since $\sigma = \lambda/[2(\lambda + \mu)]$ by (23.3).

It is clear that the deformation and stresses of a body in the state of plane strain are completely determined by the five functions $\tau_{\alpha\beta}(x_1, x_2)$ and $u_\alpha(x_1, x_2)$. We consider next the physical circumstances giving rise to the state of plane strain.

From equilibrium equations

$$\tau_{ij,j} = -F_i, \tag{15.3}$$

we conclude that the components F_1 and F_2 of the body force must be independent of x_3 inasmuch as the τ_{ij} do not depend on x_3. Also, $F_3 \equiv 0$, since $\tau_{33,3} = -F_3$ and τ_{33} is not a function of x_3. Assuming that these conditions are fulfilled by the body force F_i, we have, for the determination of the five quantities $\tau_{\alpha\beta}$, u_α, a pair of equilibrium equations

$$\tau_{\alpha\beta,\beta} = -F_\alpha(x_1, x_2), \tag{66.6}$$

and three equations (66.3). The substitution from (66.3) in (66.6) yields the appropriate Navier equations:

$$\mu\nabla^2 u_\alpha + (\lambda + \mu)\frac{\partial\vartheta}{\partial x_\alpha} = -F_\alpha(x_1, x_2), \tag{66.7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Since the deformation of a body in the state of plane strain is independent of the x_3-coordinate, we need consider only the deformation of any section of the body by a plane normal to the x_3-axis. Equation (66.7) then must be satisfied in some two-dimensional region R of the cross section of the body formed by the plane x_3 = const. If the displacements u_α of points on the boundary C of R are specified, we have a two-dimensional analog

of the second boundary-value problem formulated in Sec. 24. The uniqueness of solution of this problem for finite[1] domains follows from considerations of Sec. 27.

An analog of the first boundary-value problem of elasticity can be got by phrasing the plane-strain problem entirely in terms of stresses. We recall that if the solutions of the equilibrium equations (15.3) are to correspond to the state of stress that can exist in an elastic body, the τ_{ij} must satisfy the Beltrami-Michell compatibility equations (24.15). A specialization of these equations to the problem of plane strain leads to only one nontrivial compatibility equation in the form,[2]

$$\nabla^2 \Theta_1 = -\frac{2(\lambda + \mu)}{\lambda + 2\mu} F_{\alpha,\alpha}, \tag{66.8}$$

where

$$\Theta_1 \equiv \tau_{11} + \tau_{22}.$$

Now if the components $T_\alpha(x_1, x_2)$ of external stresses are specified along the contour C in the form

$$\tau_{\alpha\beta}\nu_\beta = T_\alpha(x_1, x_2), \tag{66.9}$$

where the ν_α are components of the exterior unit normal vector to C, the formulation of the first boundary-value problem is complete. We seek a solution of the system of Eqs. (66.6) and (66.8) in the region R, subject to the conditions (66.9) on the boundary. Again, the uniqueness of solution of this system, for finite domains, follows from the considerations of Sec. 27.

The physical situation corresponding to this problem is the following: Consider a cylinder with plane ends and with generators parallel to the x_3-axis (Fig. 49). If the lateral surface of such a cylinder is subjected to the action of surface tractions with components $T_\alpha(x_1, x_2)$, which do not vary along the axis of the cylinder, and the component $T_3 = 0$, the situation corresponds to the mathematical problem just considered, provided the tractions $T_\alpha(x_1, x_2)$ maintain the cylinder in equilibrium. If the body forces F_i are present,[3] we must further suppose that the components F_α are independent of the x_3-coordinate and that $F_3 \equiv 0$.

The state of stress in the cylinder, in this case, is determined by the

[1] Only such domains have been considered in Sec. 27. To ensure uniqueness in an infinite domain, it is necessary to impose certain restrictions on the behavior of displacements (or stresses) at infinity. These arise from the requirement that the integrals in the transformation theorems used in Sec. 27 have a meaning. See Sec. 74.

[2] We omit calculations which are entirely similar to those performed in Sec. 24 for the three-dimensional case. Since $u_3 = 0$ and the u_α are independent of x_3, the set of six Saint-Venant's compatibility equations (10.10) reduces to one nontrivial equation $e_{11,22} + e_{22,11} = 2e_{12,12}$.

[3] Of course, the external forces T_α and F_α must be assigned in such a way that the resultant force and the resultant moment acting on the cylinder as a whole vanish.

solution of the system of Eqs. (66.7) to (66.9). From Eq. (66.5) it follows that the ends of the cylinder are subjected to the action of the longitudinal force with the resultant

$$\iint_R \tau_{33}\, dx_1\, dx_2$$

The distribution of stress $\tau_{33} = \sigma(\tau_{11} + \tau_{22})$ over the ends of the cylinder may also produce a bending couple whose moment lies in the planes of the ends. Indeed, the longitudinal stresses τ_{33} are necessary to maintain the cylinder in the state of plane deformation; without their presence the displacement u_3 will not, in general, vanish. If, however, in the given physical problem the ends of the cylinder are free, the desired solution can be got by superposing, on the solution of the plane problem just

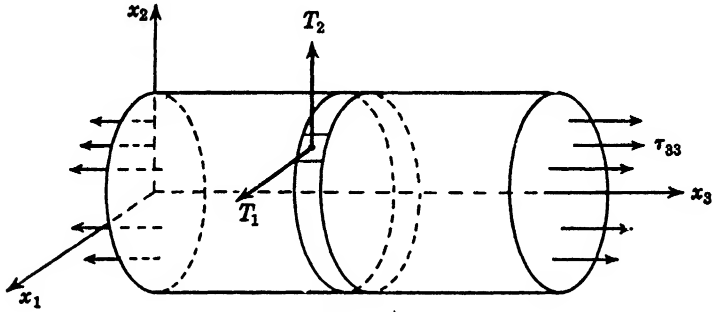

FIG. 49

considered, the solution of an auxiliary problem. This auxiliary problem concerns the deformation of a cylinder with free lateral surface by the end loads equal and opposite to those given by equation (66.5). If the cylinder is long, the auxiliary problem is in the category of simple Saint-Venant's problems[1] fully considered in Chap. 4. After superposing the solution of an auxiliary problem on the solution of the plane-deformation problem, the resulting deformation will not, in general, be plane.

67. Plane Stress. Generalized Plane Stress. A body is in the state of *plane stress* parallel to the x_1x_2-plane when the stress components τ_{13} τ_{23}, τ_{33} vanish.

If we write the stress-strain relations (22.3) in the form

(67.1) $$\tau_{ij} = \lambda\vartheta\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \qquad \vartheta = u_{i,i}$$

and let $\tau_{33} = 0$, we get

(67.2) $$u_{3,3} = -\frac{\lambda}{\lambda + 2\mu}(u_{1,1} + u_{2,2}).$$

[1] We thus have to consider the problem of extension of cylinders by longitudinal forces and the problem of pure bending. These, as we saw in Secs. 30 and 32, are quite elementary problems.

Substituting this in (67.1) yields the following expressions for the nonvanishing components $\tau_{\alpha\beta}$:

$$
(67.3) \qquad \begin{cases} \tau_{11} = \dfrac{2\lambda\mu}{\lambda + 2\mu}(u_{1,1} + u_{2,2}) + 2\mu u_{1,1}, \\ \tau_{22} = \dfrac{2\lambda\mu}{\lambda + 2\mu}(u_{1,1} + u_{2,2}) + 2\mu u_{2,2}, \\ \tau_{12} = \mu(u_{1,2} + u_{2,1}). \end{cases}
$$

If these expressions are inserted in the equilibrium equations

$$
(67.4) \qquad \tau_{\alpha\beta,\beta} + F_\alpha = 0,
$$

one obtains a pair of differential equations for the u_α, namely,

$$
(67.5) \qquad \left(\frac{2\lambda\mu}{\lambda + 2\mu} + \mu\right)\frac{\partial\vartheta_1}{\partial x_\alpha} + \mu\nabla_1^2 u_\alpha = -F_\alpha,
$$

where $\vartheta_1 \equiv u_{1,1} + u_{2,2}$ and $\nabla_1^2 \equiv \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$.

Formally, Eqs. (67.3), (67.4), (67.5) become identical with (66.3), (66.6), (66.7), if one replaces the constant $2\lambda\mu/(\lambda + 2\mu) \equiv \bar{\lambda}$ by λ, but there is a fundamental distinction in the two sets of stressed states. In the plane-strain problem, the u_α and $\tau_{\alpha\beta}$ are independent of the x_3-coordinate, whereas in the problem of plane stress these functions may depend on x_3. Since the variable x_3 may appear as a parameter in all equations of this section, the problem is not truly two-dimensional. However, following the idea of Filon,[1] it is possible to modify the system of Eqs. (67.3) to (67.5) in such a way that the resulting two-dimensional system corresponds to a physical problem of great practical interest.

Consider a cylinder with the generators parallel to the x_3-axis and with bases in the planes $x_3 = \pm h$ (Fig. 50). We shall term such cylinder a *plate* if its height $2h$ is small compared with the linear dimensions of the cross section. The bases of the cylinder are the *faces* of the plate, and the plane $x_3 = 0$ is the *middle plane* of the plate.

Let us suppose that the faces of the plate are free of applied loads and all external surface forces act on the *edge* of the plate, that is, on the lateral surface of the cylinder. Moreover, we shall suppose that the forces acting on the edge lie in the planes parallel to the middle plane and are symmetrically distributed with respect to it. If the components of external surface forces acting on an element $d\sigma \equiv 2h\,ds$ of the edge are $T_\alpha 2h\,ds$, the vector T_α is the stress vector applied to the contour C bounding the middle plane. We shall further suppose that the com-

[1] L. N. G. Filon, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 201 (1903), pp. 63–155; *Quarterly Journal of Applied Mathematics*, Oxford Series, I (1930), pp. 289–299.

ponent F_3 of the body force vanishes and the components F_α are symmetrical with respect to the middle plane. Under these hypotheses, the points of the middle plane will undergo no displacement in the direction of the x_3-axis, and if the plate is thin, the displacement u_3 will be small. Indeed, the symmetry of distribution of external forces implies that the mean value of u_3 with respect to the thickness of the plate is precisely zero. For thin plates the mean values $\bar{u}_i$ of the displacements u_i give as

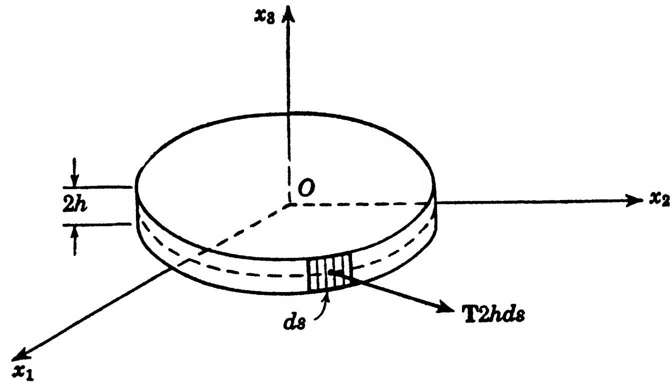

FIG. 50

useful information as that furnished by the u_i. This suggests dealing with the average values

$$\bar{u}_i(x_1, x_2) \equiv \frac{1}{2h} \int_{-h}^{h} u_i(x_1, x_2, x_3)\, dx_3, \tag{67.6}$$

where, as we already noted, $\bar{u}_3 = 0$.

Since the faces of the plate are assumed free of external loads,

$$\tau_{13}(x_1, x_2, \pm h) = \tau_{23}(x_1, x_2, \pm h) = \tau_{33}(x_1, x_2, \pm h) = 0, \tag{67.7}$$

and these equations together with the equilibrium equation

$$\tau_{13,1} + \tau_{23,2} + \tau_{33,3} = 0, \tag{67.8}$$

demand[1] that $\tau_{33,3}(x_1, x_2, \pm h) = 0$. The fact that τ_{33} *and* its derivative with respect to x_3 vanish on the faces of the plate suggests that τ_{33} can differ from zero but slightly throughout the plate if h is small. This justifies us in assuming that $\tau_{33} \equiv 0$.

The remaining equilibrium equations

$$\tau_{\alpha 1,1} + \tau_{\alpha 2,2} + \tau_{\alpha 3,3} + F_\alpha = 0,$$

[1] From (67.7) we conclude that $\tau_{13,1}(x_1, x_2, \pm h) = \tau_{23,2}(x_1, x_2, \pm h) = 0$, and since (67.8) is valid throughout the plate, it follows upon setting $x_3 = \pm h$ in (67.8) that $\tau_{33,3}(x_1, x_2, \pm h) = 0$.

upon integration with respect to x_3 between the limits $-h$ and $+h$, yield

$$\frac{1}{2h}\int_{-h}^{h}(\tau_{\alpha 1,1}+\tau_{\alpha 2,2}+\tau_{\alpha 3,3}+F_\alpha)\,dx_3=0. \tag{67.9}$$

Since

$$\int_{-h}^{h}\tau_{\alpha 3,3}\,dx_3=\tau_{\alpha 3}(x_1,x_2,\pm h)=0,$$

by (67.7), we can write (67.9) in the form

$$\bar\tau_{\alpha 1,1}+\bar\tau_{\alpha 2,2}+\bar F_\alpha=0, \tag{67.10}$$

where

$$\left|\begin{aligned}\bar\tau_{\alpha\beta}(x_1,x_2)&\equiv\frac{1}{2h}\int_{-h}^{h}\tau_{\alpha\beta}(x_1,x_2,x_3)\,dx_3,\\ \bar F_\alpha(x_1,x_2)&\equiv\frac{1}{2h}\int_{-h}^{h}F_\alpha(x_1,x_2,x_3)\,dx_3,\end{aligned}\right. \tag{67.11}$$

are the mean values of $\tau_{\alpha\beta}$ and F_α.

If we form the mean values in the stress-strain relations (67.3) we get three equations,

$$\bar\tau_{\alpha\beta}=\bar\lambda\bar\vartheta\delta_{\alpha\beta}+\mu(\bar u_{\alpha,\beta}+\bar u_{\beta,\alpha}), \tag{67.12}$$

with $\bar\lambda\equiv 2\lambda\mu/(\lambda+2\mu)$ and $\bar\vartheta\equiv\bar u_{\alpha,\alpha}$. These, together with two equations (67.10), serve to determine the five unknown mean values $\bar u_\alpha(x_1,x_2)$ and $\bar\tau_{\alpha\beta}(x_1,x_2)$.

The substitution from (67.12) in (67.10) yields two equations of the Navier type,

$$\mu\nabla^2\bar u_\alpha+(\bar\lambda+\mu)\frac{\partial\bar\vartheta}{\partial x_\alpha}+\bar F_\alpha(x_1,x_2)=0, \tag{67.13}$$

from which the average displacements $\bar u_\alpha$ can be determined when the values of the $\bar u_\alpha$ are specified on the contour.

The system of equations involving the average stresses $\bar\tau_{\alpha\beta}$ can be got by deducing the corresponding Beltrami-Michell compatibility equations. It turns out to be[1]

$$\nabla^2\bar\Theta_1=-\frac{2(\bar\lambda+\mu)}{\bar\lambda+2\mu}\bar F_{\alpha,\alpha}, \tag{67.14}$$

where $\bar\Theta_1=\bar\tau_{11}+\bar\tau_{22}$.

This equation, together with the equilibrium equations (67.10), suffices to determine the mean stresses $\bar\tau_{\alpha\beta}$ when the boundary conditions on the edge are given in the form,

$$\tau_{\alpha\beta}\nu_\beta=T_\alpha.$$

For integrating these equations with respect to x_3 between the limits $-h$

[1] Compare with Eq. (66.8).

and $+h$ and dividing by $2h$ yields,

$$\bar{\tau}_{\alpha\beta}\nu_\beta = \bar{T}_\alpha(s) \qquad \text{on } C, \tag{67.15}$$

where $\bar{T}_\alpha(s)\, ds$ are the components of applied force acting on the element of arc ds of the contour C.

The two-dimensional boundary-value problem consisting of the system of Eqs. (67.10), (67.14), and (67.15) is known as the problem in *generalized* plane stress.[1]

68. Plane Elastostatic Problems. The discussion of the plane-deformation problem in Sec. 66, and of the generalized plane-stress problem in Sec. 67, shows that their mathematical formulations are identical. The relevant differential equations and boundary conditions in Sec. 67 differ from those in Sec. 66 only in the appearance of the barred symbols: $\bar{u}_\alpha$, $\bar{\tau}_{\alpha\beta}$, $\bar{\lambda}$, $\bar{T}_\alpha$, etc. Henceforth we shall refer to problems of these two types as *plane elastostatic problems*.

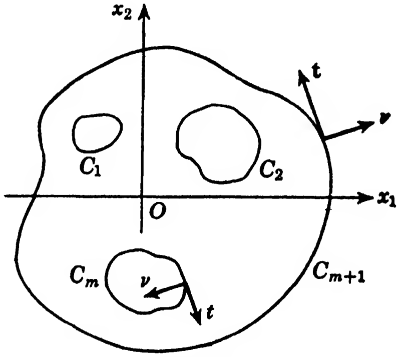

FIG. 51

In the formulation of these plane problems no restrictions on the connectivity of the region R was introduced. If the region R is multiply connected and finite, we shall suppose that its boundary C consists of $m + 1$ simple closed contours C_i, such that the exterior contour C_{m+1} contains within it m contours C_i (Fig. 51). We shall suppose that the contours C_i, with the possible exception of a finite number of points, are smooth. This means that a smooth contour C_i can be represented parametrically by equations of the form $x_\alpha = x_\alpha(s)$, where the functions $x_\alpha(s)$ have continuous derivatives that do not vanish for the same value of the arc parameter s. We shall agree that the positive direction of description of the contours is such that the region R remains on the left in the course of tracing the contour C_i. The positive direction for the tangent vector $\mathbf{t}$ along C_i is that of the positive direction of description of C_i and the positive unit normal $\boldsymbol{\nu}$ at any point of the contour is directed outward relative to the region R.

In the boundary conditions,

$$\begin{cases} \tau_{\alpha\beta}\nu_\beta = T_\alpha & \text{on } C, \\ u_\alpha = f_\alpha & \text{on } C, \end{cases} \tag{68.1}$$

[1] This terminology was introduced by A. E. H. Love, but such states of stress were first investigated by Filon in the study of bending of a beam with rectangular cross section. See L. N. G. Filon, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 201 (1903), pp. 63–155.

the boundary C is interpreted to mean $C_{m+1} + C_1 + \cdots + C_m$, so that the functions T_α and f_α are specified on each of the contours C_i forming the boundary of R. The functions T_α must clearly be such that the resultant force and the resultant moment of all external forces applied to C vanish, since the region R is in equilibrium.

If the exterior contour C_{m+1} is allowed to recede to infinity, an infinite region R is obtained. This region is bounded by the contours $C_1, C_2, \ldots, C_m$, and it corresponds to an infinite plate with m holes bounded by the C_i ($i = 1, 2, \ldots, m$), or, alternatively, to an infinite solid with longitudinal cavities. To ensure the existence and uniqueness of the solution of plane elastostatic problems for infinite regions, it is necessary to impose certain restrictions on the behavior of stresses and displacements at infinity.[1]

The treatment of plane problems of elasticity simplifies somewhat when the body forces F_α do not appear in the differential equations. But since these equations are linear, it is always possible to reduce them to a homogeneous form by finding one of the infinitely many particular integrals. Thus, if $\tau_{\alpha\beta}^{(0)}$ is any set of functions satisfying the equilibrium equations (67.4), then the functions $\tau_{\alpha\beta}^{(1)}$ defined by

$$\tau_{\alpha\beta} = \tau_{\alpha\beta}^{(0)} + \tau_{\alpha\beta}^{(1)}$$

satisfy the homogeneous system

$$\tau_{\alpha\beta,\beta}^{(1)} = 0.$$

Similar considerations apply to Eqs. (66.7).

In the following sections we shall suppose that the differential equations have been reduced to a form in which the body forces are not present. Since in technical applications constant gravitational forces (the weight of the body) and uniform centrifugal forces frequently arise, we record here suitable particular integrals for these types of body forces:

1. Let the constant gravitational force F_α be directed along the x_2-axis. Then

$$F_1 = 0, \qquad F_2 = -g\rho,$$

where ρ is the density per unit volume and g the gravitational acceleration. The particular integrals $\tau_{\alpha\beta}^{(0)}$ and $u_\alpha^{(0)}$ of Eqs. (66.3) and (66.7), respectively, are:

$$(68.2)\qquad \begin{cases} \tau_{11}^{(0)} = \tau_{12}^{(0)} = 0, \qquad \tau_{22}^{(0)} = \rho g x_2, \\ u_1^{(0)} = -\dfrac{\lambda}{4\mu(\lambda+\mu)}\rho g x_1 x_2, \\ u_2^{(0)} = \dfrac{\lambda+2\mu}{8\mu(\lambda+\mu)}\rho g x_2^2 + \dfrac{\lambda}{8\mu(\lambda+\mu)}\rho g x_1^2. \end{cases}$$

[1] See Secs. 72 and 74.

2. If a uniform centrifugal force acts on a body, rotating with constant angular velocity ω about the x_3-axis, then $F_\alpha = \rho\omega^2 x_\alpha$ and we can take

$$(68.3)\qquad \begin{cases} \tau_{\alpha\beta}^{(0)} = -\dfrac{2\lambda+\mu}{4(\lambda+2\mu)}\,\rho\omega^2(x_1^2+x_2^2)\delta_{\alpha\beta} - \dfrac{\mu}{2(\lambda+2\mu)}\,\rho\omega^2 x_\alpha x_\beta, \\ u_\alpha^{(0)} = -\dfrac{\rho\omega^2}{8(\lambda+2\mu)}\,(x_1^2+x_2^2)x_\alpha. \end{cases}$$

Clearly, the boundary conditions (68.1), upon setting

$$\tau_{\alpha\beta} = \tau_{\alpha\beta}^{(0)} + \tau_{\alpha\beta}^{(1)},$$
$$u_\alpha = u_\alpha^{(0)} + u_\alpha^{(1)},$$

will yield new boundary conditions, associated with the homogeneous equations. They will be in the form:

$$(68.4)\qquad \begin{cases} \tau_{\alpha\beta}^{(1)}\nu_\beta = T_\alpha^{(1)}, \\ u_\alpha^{(1)} = f_\alpha^{(1)}, \qquad \text{on } C. \end{cases}$$

69. Airy's Stress Function. We noted in the preceding section that the boundary-value problems in plane elasticity can always be reduced to the study of the case in which the body forces are absent. Accordingly, we consider the equilibrium equations in the form

$$(69.1)\qquad \tau_{\alpha\beta,\beta} = 0,$$

in which the $\tau_{\alpha\beta}$, as follows from Eqs. (66.8) and (67.14), satisfy in the region R the compatibility condition

$$(69.2)\qquad \nabla^2(\tau_{11} + \tau_{22}) = 0.$$

On the boundary C of R, the functions $\tau_{\alpha\beta}$ must be chosen so that

$$(69.3)\qquad \tau_{\alpha\beta}\nu_\beta = T_\alpha(s),$$

where the $T_\alpha(s)$ are known functions of the arc parameter s on C.

The equilibrium equations (69.1), as first noted by a British astronomer G. B. Airy, imply the existence of a function $U(x_1, x_2)$ such that

$$(69.4)\qquad \tau_{22} = U_{,11}, \qquad \tau_{12} = -U_{,12}, \qquad \tau_{11} = U_{,22}.$$

Indeed, the $\tau_{\alpha\beta}$ thus related to the Airy function $U(x_1, x_2)$, satisfy Eqs. (69.1) identically, while the compatibility equation (69.2) demands that $U(x_1, x_2)$ satisfies the *biharmonic equation*

$$\nabla^2\nabla^2 U = 0,$$

or

$$(69.5)\qquad \nabla^4 U \equiv U_{,1111} + 2U_{,1122} + U_{,2222} = 0,$$

in the region R.

Every solution of this equation of class C^4 is termed a *biharmonic function,* but inasmuch as we are interested in those states of stress for which

the $\tau_{\alpha\beta}$ are single-valued, we need consider only biharmonic functions with single-valued second partial derivatives [see (69.4)]. The function U and its first derivatives may be multiple-valued.

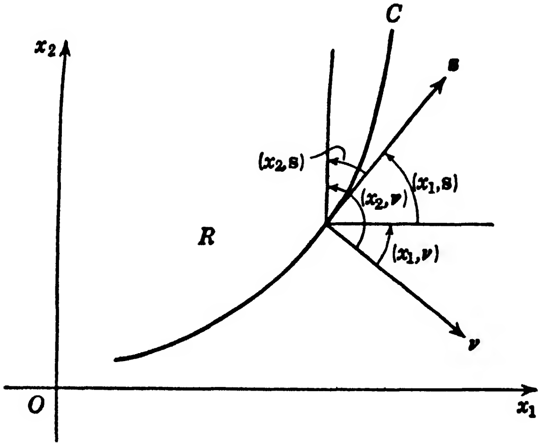

FIG 52

The boundary conditions (69.3) impose a restriction on the choice of U. Substituting in (69.3) from (69.4), we get

$$\text{(69.6)} \qquad \begin{cases} U_{,22}\nu_1 - U_{,12}\nu_2 = T_1(s), \\ -U_{,12}\nu_1 + U_{,11}\nu_2 = T_2(s), \end{cases}$$

but, from Fig. 52,

$$\text{(69.7)} \qquad \begin{cases} \nu_1 = \cos(x_1, \nu) = \cos(x_2, s) = \dfrac{dx_2}{ds}, \\ \nu_2 = \cos(x_2, \nu) = -\cos(x_1, s) = -\dfrac{dx_1}{ds}, \end{cases}$$

so that (69.6) can be written in the form

$$\text{(69.8)} \qquad \begin{cases} \dfrac{d}{ds}(U_{,2}) = T_1(s), \\ -\dfrac{d}{ds}(U_{,1}) = T_2(s). \end{cases}$$

Integrating these equations along C from some fixed point s_0 to a variable point s, we get

$$\text{(69.9)} \qquad \begin{cases} U_{,1}(s) = -\displaystyle\int_{s_0}^{s} T_2(s)\,ds \equiv f_1(s) + c_1, \\ U_{,2}(s) = \displaystyle\int_{s_0}^{s} T_1(s)\,ds \equiv f_2(s) + c_2. \end{cases}$$

It is clear from (69.9) that the derivatives of U along C are not determined uniquely. Moreover, if the region is multiply connected, the

integration has to be performed over each contour C_i forming the boundary of R and the resulting functions $f_\alpha(s)$ need not be single-valued. Some degree of arbitrariness in the choice of U and its derivatives is to be expected, however, inasmuch as the stresses $\tau_{\alpha\beta}$ are determined by the second derivatives of U.

We see that the boundary-value problem characterized by the system of Eqs. (69.1), (69.2), (69.3) is intimately related to the boundary-value problem of the type:

$$\left\{\begin{aligned} \nabla^4 U &= 0 \qquad \text{in } R, \\ U_{,\alpha} &= f_\alpha(s) \qquad \text{on } C, \end{aligned}\right. \tag{69.10}$$

wherein the $f_\alpha(s)$ are certain known functions. The problem (69.10) was the subject of numerous investigations that have led to developments of cardinal importance in the theory of differential and integral equations, in the calculus of variations, and in several other branches of analysis.[1] It is known as the *fundamental biharmonic boundary-value problem.*

This problem can be phrased in a somewhat different form, by observing that the knowledge of the $U_{,\alpha}(s)$ on C permits one to compute the value of $U(s)$ and of its normal derivative $\dfrac{dU}{d\nu}$ on C.

Indeed

$$\begin{aligned} \frac{dU}{d\nu} = U_{,\alpha}\nu_\alpha &= U_{,1}(s)\frac{dx_2}{ds} - U_{,2}(s)\frac{dx_1}{ds} \\ &\equiv g(s), \end{aligned}$$

and since

$$\begin{aligned} dU &= U_{,\alpha}\, dx_\alpha = f_\alpha\, dx_\alpha, \\ U(s) &= \int_{s_0}^{s} f_\alpha \frac{dx_\alpha}{ds}\, ds \equiv f(s) + \text{const.} \end{aligned}$$

Conversely, if U and $\dfrac{dU}{d\nu}$ are known on C, we can compute the $U_{,\alpha}(s)$. Accordingly, the problem (69.10) can be written in an equivalent form,

$$\left\{\begin{aligned} \nabla^4 U &= 0 \qquad \text{in } R, \\ U &= f(s) + \text{const} \\ \frac{dU}{d\nu} &= g(s) \end{aligned}\right\} \quad \text{on } C, \tag{69.11}$$

which is more convenient in some investigations.

A simple modification of computations of this section permits one to extend the results to problems in which certain types of body forces are present. If the body forces have a potential V determined by equations[2]

$$F_\alpha = -V_{,\alpha},$$

[1] A special case of this problem first arose in the study of the transverse deflections of clamped elastic plates.

[2] This is the case with the gravitational and centrifugal forces discussed in Sec. 68.

we consider the nonhomogeneous equations (67.4) and write

$$\tau_{11} = U_{,22} + V, \qquad \tau_{12} = -U_{,12}, \qquad \tau_{22} = U_{,11} + V.$$

The compatibility equation (66.8) now yields the equation

$$\nabla^4 U = -\frac{2\mu}{\lambda + 2\mu} \nabla^2 V.$$

If V is harmonic, then, as above, U is a biharmonic function. Otherwise, we are led to consider equations of the form

$$\nabla^4 U = F(x_1, x_2) \qquad \text{in } R.$$

70. General Solution of the Biharmonic Equation. The solution of the fundamental biharmonic boundary-value problem can be made to depend on a certain general representation of the biharmonic function by means of two analytic functions of a complex variable.[1] We consider the biharmonic equation

$$\nabla^2\nabla^2 U = 0 \qquad \text{in } R, \tag{70.1}$$

and if we let $\nabla^2 U \equiv P_1(x_1, x_2)$, the function P_1 is, clearly, harmonic in R. Consequently[2] we can construct an analytic function

$$F(z) \equiv P_1 + iP_2$$

of a complex variable $z = x_1 + ix_2$ by computing from P_1 the conjugate

[1] This representation was first obtained by E. Goursat, *Bulletin de la société mathématique de France*, vol. 26 (1898), p. 236, who assumed that the biharmonic function is analytic. A derivation given here is due to N. I. Muskhelishvili, *Izvestiya (Bulletin) Akademiya Nauk SSSR* (1919), pp. 663–686. The analyticity of the biharmonic function is not hypothesized here, and, indeed, it follows from the representation itself.

[2] We use the term *harmonic function* only for single-valued functions of class C^2 which satisfy Laplace's equation in the given region. Since the second derivatives of the biharmonic function U are related to stresses by (69.4), the function $P_1 \equiv \nabla^2 U$ is necessarily single-valued. If $P_1(x_1, x_2)$ is known, its conjugate $P_2(x_1, x_2)$ is determined by integrating

$$\begin{aligned} dP_2 &= P_{2,1}\,dx_1 + P_{2,2}\,dx_2 \\ &= -P_{1,2}\,dx_1 + P_{1,1}\,dx_2, \end{aligned}$$

since Cauchy-Riemann equations demand that $P_{2,1} = -P_{1,2}$ and $P_{2,2} = P_{1,1}$. Then

$$P_2(x_1, x_2) = \int_{M_0}^{M} (-P_{1,2}\,dx_1 + P_{1,1}\,dx_2)$$

is independent of the path joining an arbitrary point $M_0(x_1^0, x_2^0)$, with the point $M(x_1, x_2)$, for P_1 is harmonic. It follows that $P_2(x_1, x_2)$ is determined to within an arbitrary constant C and, hence, $F(z) \equiv P_1 + iP_2$, to within a pure imaginary constant Ci. If the region R is simply connected, $P_2(x_1, x_2)$, and hence $F(z)$, is single-valued. In a multiply connected region, $F(z)$ is, in general, multiple-valued, and we can confine our attention to some single-valued branch of $F(z)$. The same considerations apply to $\varphi(z) \equiv \frac{1}{4} \int F(z)\,dz$.

harmonic function P_2. The function $\varphi(z)$, defined by

$$\begin{aligned}\varphi(z) &\equiv \tfrac{1}{4}\int F(z)\, dz \\ &= p_1 + ip_2,\end{aligned} \tag{70.2}$$

is surely analytic, and therefore

$$\varphi'(z) = \frac{\partial p_1}{\partial x_1} + i\frac{\partial p_2}{\partial x_1} = \frac{1}{4}(P_1 + iP_2).$$

It follows from this, upon noting the Cauchy-Riemann equations $p_{1,1} = p_{2,2}$, $p_{1,2} = -p_{2,1}$, that

$$\begin{aligned} p_{1,1} &= p_{2,2} = \tfrac{1}{4}P_1, \\ p_{1,2} &= -p_{2,1} = -\tfrac{1}{4}P_2. \end{aligned}$$

Using these results and the fact that p_1 and p_2 are harmonic in R, we readily verify that

$$\nabla^2(U - p_1x_1 - p_2x_2) \equiv 0 \qquad \text{in } R.$$

Hence U has the structure,

$$U = p_1x_1 + p_2x_2 + q_1(x_1, x_2), \tag{70.3}$$

where $q_1(x_1, x_2)$ is harmonic in R.

Now if $\chi(z) \equiv q_1 + iq_2$ is an analytic function of z whose real part is q_1, the formula (70.3) can be written as

$$U = \mathcal{R}[\bar{z}\varphi(z) + \chi(z)], \tag{70.4}$$

where $\bar{z} \equiv x_1 - ix_2$ and $\mathcal{R}$ denotes the real part of the bracketed expression.

Since $\varphi(z)$ and $\chi(z)$ are analytic functions, it follows from (70.4) that $U(x_1, x_2)$ is of class C^∞ in R. The important representation (70.4) was first deduced by Goursat by different means.

If we denote the conjugate complex values by bars, so that, for example, $\overline{\varphi(z)} \equiv p_1 - ip_2$, then (70.4) can be written as

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}. \tag{70.5}$$

We shall make frequent use of this result in the sequel.

71. Formulas for Stresses and Displacements. The components $\tau_{\alpha\beta}$ of the stress tensor can be expressed in terms of the functions $\varphi(z)$ and $\chi(z)$, introduced in Sec. 70, by substituting in the relations

$$\tau_{11} = U_{,22}, \qquad \tau_{12} = -U_{,12}, \qquad \tau_{22} = U_{,11}, \tag*{[69.4]}$$

from the Goursat formula,

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}. \tag*{[70.5]}$$

To simplify calculations, we rewrite (69.4) as

$$(71.1)\qquad \begin{cases} \tau_{11} + i\tau_{12} = U_{,22} - iU_{,12} \equiv -i(U_{,1} + iU_{,2})_{,2}, \\ \tau_{22} - i\tau_{12} = U_{,11} + iU_{,12} \equiv (U_{,1} + iU_{,2})_{,1}, \end{cases}$$

and compute first the expression $U_{,1} + iU_{,2}$ from (70.5). We easily find[1] that

$$(71.2)\qquad U_{,1} + iU_{,2} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)},$$

where we have set

$$(71.3)\qquad \psi(z) \equiv \chi'(z).$$

Calculating the derivatives of (71.2) with respect to x_1 and x_2 and inserting the results in the right-hand members of (71.1) yield

$$\begin{aligned} \tau_{11} + i\tau_{12} &= \varphi'(z) + \overline{\varphi'(z)} - z\overline{\varphi''(z)} - \overline{\psi'(z)}, \\ \tau_{22} - i\tau_{12} &= \varphi'(z) + \overline{\varphi'(z)} + z\overline{\varphi''(z)} + \overline{\psi'(z)}, \end{aligned}$$

which can be written more compactly as[2]

$$(71.4)\qquad \begin{cases} \tau_{11} + \tau_{22} = 2[\varphi'(z) + \overline{\varphi'(z)}] \equiv 4\Re[\varphi'(z)], \\ \tau_{22} - \tau_{11} + 2i\tau_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)]. \end{cases}$$

The formulas for displacements can be obtained by integrating the stress-strain relations (66.3), which we can write as

$$(71.5)\qquad \begin{cases} \tau_{11} = U_{,22} = \lambda\vartheta + 2\mu u_{1,1}, \\ \tau_{22} = U_{,11} = \lambda\vartheta + 2\mu u_{2,2}, \\ \tau_{12} = -U_{,12} = \mu(u_{1,2} + u_{2,1}). \end{cases}$$

Solving the first two of these for $u_{1,1}$ and $u_{2,2}$, we get

$$\begin{aligned} 2\mu u_{1,1} &= -U_{,11} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} \nabla^2 U, \\ 2\mu u_{2,2} &= -U_{,22} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} \nabla^2 U, \end{aligned}$$

and, recalling the definitions

$$\nabla^2 U = P_1 = 4p_{1,1} = 4p_{2,2}$$

[1] We omit the details of elementary calculations making use of the formula

$$U_{,i} = \frac{\partial U}{\partial z}\frac{\partial z}{\partial x_i} + \frac{\partial U}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_i},$$

where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$.

[2] These useful formulas were deduced first by G. V. Kolossoff in references given in Sec. 65. The derivation sketched above is due to N. I. Muskhelishvili. See, for example, Sec. 32 of his book Some Basic Problems of the Mathematical Theory of Elasticity (1953).

in Sec. 70, we obtain

$$2\mu u_{1,1} = -U_{,11} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} p_{1,1},$$
$$2\mu u_{2,2} = -U_{,22} + \frac{2(\lambda + 2\mu)}{\lambda + \mu} p_{2,2}.$$

The integration of these equations yields,

$$\text{(71.6)} \qquad \begin{cases} 2\mu u_1 = -U_{,1} + \dfrac{2(\lambda + 2\mu)}{\lambda + \mu} p_1 + f(x_2), \\ 2\mu u_2 = -U_{,2} + \dfrac{2(\lambda + 2\mu)}{\lambda + \mu} p_2 + g(x_1), \end{cases}$$

where $f(x_2)$ and $g(x_1)$ are, as yet, arbitrary functions. The third of Eqs. (71.5) serves to determine f and g. Since $p_{1,2} = -p_{2,1}$, we easily find that

$$f'(x_2) + g'(x_1) = 0,$$

and hence

$$f(x_2) = \alpha x_2 + \beta,$$
$$g(x_1) = -\alpha x_1 + \gamma,$$

where α, β, and γ are constants. The forms of f and g indicate that they represent a rigid displacement and can thus be disregarded in the analysis of deformation.

If we set $f = g = 0$ in (71.6), recall that $\varphi = p_1 + ip_2$, and make use of (71.2), we easily deduce the compact formula

$$\text{(71.7)} \qquad 2\mu(u_1 + iu_2) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)},$$

where

$$\text{(71.8)} \qquad \varkappa \equiv \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\sigma,$$

σ being the Poisson ratio.

The formula (71.8) for $\varkappa$ is that corresponding to the state of plane strain. In the generalized plane-stress problems, λ must be replaced by $\bar{\lambda} = (2\lambda\mu)/(\lambda + 2\mu)$, and if the corresponding value of $\varkappa$ in (71.7) is denoted by $\bar{\varkappa}$, we find

$$\bar{\varkappa} = \frac{\bar{\lambda} + 3\mu}{\bar{\lambda} + \mu} = \frac{5\lambda + 6\mu}{3\lambda + 2\mu} = \frac{3 - \sigma}{1 + \sigma}.$$

We note that both $\varkappa$ and $\bar{\varkappa}$ are greater than 1.

Inasmuch as the functions $\varphi(z)$ and $\psi(z)$ are analytic in the interior of the region R, it follows from formulas (71.4) and (71.7) that the $\tau_{\alpha\beta}$ and u_α are analytic functions of the real variables x_1 and x_2 throughout the interior of the region occupied by the body.

As a consequence of this it is possible to prove that, if on any part of the boundary, however small,

$$T_\alpha = u_\alpha = 0,$$

then the $\tau_{\alpha\beta}$ vanish throughout the region R. This result is due to Almansi,[1] who proved this theorem for the three-dimensional case.

72. The Structure of Functions $\varphi(z)$ and $\psi(z)$. The considerations of Sec. 70 indicate that there is some freedom in the choice of functions appearing in the representation (70.5) of the general solution of the biharmonic equation. This implies some arbitrariness in the selection of functions $\varphi(z)$ and $\psi(z)$ in the representation of stresses and displacements by formulas (71.4) and (71.7). In this section we discuss the precise extent of this arbitrariness and record the structures of $\varphi(z)$ and $\psi(z)$ for several domains of interest in applications.

We begin with a finite simply connected domain R bounded by a contour C and raise the question: What is the difference in the forms of two sets of functions (φ, ψ) and (φ_0, ψ_0) that correspond to the same stress distribution in R?

If the stress distribution specified by φ and ψ is to be identical with that given by φ_0 and ψ_0, the formulas (71.4) demand that

$$\mathcal{R}[\varphi'(z)] = \mathcal{R}[\varphi_0'(z)] \tag{72.1}$$

and

$$\bar{z}\varphi''(z) + \psi'(z) = \bar{z}\varphi_0''(z) + \psi_0'(z). \tag{72.2}$$

From (72.1) we conclude that $\varphi_0'(z) = \varphi'(z) + ci$, where c is a real constant. Consequently,

$$\varphi_0(z) = \varphi(z) + c iz + \alpha, \tag{72.3}$$

where α is an arbitrary complex number.

Inserting this in (72.2) yields

$$\psi'(z) = \psi_0'(z),$$

so that

$$\psi_0(z) = \psi(z) + \beta, \tag{72.4}$$

where β is a complex constant.

Thus, if the state of stress in R is specified, the single-valued analytic functions $\varphi(z)$ and $\psi(z)$ are determined to within a linear function $ciz + \alpha$ and a constant β, respectively. Conversely, the state of stress in R will be unaltered if φ is replaced by $\varphi + ciz + \alpha$ and ψ by $\psi + \beta$.

Consider now the situation in which the displacements throughout R

[1] E. Almansi, *Atti della reale accademia dei nazionale Lincei*, vol. 16 (1907) pp. 865–868.

are specified. Inasmuch as the specification of displacements uniquely determines the state of stress, the extent of arbitrariness in choosing $\varphi_0(z)$ and $\psi_0(z)$ cannot be greater than that indicated by (72.3) and (72.4).

This time, however, the equality of displacements, as follows from (71.7), requires that

$$\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = \varkappa\varphi_0(z) - z\overline{\varphi_0'(z)} - \overline{\psi_0(z)},$$

and the substitution from (72.3) and (72.4) shows that

$$c = 0 \quad \text{and} \quad \varkappa\alpha = \bar{\beta}. \tag{72.5}$$

Hence, if the displacements u_i are known in R, the function φ is determined to within a complex constant α and the specification of this constant completely determines the constant β in $\psi_0(z)$.

If the origin of coordinates is taken within R, the functions $\varphi(z)$ and $\psi(z)$ will be determined uniquely for the given state of stress, if c, α, and β are chosen so that

$$\varphi(0) = 0, \qquad \mathscr{I}\varphi'(0) = 0, \qquad \psi(0) = 0. \tag{72.6}$$

If the displacements are known, c is necessarily zero and we can choose α so that $\varphi(0) = 0$. This choice fixes the value of β. We emphasize the fact that, in a finite simply connected region R, $\varphi(z)$ and $\psi(z)$ are single-valued analytic functions of z, and hence they have the power-series representations $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$, $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ in R.

If R is not simply connected, φ and ψ need not be single-valued, but it is not difficult to determine their structures if the stresses and displacements are assumed to be single-valued. Without going into details of the analysis, we record here the forms of φ and ψ in finite and infinite multiply connected domains.

We sketch the argument leading to the determination of the structures of these functions. As noted in Sec. 70, the real part $P_1(x_1, x_2) = \tau_{11} + \tau_{22}$ of the analytic function $F(z) = P_1 + iP_2$ is single-valued, but, in describing once each interior contour C_k $(k = 1, 2, \ldots, m)$, the imaginary part P_2 acquires a constant increment. If this increment is denoted by $8\pi A_k$, then the function $\varphi'(z) = \frac{1}{4}F(z)$ acquires an increment $2\pi i A_k$. But the function $A_k \log (z - z_k)$, where z_k is a point in the simply connected region R_k bounded by C_k, acquires precisely the increment $2\pi i A_k$ in going around the contour C_k. Hence

$$\varphi'(z) = \sum_{k=1}^{m} A_k \log (z - z_k) + f(z),$$

where $f(z)$ is single-valued and analytic in R. The integration then yields

$$\varphi(z) = \sum A_k z \log (z - z_k) + \sum^{m} B_k \log (z - z_k) + \varphi_0(z), \tag{a}$$

since the indefinite integral of $f(z)$ has the structure

$$\int f(z)\,dz = \sum_{k=1}^{m} B_k \log (z - z_k) + \varphi_0(z),$$

where $\varphi_0(z)$ is analytic and single-valued in R. It is clear from (a) that $\varphi''(z)$ is a single-valued function, and since the left-hand members of (71.4) are single-valued, it follows that $\psi'(z)$ is also single-valued. Therefore

$$(b) \qquad \psi(z) = \sum_{k=1}^{m} C_k \log (z - z_k) + \psi_0(z),$$

where $\psi_0(z)$ is analytic and single-valued in R. If we further suppose that the displacements u_α are single-valued functions in R, then the increment acquired by $2\mu(u_1 + iu_2)$ in describing the contour C_k is zero. Using this condition in (71.7), with φ and ψ in the forms (a) and (b), we find

$$2\pi i[(\varkappa + 1)A_k z + \varkappa B_k + \bar{C}_k] = 0.$$

Hence $A_k = 0$, and $C_k = -\varkappa \bar{B}_k$. The B_k have a simple physical meaning explained in formulas (72.7) and (72.8). These follow from the calculation of the resultant force acting on each contour C_k with the aid of formulas (71.1). The details of this argument will be found in Secs. 35, 36 of N. I. Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953).

If R is a finite multiply connected domain bounded by the exterior contour C_{m+1} and by m interior contours C_k ($k = 1, 2, \ldots, m$) (Fig. 51), and if the *displacements and stresses* are single-valued functions throughout R, then φ and ψ have the following structures:

$$(72.7) \qquad \begin{cases} \varphi(z) = -\dfrac{1}{2\pi(1 + \varkappa)} \displaystyle\sum_{k=1}^{m} (X_1^{(k)} + iX_2^{(k)}) \log (z - z_k) + \varphi_0(z), \\ \psi(z) = \dfrac{\varkappa}{2\pi(1 + \varkappa)} \displaystyle\sum_{k=1}^{m} (X_1^{(k)} - iX_2^{(k)}) \log (z - z_k) + \psi_0(z), \end{cases}$$

where $(X_1^{(k)}, X_2^{(k)})$ is the resultant vector of external forces applied to the contour C_k and z_k is an arbitrary point in the simply connected region R_k bounded by C_k. The functions $\varphi_0(z)$ and $\psi_0(z)$ are single-valued analytic functions in R.

If R is an infinite region, bounded by several simple closed contours C_k ($k = 1, 2, \ldots, m$), and if the stress components τ_{ij} are bounded in the neighborhood of the point at infinity, then[1] it is not difficult to prove that for sufficiently large $|z|$,

[1] The region R in this case can be thought to be obtained from the region R of Fig. 51, by making the contour C_{m+1} expand to infinity. It corresponds to an infinite plate with m holes bounded by the C_k.

$$
(72.8) \qquad \begin{cases} \varphi(z) = -\dfrac{X_1 + iX_2}{2\pi(1+\varkappa)} \log z + (B + iC)z + \varphi_0(z), \\[2ex] \psi(z) = \dfrac{\varkappa(X_1 - iX_2)}{2\pi(1+\varkappa)} \log z + (B' + iC')z + \psi_0(z), \end{cases}
$$

provided the origin of coordinates is taken outside R, that is, within one of the contours C_k. The X_1 and X_2 are the components of the resultant vector of all external forces acting on the boundary $C_1 + \cdots + C_m$, so that

$$
X_1 + iX_2 = \sum_{k=1}^{m} (X_1^{(k)} + iX_2^{(k)});
$$

$\varphi_0(z)$ and $\psi_0(z)$ are single-valued analytic functions in R including the point at infinity.[1] The constants B, B', C' are related to the state of stress at infinity as follows:

$$
(72.9) \quad 2B - B' = \tau_{11}(\infty), \qquad 2B + B' = \tau_{22}(\infty), \qquad \tau_{12}(\infty) = C',
$$

where $\tau_{\alpha\beta}(\infty)$ represents the limiting value of $\tau_{\alpha\beta}(x)$ as the point x recedes to infinity. The constant C has no effect on the state of stress and is related to the rigid rotation $\omega \equiv \lim\limits_{|z|\to\infty} \frac{1}{2}(u_{2,1} - u_{1,2})$ at infinity by the formula

$$
C = \frac{2\mu}{1+\varkappa}\,\omega.
$$

In the analysis of stress C can always be set equal to zero.

It is worth noting that the requirement for the $\tau_{\alpha\beta}$ to be bounded at infinity does not imply that the displacements u_α remain bounded. If the displacements are to be bounded at infinity, then[2] $\tau_{\alpha\beta}(\infty) = 0$, $X_1 + iX_2 = 0$, and $C = 0$.

If R is an infinite region bounded by a single contour C, the representation (72.8) is valid throughout the region.

73. First and Second Boundary-value Problems in Plane Elasticity. We are now in a position to show that the fundamental boundary-value problems in plane elasticity can be reduced to the determination of $\varphi(z)$ and $\psi(z)$ from prescribed values of certain combinations of these functions on the boundary of the region.

[1] This means that, for sufficiently large $|z|$, $\varphi_0(z)$ and $\psi_0(z)$ can be represented in the forms

$$
\varphi_0(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \qquad \psi_0(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n}.
$$

[2] See the concluding paragraphs of Sec. 36, in N. I. Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953).

We begin with the first boundary-value problem in which the $\tau_{\alpha\beta}$ mus be such that

$$\tau_{\alpha\beta}\nu_\beta = T_\alpha(s),$$

where the stress vector T_α is specified on the boundary C.

Formulas (69.9) yield at once the result

$$U_{,1} + iU_{,2} = f_1(s) + if_2(s) + \text{const} \qquad \text{on } C, \tag{73.1}$$

where

$$f_1 + if_2 = i\int^s [T_1(s) + iT_2(s)]\,ds.$$

But if we recall the formula (71.2), we can write the condition (73.1) as

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = f_1 + if_2 + \text{const} \qquad \text{on } C. \tag{73.2}$$

The constant in the right-hand member of this formula is, in general, complex and has a different value on each of the contours forming the boundary C. However, we saw in the preceding section that there is some freedom in the choice of $\varphi(z)$ and $\psi(z)$ corresponding to the same state of stress. This freedom can be utilized to fix the values of some constants in (73.2).

Thus, if the region R is finite and simply connected, the replacement of φ by $\varphi + ciz + \alpha$ and of ψ by $\psi + \beta$ does not change the state of stress in R. Accordingly, the left-hand member in (73.2) can be replaced by $\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha + \bar{\beta}$, and hence, by choosing suitably the value of $\alpha + \bar{\beta}$, the constant in (73.2) can be fixed in an arbitrary manner. If, for example, we set this constant equal to zero, we can no longer take[1] $\varphi(0) = 0$, $\mathcal{I}\varphi'(0) = 0$, and $\psi(0) = 0$, but we can still choose $\varphi(0) = 0$ and $\mathcal{I}\varphi'(0) = 0$. The condition $\varphi(0) = 0$ fixes α, and $\mathcal{I}\varphi'(0) = 0$ determines c. But if α is known, the value of $\alpha + \bar{\beta}$ fixing the constant in (73.2) determines $\bar{\beta}$ and hence we no longer have control over the choice of β.

The situation with an infinite region bounded by one contour C is similar. In this case φ and ψ have the structures shown in (72.8). If the constant in (73.2) is fixed, we can consider $\varphi_0(\infty) = 0$, $C = 0$. This choice, together with the choice of the constant in (73.2), determines φ_0 and ψ_0 completely.

If the region is multiply connected (finite or infinite), the constant in (73.2) may have a different value on each contour C_i forming the boundary of R and only on one of these contours can it be fixed arbitrarily. On the remaining contours the integration constants are determined from the requirement of continuity and single-valuedness of displacements and stresses.[2]

The determination of the corresponding boundary conditions in the

[1] See (72.6).

[2] See an analogous problem in Sec. 47.

second boundary-value problem is just as simple. This time we have the conditions

$$u_\alpha = g_\alpha(s) \qquad \text{on } C,$$

where the $g_\alpha(s)$ are known functions. We recall the formula (71.7) and deduce at once the boundary condition

$$\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = 2\mu[g_1(s) + ig_2(s)] \qquad \text{on } C. \tag{73.3}$$

The displacements $g_\alpha(s)$ specified on C determine completely the states of stress and deformation in R. Hence we can no longer specify the values of both[1] φ_0 and ψ_0 at a given point of R in formulas (72.7). If the region is finite, we can take the origin in R and set $\varphi_0(0) = 0$. If the region is infinite, we can consider that φ_0 in (72.8) is chosen so that $\varphi_0(\infty) = 0$.

We conclude this section by recording another form of the boundary condition for the first boundary-value problem when the normal and tangential components N and T of the stress vector are prescribed on the boundary instead of the cartesian components T_α. We take the positive direction of the normal component N along the normal $\boldsymbol{\nu}$ and the tangential component T as shown in Fig. 53. Then[2]

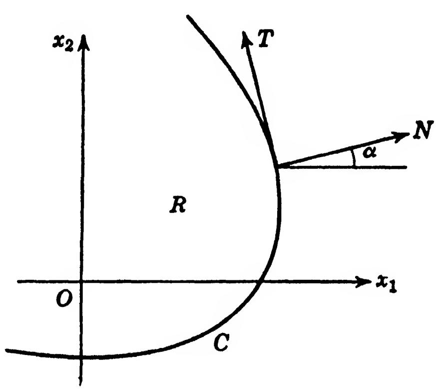

FIG. 53

$$2(N - iT) = \tau_{11} + \tau_{22} - (\tau_{22} - \tau_{11} + 2i\tau_{12})e^{2i\alpha}, \tag{73.4}$$

where α is the angle measured from the positive direction of the x_1-axis to the normal $\boldsymbol{\nu}$.

The substitution in (73.4) from (71.4) yields the desired boundary con-

[1] See (72.5).

[2] This formula is easily checked by using the transformation formulas (16.4) upon taking the direction of N along the x_1'-axis and that of T along the x_2'-axis. Then $N = \tau_{11}'$, $T = \tau_{12}'$, and the transformations connecting the coordinate systems are:

$$x_1 = x_1' \cos\alpha - x_2' \sin\alpha,$$
$$x_2 = x_1' \sin\alpha + x_2' \cos\alpha.$$

It is also easy to verify that,

$$\tau_{11}' + \tau_{22}' = \tau_{11} + \tau_{22},$$
$$\tau_{22}' - \tau_{11}' + 2i\tau_{12}' = (\tau_{22} - \tau_{11} + 2i\tau_{12})e^{2i\alpha}.$$

The first of these formulas is obvious, since $\Theta_1 = \tau_{\alpha\alpha}$ is an invariant.

dition in the form

$$(73.5) \qquad \varphi'(z) + \overline{\varphi'(z)} - e^{2i\alpha}[\bar{z}\varphi''(z) + \psi'(z)] = N - iT \qquad \text{on } C,$$

where N and T are specified along C.

74. Remarks on the Existence and Uniqueness of Solutions. The boundary conditions (73.2) and (73.3) for the basic two-dimensional elastostatic problems can be written in the form

$$(74.1) \qquad \alpha\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \qquad \text{on } C,$$

where we use the symbol t to represent the values of z on the contour C of the region. In the first boundary-value problem $\alpha = 1$ and

$$f = f_1 + if_2 + \text{const},$$

while, in the second, $\alpha = -\varkappa$ and $f = -2\mu(g_1 + ig_2)$.

The existence of solutions of the first and second boundary-value problems in plane elasticity can thus be made to depend on the demonstration of the existence of functions $\varphi(z)$ and $\psi(z)$ which satisfy on the boundary C the conditions of the form (74.1). We shall see in Secs. 83 and 86 that the boundary conditions (74.1) can be used to construct a Fredholm integral equation of the second kind for the determination of these functions, and their existence would then follow directly from the fact that the associated homogeneous integral equation has no solution other than the trivial (zero) solution.[1]

As regards the uniqueness of solution, it should be noted that the Kirchhoff proof (Sec. 27), for finite three-dimensional domains, is clearly valid for finite two-dimensional regions. Instead of formula (27.1) we now have the equation

$$(74.2) \qquad \iint\limits_R F_\alpha u_\alpha \, d\sigma + \int_C T_\alpha u_\alpha \, ds = 2 \iint\limits_R W \, d\sigma.$$

[1] The first boundary-value problem, as was shown in Sec. 69, is equivalent to the fundamental biharmonic boundary-value problem. The existence of the solution of it, for finite simply connected domains, was established by: J. Hadamard [*Mémoires des savants etrangers*, vol. 33, No. 4 (1908)], G. Lauricella [*Atti della reale accademia nazionale dei Lincei*, vol. 15 (1906), pp. 426–432], T. Boggio [*Atti della accademia delle scienze di Torino*, vol. 35 (1900), pp. 219–239; *Atti del reale institute Veneto di scienze, lattere ed arti*, vol. 61 (1901–1902), pp. 619–636], and A. Korn [*Annales de l'école normale supérieure*, vol. 25 (1908), pp. 529–583].

The matter of the existence of solutions of the first, second, and mixed problems for finite and infinite multiply connected domains (including anisotropic media) was settled principally by: N. I. Muskhelishvili [*Mathematische Annalen*, vol. 107 (1932), pp. 282–312], S. G. Mikhlin [*Matematicheski Sbornik*, vol. 41 (1934), pp. 284–291, 408–420; *Trudy Seismological Institute Akademii Nauk SSSR*, No. 65 (1935), No. 66 (1935), No. 76 (1936)], and D. I. Sherman [*Trudy Seismological Institute Akademii Nauk SSSR*, No. 54 (1935), No. 86 (1938), No. 88 (1938), No. 100 (1940); *Doklady Akademii Nauk SSSR*, vol. 27 (1940), pp. 911–913; vol. 28 (1940), pp. 29–32; vol. 32 (1941), pp. 314–315; *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 7 (1943), pp. 341–360, 413–420].

where

$$W = \tfrac{1}{2}\lambda(e_{11} + e_{22})^2 + \mu(e_{11}^2 + e_{22}^2 + 2e_{12}^2).$$

If the region is infinite, the proof of Sec. 27 is easily extended. We apply, first, formula (74.2) to a finite domain bounded by the contour

$$C = C_1 + \cdot \cdot \cdot + C_m$$

and by the circle C_ρ with center at $z = 0$ and with radius ρ so large that C_ρ contains C within it. Then if the integral

$$\int_{C_\rho} T_\alpha u_\alpha \, ds \to 0 \qquad \text{as} \qquad \rho \to \infty, \tag{74.3}$$

the argument in Sec. 27 establishes the uniqueness of solution in the two-dimensional infinite region. The fact that (74.3) is, indeed, true follows from (71.4) and (71.7) if we recall[1] that, for sufficiently large $|z|$,

$$\varphi_0(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \qquad \psi_0(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n}.$$

For, in this case, the integrand $T_\alpha u_\alpha$ is at least of the order $1/\rho^2$, and hence the integral tends to zero as ρ becomes infinite.

75. The Role of Conformal Representation in Plane Problems of Elasticity. We have indicated in the preceding chapter how effectively conformal mapping can be used in solving the Dirichlet problem for simply connected domains. Techniques, similar to those used in calculating the complex torsion and flexure functions, can be applied to the boundary-value problems in plane elasticity. We suppose that the given region R (finite or infinite) is simply connected and map it conformally on the unit circle $|\zeta| \leq 1$ by the analytic function

$$z = \omega(\zeta). \tag{75.1}$$

If the correspondence of points specified by (75.1) is one-to-one, then, as noted in Sec. 43, $\omega'(\zeta)$ does not vanish at any point of the region. To ensure the nonvanishing of $\omega'(\zeta)$ throughout the closed region $|\zeta| \leq 1$, it suffices to assume that the boundary C of R has continuously changing curvature.[2] We shall suppose that such is the case. Then, if the region R is finite and the origin $z = 0$ is taken in the interior, we can represent (75.1) in the power series

$$z = \omega(\zeta) = \sum_{n=1}^{\infty} k_n \zeta^n, \qquad |\zeta| \leq 1, \tag{75.2}$$

[1] See (72.8), where we have set $X_1 = X_2 = B = C = B' = C' = 0$, since we suppose that the displacements remain bounded at infinity.

[2] This is a special case of a theorem due to V. Smirnoff, *Mathematische Annalen*, vol. 107 (1932), pp. 313–323.

by making the point $z = 0$ correspond to $\zeta = 0$. If the region R is infinite, we shall suppose that $z = 0$ is an exterior point and represent $\omega(\zeta)$ in the form

$$\omega(\zeta) = \frac{c}{\zeta} + \sum_{n=0}^{\infty} k_n \zeta^n, \qquad |\zeta| \leq 1, \tag{75.3}$$

by taking $z = \infty$ and $\zeta = 0$ as the corresponding points.[1]

Let us determine next how the essential formulas (71.2) and (71.7) and the boundary conditions transform under (75.1). We denote the results of the substitution $z = \omega(\zeta)$ in $\varphi(z)$ and $\psi(z)$ by $\varphi_1(\zeta)$ and $\psi_1(\zeta)$, respectively, so that

$$\varphi[\omega(\zeta)] \equiv \varphi_1(\zeta), \qquad \psi[\omega(\zeta)] \equiv \psi_1(\zeta). \tag{75.4}$$

Since

$$\varphi'(z) = \frac{d\varphi_1}{d\zeta}\frac{d\zeta}{dz} = \varphi_1'(\zeta)\frac{1}{\omega'(\zeta)},$$

formulas (71.2) and (71.7) assume the forms

$$U_{,1} + iU_{,2} = \varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} + \overline{\psi_1(\zeta)}, \qquad |\zeta| \leq 1, \tag{75.5}$$

$$2\mu(u_1 + iu_2) = \kappa\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} - \overline{\psi_1(\zeta)}, \qquad |\zeta| \leq 1. \tag{75.6}$$

Hence the boundary conditions (73.2) and (73.3) become

$$\varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} + \overline{\psi_1(\zeta)} = F(\vartheta), \qquad \text{on } |\zeta| = 1, \tag{75.7}$$

$$\kappa\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} - \overline{\psi_1(\zeta)} = G(\vartheta), \qquad \text{on } |\zeta| = 1, \tag{75.8}$$

where $F(\vartheta)$ and $G(\vartheta)$ are uniquely determined by (75.1) on the boundary γ of the unit circle from known values, $f_1 + if_2 +$ const and $2\mu(g_1 + ig_2)$, specified[2] in the contour C of R.

The structure of the left-hand members in (75.7) and (75.8) suggests that we impose on $\varphi_1(\zeta)$, $\varphi_1'(\zeta)$, and $\psi_1(\zeta)$ the requirement of continuity in the closed region $|\zeta| \leq 1$. Moreover, if the domain is bounded, $\varphi_1(\zeta)$

[1] Occasionally it proves convenient to map an infinite region R on the region $|\zeta| \geq 1$ and make the point at infinity in the z-plane correspond to the point $\zeta = \infty$. The appropriate mapping function is obtained then from (75.3) by replacing ζ by $1/\zeta$.

[2] We suppose that the value of the integration constant in (73.2) is fixed in some definite way, say, by setting it equal to zero. This can be done by utilizing the available freedom in the choice of $\psi(z)$. The transform of $f_1(s) + if_2(s)$, which we denoted by $F(\vartheta)$, is then a known function $f_1(\vartheta) + if_2(\vartheta)$ of the angular variable ϑ in the ζ-plane. The functional forms of $f_1(\vartheta) + if_2(\vartheta)$ will, in general, differ from $f_1(s) + if_2(s)$, but the values of these functions at the corresponding points on γ and C are the same.

and $\psi_1(\zeta)$ are analytic in the region $|\zeta| < 1$. In the first boundary-value problem for such domains, we are free to assign arbitrary values to $\varphi(0)$ and to $\mathscr{I}[\varphi'(0)]$. Hence[1] the values of $\varphi_1(0)$ and $\mathscr{I}[\varphi_1'(0)/\omega'(0)]$ can be specified arbitrarily.

In the second boundary-value problem we can assign the value to either $\varphi(z)$ or $\psi(z)$ at some point of R, and in the sequel we shall choose to assign an arbitrary value to $\varphi_1(\zeta)$ at some point $\zeta = \zeta_0$ of the transformed region.

If R is a bounded region, $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ have the representations

$$\varphi_1(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n, \qquad \psi_1(\zeta) = \sum_{n=0}^{\infty} b_n\zeta^n, \qquad |\zeta| \leq 1.$$

The substitution of these series in the boundary conditions (75.7) or (75.8) leads to a system of equations for the coefficients a_n and b_n. We shall use this elementary method of solving the boundary-value problems

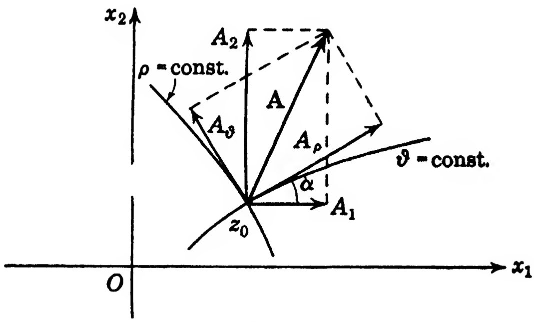

Fig. 54

in the following section. In general it is cumbersome, and the procedure resulting from the conversion of the boundary conditions into certain functional equations, presented in Secs. 82 to 86, leads to considerably more effective methods of solution.

Occasionally it is convenient to use the boundary condition in the form (73.5), and we deduce next the corresponding expression in the ζ-plane.

The relationship $z = \omega(\zeta)$ determines an orthogonal curvilinear net ρ = const, ϑ = const in the z-plane corresponding to the families of circles and radial lines in the ζ-plane. If α (Fig. 54) is the angle made at $z = z_0$ by the coordinate line ϑ = const with the x_1-axis, then the cartesian components A_α of an arbitrary vector **A** at $z = z_0$ are related to the components A_ρ, A_ϑ along the coordinate lines ρ = const, ϑ = const by the formula

$$A_\rho + iA_\vartheta = e^{-i\alpha}(A_1 + iA_2).$$

[1] See Sec. 73.

It is not difficult to express $e^{-i\alpha}$ in terms of the mapping function $z = \omega(\zeta)$. For, if $d\zeta$ represents a displacement of $\zeta = \rho e^{i\vartheta}$ along the radius, the corresponding displacement dz in the z-plane will be along the line $\vartheta =$ const. Hence

$$dz = e^{i\alpha}\,|dz| \qquad \text{and} \qquad d\zeta = e^{i\vartheta}\,|d\zeta|,$$

and we find

$$e^{-i\alpha} = \frac{\bar{\zeta}}{\rho}\,\frac{\omega'(\zeta)}{|\omega'(\zeta)|}. \tag{75.9}$$

Thus, the components u_ρ, u_ϑ of the displacement vector in the z-plane are related to the cartesian components u_α by the formula

$$u_\rho + iu_\vartheta = \frac{\bar{\zeta}}{\rho}\,\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\,(u_1 + iu_2). \tag{75.10}$$

Replacing z by $\omega(\zeta)$ in (73.5) and noting (75.9), we get the boundary condition in the form

$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}\,[\overline{\omega(\sigma)}\Phi'(\sigma) + \omega'(\sigma)\Psi(\sigma)] = N - iT \qquad \text{on } |\zeta| = 1, \tag{75.11}$$

where

$$\Phi(\zeta) \equiv \frac{\varphi_1'(\zeta)}{\omega'(\zeta)}, \qquad \Psi(\zeta) = \frac{\psi_1'(\zeta)}{\omega'(\zeta)},$$

and $\sigma = e^{i\vartheta}$.

If we let $\tau'_{11} = \tau_{\rho\rho}$, $\tau'_{22} = \tau_{\vartheta\vartheta}$, $\tau'_{12} = \tau_{\rho\vartheta}$ in the formulas in the footnote on page 271 and recall formulas (71.4), we get the useful expressions,[1]

$$\begin{cases} \tau_{\rho\rho} + \tau_{\vartheta\vartheta} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \\ \tau_{\vartheta\vartheta} - \tau_{\rho\rho} + 2i\tau_{\rho\vartheta} = \dfrac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\,[\overline{\omega(\zeta)}\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)] \end{cases} \tag{75.12}$$

76. An Elementary Method of Solution of the Basic Problems for Simply Connected Domains. The boundary conditions (75.7) and (75.8) have the form

$$\alpha\varphi_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi_1'(\sigma)} + \overline{\psi_1(\sigma)} = H(\sigma), \tag{76.1}$$

where $\sigma = e^{i\vartheta}$ is the value of ζ on the boundary of the unit circle. In the first boundary-value problem $\alpha = 1$ and $H(\sigma) \equiv f_1(\vartheta) + if_2(\vartheta)$, and in the second problem $\alpha = -\varkappa$ and $H(\sigma) = -2\mu[g_1(\vartheta) + ig_2(\vartheta)]$. If the region R is finite and simply connected, the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ can be represented in the power series

[1] Note that $e^{2i\alpha} = \dfrac{\zeta^2}{\rho^2}\,\dfrac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}$.

$$\varphi_1(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k, \qquad \psi_1(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^k, \tag{76.2}$$

and it is natural to attempt to calculate the a_k and b_k by the method of undetermined coefficients. To this end we expand the right-hand member of (76.1) in the Fourier series (39.6) to obtain

$$H(\sigma) = \sum_{k=-\infty}^{\infty} C_k e^{ik\vartheta} \equiv \sum_{-\infty}^{\infty} C_k \sigma^k, \tag{76.3}$$

and write the complex Fourier series for the known function

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \sum_{k=-\infty}^{\infty} c_k e^{ik\vartheta} \equiv \sum_{-\infty}^{\infty} c_k \sigma^k. \tag{76.4}$$

The insertion from (76.2), (76.3), and (76.4) in (76.1) yields the equation

$$\alpha \sum_{k=1}^{\infty} a_k \sigma^k + \sum_{k=-\infty}^{\infty} c_k \sigma^k \sum_{k=1}^{\infty} k \bar{a}_k \sigma^{-k+1} + \sum_{k=0}^{\infty} \bar{b}_k \sigma^{-k} = \sum_{k=-\infty}^{\infty} C_k \sigma^k, \tag{76.5}$$

if we take $\varphi_1(0) = a_0 = 0$ and note that $\bar{\sigma} = e^{-i\vartheta} = \sigma^{-1}$.

On performing the indicated operations, which are surely legitimate if the involved series are absolutely convergent, we get

$$\alpha \sum_{k=1}^{\infty} a_k \sigma^k + \sum_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} m \bar{a}_m c_{m+k-1} \right) \sigma^k + \sum_{k=0}^{\infty} \left(\sum_{m=1}^{\infty} m \bar{a}_m c_{m-k-1} \right) \sigma^{-k} + \sum_{k=0}^{\infty} \bar{b}_k \sigma^{-k} = \sum_{k=-\infty}^{\infty} C_k \sigma^k,$$

and, on comparing the coefficients of like powers of σ, we obtain:

$$\alpha a_k + \sum_{m=1}^{\infty} m \bar{a}_m c_{m+k-1} = C_k, \qquad (k = 1, 2, \ldots), \tag{76.6}$$

$$\bar{b}_k + \sum_{m=1}^{\infty} m \bar{a}_m c_{m-k-1} = C_{-k}, \qquad (k = 0, 1, 2, \ldots). \tag{76.7}$$

If the system of Eqs. (76.6) can be solved for the a_k, the b_k are determined at once from formula (76.7). In the first boundary-value problem the system (76.6) cannot be expected to yield a unique solution if the imaginary part of a_1 is left unspecified, since the function $\varphi_1(\zeta)$ is not determined uniquely unless the value of $\mathscr{I}[\varphi_1'(0)/\omega'(0)] \equiv \mathscr{I}[a_1/\omega'(0)]$ is assigned. No such supplementary condition is needed for the second boundary-value problem, inasmuch as the condition $\varphi_1(0) = 0$ completely determines both $\varphi_1(\zeta)$ and $\psi_1(\zeta)$

Further, to ensure the existence of solution of the first boundary-value problem for finite domains, the resultant force and the resultant moment of assigned stresses $T_\alpha(s)$ must vanish. It is not difficult to show[1] that the vanishing of the resultant force implies that the function $H(\sigma)$ is single-valued on the unit circle, while the moment condition imposes a restriction on the coefficients C_k in the representation (76.3).

An important special case arises when $\omega(\zeta)$ is a polynomial of degree n, because, as we shall presently see, the determination of a_k's, in this case, reduces to the solution of the system of n linear equations in n unknowns. The practical importance of this becomes obvious if it is noted that the mapping function for a finite domain can be approximated with arbitrary accuracy by a polynomial.

We note first that, when $\omega(\zeta)$ is a polynomial of degree n, the function $\omega(\sigma)/\overline{\omega'(\sigma)}$ has the representation[2]

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \sum_{k=0}^{n} c_k\sigma^k + \sum_{k=1}^{\infty} c_{-k}\sigma^{-k}. \tag{76.8}$$

Consequently, on setting $c_k = 0$ for $k \geq n + 1$ in (76.6) and (76.7), we get:

[1] If the resultant force vanishes, then $\int_C (T_1 + iT_2)\, ds = 0$. But

$$f_1(s) + if_2(s) = i\int_{s_0}^{s} (T_1 + iT_2)\, ds,$$

and hence the increment in $f_1 + if_2$ as the contour C is traversed is zero. This is another way of saying that $H(\sigma) = f_1(\vartheta) + if_2(\vartheta)$ is single-valued on $|\zeta| = 1$. The vanishing of the resultant moment requires that

$$\int_C (x_1T_2 - x_2T_1)\, ds = -\int_C (x_1\, df_1 + x_2\, df_2) = 0.$$

Integration by parts gives

$$[x_1f_1(s) + x_2f_2(s)]_C - \int_C [f_1(s)\, dx_1 + f_2(s)\, dx_2] = 0,$$

and since the function in the brackets is single valued, the bracketed term vanishes. The integral can be written as $\mathcal{R}\int_C [f_1(s) + if_2(s)]\, \overline{dz} = 0$. Under the transformation $z = \omega(\zeta)$, $f_1(s) + if_2(s)$ goes over into $H(\sigma) \equiv f_1(\vartheta) + if_2(\vartheta)$, and since $dz = \omega'(\zeta)\, d\zeta$,

$$\mathcal{R}\int_C [f_1(s) + if_2(s)]\, \overline{dz} = \mathcal{R}\int_\gamma H(\sigma)\overline{\omega'(\sigma)}\, \overline{d\sigma} = 0.$$

The last of these equalities implies a restriction on the choice of the C_k in (76.3).

[2] The left-hand member of (76.8), viewed as a function of a complex variable σ, has a pole of order n at infinity and no other singularities in the region $|\sigma| \geq 1$. Hence it has a Laurent series representation in the region $|\sigma| \geq 1$, which, for $|\sigma| = 1$, has the form (76.8).

$$
(76.9)\quad \begin{cases} \alpha a_1 + \bar{a}_1 c_1 + 2\bar{a}_2 c_2 + \cdots + n\bar{a}_n c_n = C_1, \\ \alpha a_2 + \bar{a}_1 c_2 + 2\bar{a}_2 c_3 + \cdots + (n-1)\bar{a}_{n-1} c_n = C_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha a_n + \bar{a}_1 c_n \qquad\qquad\qquad\qquad = C_n; \\ \alpha a_k = C_k, \qquad k \geq n+1. \end{cases}
$$

$$
(76.10)\quad \bar{b}_k = -\sum_{m=1}^{n+k+1} m\bar{a}_m c_{m-k-1} + C_{-k}, \qquad (k = 0, 1, 2, \ldots).
$$

It follows from these formulas that, if $|C_k| < M/k^3$, then the series (76.2) define the analytic functions $\varphi_1(\zeta)$, $\varphi_1'(\zeta)$, and $\psi_1(\zeta)$ in the region $|\zeta| < 1$, which satisfy the boundary condition (76.1). The Fourier coefficients C_k will surely be of this order if the second derivatives of $H(\sigma)$ are of bounded variation. To ensure this, it would suffice to suppose that, in the first boundary-value problem, the functions $T_\alpha(s)$ have first derivatives of bounded variation and, in the second problem, the second derivatives of the displacements $g_\alpha(s)$ are of bounded variation.

If the domain R is infinite, the mapping function has the form (75.3), and it follows from (72.8) that $\varphi_1(\zeta) = \varphi[\omega(\zeta)]$ and $\psi_1(\zeta) = \psi[\omega(\zeta)]$ have the representations,

$$
(76.11)\quad \begin{cases} \varphi_1(\zeta) = \dfrac{X_1 + iX_2}{2\pi(1+\varkappa)} \log \zeta + (B + iC)\dfrac{c}{\zeta} + \varphi^0(\zeta), \\ \psi_1(\zeta) = -\dfrac{\varkappa(X_1 - iX_2)}{2\pi(1+\varkappa)} \log \zeta + (B' + iC')\dfrac{c}{\zeta} + \psi^0(\zeta), \end{cases}
$$

where $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ are analytic and single-valued for $|\zeta| < 1$.

The constants B, B', and C' are related to the stress distribution at infinity. They are,

$$
B = \frac{\tau_{11}(\infty) + \tau_{22}(\infty)}{4}, \qquad B' = \frac{\tau_{22}(\infty) - \tau_{11}(\infty)}{2}, \qquad C' = \tau_{12}(\infty).
$$

As noted in Sec. 72, the constant C can be set equal to zero. To obtain the boundary conditions for $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ in the first boundary-value problem, we substitute from (76.11) in (76.1) and find

$$
(76.12)\quad \varphi^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\, \overline{\varphi^{0\prime}(\sigma)} + \overline{\psi^0(\sigma)} = F^0(\sigma),
$$

where

$$
(76.13)\quad F^0(\sigma) \equiv F(\sigma) - \frac{X_1 + iX_2}{2\pi} \log \sigma - \frac{Bc}{\sigma} - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \left[\frac{X_1 - iX_2}{2\pi(1+\varkappa)}\, \sigma - B\bar{c}\sigma^2 \right] - (B' - iC')\bar{c}\sigma,
$$

with $F(\sigma) = f_1(\vartheta) + if_2(\vartheta)$.

The function $F^0(\sigma)$ is clearly single-valued when the components X_1, X_2

of the resultant force acting on C vanish. In this case we have for the determination of $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ the boundary condition similar to (76.1). If the resultant external force does not vanish, $F(\sigma)$ acquires an increment $i(X_1 + iX_2)$ as the point σ traverses the circle $|\zeta| = 1$ once in the counterclockwise direction.[1] But as σ traverses the circle, $\log \sigma$ acquires an increment $2\pi i$ and thus the contribution from the second term in the right-hand member of (76.13) annuls the contribution from the first term It follows that $F^0(\sigma)$ is single-valued in this case also. Thus, in either circumstance, we have a problem of the same type as for the finite domain.

The treatment of the second boundary-value problem for an infinite domain when the values of X_1, X_2, B_1, C_1, B', and C' are specified in advance obviously leads to the boundary condition

$$\kappa\varphi^0(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi^{0\prime}(\sigma)} - \overline{\psi^0(\sigma)} = G^0(\sigma), \tag{76.14}$$

where $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ are single-valued and analytic functions for $|\zeta| < 1$ and $G^0(\sigma)$ is a known single-valued function.

PROBLEM

Show that whenever the function $\omega(\zeta)$, mapping an infinite simply connected domain on a unit circle $|\zeta| = 1$, has the form

$$\omega(\zeta) = \frac{\gamma}{\zeta} + \sum_{k=1}^{n} \gamma_k \zeta^k,$$

then

$$\frac{\omega(\sigma)}{\overline{\omega'(1/\sigma)}} = c_{n-2}\sigma^{n-2} + \cdots + c_1\sigma + \sum_{k=0}^{\infty} c_{-k}\sigma^{-k},$$

and hence the system of equations for the coefficients a_k in $\varphi^0(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^k$, from (76.12), is:

$$\left\{\begin{array}{ll} a_1 + \bar{a}_1c_1 + 2\bar{a}_2c_2 + \cdots + (n-2)\bar{a}_{n-2}c_{n-2} = A_1^0, \\ a_2 + \bar{a}_1c_2 + 2\bar{a}_2c_3 + \cdots + (n-3)\bar{a}_{n-3}c_{n-2} = A_2^0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n-2} + \bar{a}_1c_{n-2} \qquad\qquad\qquad\qquad = A_{n-2}^0, \\ a_k = A_k^0, \qquad k \geq n-1, \end{array}\right. \tag{76.15}$$

the A_k^0 being the Fourier coefficients in the representation, $F^0(\sigma) = \sum_{k=-\infty}^{\infty} A_k^0\sigma^k$.

77. Solution of Basic Problems for a Circular Region. We specialize the formulas of the preceding section to obtain the solutions of the first

[1] Recall that $f_1(s) + if_2(s) = i\int_{s_0}^{s}(T_1 + iT_2)\,ds$.

and second boundary-value problems for the circular region $|z| \leq R$. The appropriate mapping function $\omega(\zeta)$ in this case is

$$z = R\zeta, \tag{77.1}$$

so that $\omega(\sigma)/\overline{\omega'(\sigma)} = \sigma$. Thus all c_k in the expansion (76.8), with the exception $c_1 = 1$, vanish.

If we represent the function $f_1(\vartheta) + if_2(\vartheta)$, characterizing the stress distribution on the boundary, in the form

$$F(\vartheta) = \sum_{k=-\infty}^{\infty} A_k\sigma^k, \tag{77.2}$$

where

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} F(\vartheta)e^{-ik\vartheta}\,d\vartheta, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

the systems (76.9), (76.10), upon setting $C_k = A_k$, $\alpha = 1$, yield:

$$\begin{cases} a_1 + \bar{a}_1 = A_1 \\ a_k = A_k, & k \geq 2, \\ b_k = \bar{A}_{-k} - (k+2)A_{k+2}, & k = 0, 1, 2, \ldots. \end{cases} \tag{77.3}$$

The first of these equations requires A_1 to be real, and it is easy to check that it is the consequence of the vanishing of the resultant moment of forces applied to the boundary. In order to determine $\varphi_1(\zeta)$ uniquely we take $\mathfrak{I}(a_1/c_1) = \mathfrak{I}(a_1) = 0$. Then $a_1 = A_1/2$, and we have

$$\varphi_1(\zeta) \equiv \sum_{k=1}^{\infty} a_k\zeta^k = \frac{A_1}{2}\zeta + \sum_{k=2}^{\infty} A_k\zeta^k, \tag{77.4}$$

and

$$\psi_1(\zeta) \equiv \sum_{k=0}^{\infty} b_k\zeta^k = \sum_{k=0}^{\infty} [\bar{A}_{-k} - (k+2)A_{k+2}]\zeta^k. \tag{77.5}$$

Setting $\zeta = z/R$, we obtain $\varphi(z)$ and $\psi(z)$.

The displacements and stresses in cartesian coordinates are then determined from formulas (71.4) and (71.7). The corresponding components in polar coordinates are given by:

$$\begin{cases} 2\mu(u_r + iu_\theta) = e^{-i\theta}[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}] \\ \tau_{rr} + \tau_{\theta\theta} = 4\Re[\varphi'(z)], \\ \tau_{\theta\theta} - \tau_{rr} + 2i\tau_{r\theta} = 2[\bar{z}\varphi''(z) + \psi'(z)]e^{2i\theta}, \end{cases} \tag{77.6}$$

as follows from (75.10) and (75.12).

When the displacements are specified on the boundary, we represent the function $G(\vartheta) = 2\mu[g_1(\vartheta) + ig_2(\vartheta)]$ in Fourier series,

$$G(\vartheta) = \sum_{k=-\infty}^{\infty} B_k\sigma^k$$

and, upon setting $B_k = -C_k$ and $\alpha = -\varkappa$ in (76.9). find that

$$\text{(77.7)} \qquad \begin{cases} \varkappa a_1 - \bar{a}_1 = B_1, & \\ \varkappa a_k = B_k, & k > 1, \\ \bar{b}_k = -B_{-k} - (k+2)\bar{a}_{k+2}, & k \geq 0. \end{cases}$$

These completely determine the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$.

As an illustration of the use of these formulas we consider several special problems.

a. Uniform Pressure. If a uniform pressure of intensity P acts on the boundary of the circle, we take $T_1 = -P \cos \theta$, $T_2 = -P \sin \theta$ and compute

$$\begin{aligned} f_1(s) + if_2(s) &= i \int^s (T_1 + iT_2)\, ds \\ &= -i \int^\theta Pe^{i\theta} R\, d\theta \\ &= -PRe^{i\theta}. \end{aligned}$$

Thus,

$$f_1(\vartheta) + if_2(\vartheta) = -PR\sigma,$$

and hence all A_k, in the expansion (77.2), with the exception of

$$A_1 = -PR,$$

vanish.

The substitution of $A_1 = -PR$ and $A_k = 0$, $k \neq 1$, in (77.4) and (77.5) then gives

$$\varphi_1(\zeta) = -\frac{PR}{2}\zeta, \qquad \psi_1(\zeta) = 0,$$

so that

$$\varphi(z) = -\frac{Pz}{2}, \qquad \psi(z) = 0.$$

Making use of these results in (77.6), we easily find that

$$u_r = \frac{P(1-\varkappa)}{4\mu} r, \qquad u_\theta = 0,$$
$$\tau_{rr} = \tau_{\theta\theta} = -P, \qquad \tau_{r\theta} = 0.$$

b. Uniform Radial Displacement If a uniform radial displacement $u_r = -u_0$ is specified on the boundary,

$$u_1 = -u_0 \cos \theta, \qquad u_2 = -u_0 \sin \theta,$$

and we find

$$2\mu(g_1 + ig_2) = -2\mu u_0 \sigma.$$

Hence the coefficients B_k in (77.7) are:

$$B_1 = -2\mu u_0, \qquad B_k = 0, \qquad k \neq 1.$$

Thus,

$$a_1 = \frac{\varkappa B_1 + \bar{B}_1}{\varkappa^2 - 1} = -\frac{2\mu u_0}{\varkappa - 1}, \qquad a_k = 0, k \neq 1,$$
$$b_k = 0, \qquad k \geq 0,$$

and, hence,

$$\varphi_1(\zeta) = -\frac{2\mu u_0}{\varkappa - 1}\zeta, \qquad \psi_1(\zeta) = 0.$$

Using

$$\varphi(z) = -\frac{2\mu u_0}{\varkappa - 1}\frac{z}{R}, \qquad \psi(z) = 0$$

in (77.6), we easily find that

$$u_r = -\frac{r u_0}{R}, \qquad u_\theta = 0.$$

c. Concentrated Loads. Let the concentrated force with components $(0, -P)$ act at the point $z_0 = Re^{i\alpha}$ of the circular boundary and an equal and opposite force with components $(0, P)$ act at $\bar{z}_0 = Re^{-i\alpha}$ (Fig. 55).

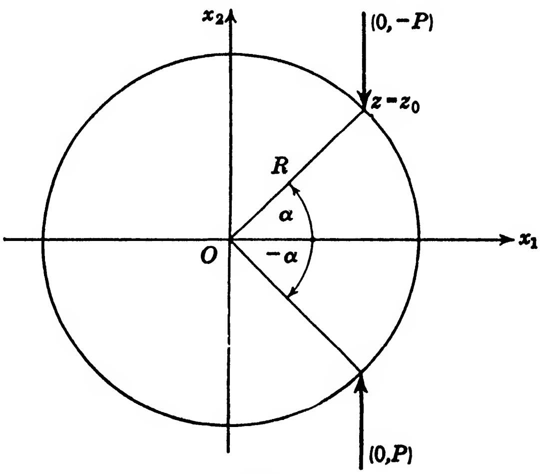

FIG. 55

The concentrated force P can be regarded as the limiting case of the uniform distribution of stress T applied to a small segment L of the boundary, wherein T is allowed to increase as $L \to 0$ in such a way that $TL = P$. With this interpretation, the function

$$f_1(s) + if_2(s) = i\int^s (T_1 + iT_2)\, ds$$

will be constant along the part of the boundary where no load is applied, but as the variable point passes the point of application of the load, $f_1 + if_2$ suffers a discontinuity of magnitude P.

In our case,

$$F(\theta) \equiv f_1 + if_2 = 0, \qquad 0 \le \theta < \alpha,$$
$$= P, \qquad \alpha < \theta < 2\pi - \alpha,$$
$$= 0, \qquad 2\pi - \alpha < \theta \le 2\pi.$$

Hence,

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{-ik\theta}\,d\theta = \frac{P}{2\pi}\int_\alpha^{2\pi-\alpha} e^{-ik\theta}\,d\theta$$
$$= \frac{P}{\pi}(\pi - \alpha), \qquad k = 0,$$
$$= \frac{Pi}{2\pi k}(e^{ik\alpha} - e^{-ik\alpha}), \qquad k \ne 0.$$

The substitution of these values in (77.4) and (77.5) gives:

$$\varphi_1(\zeta) = \frac{Pi}{2\pi}\sum_{k=1}^{\infty}\frac{e^{ik\alpha} - e^{-ik\alpha}}{k}\zeta^k - \frac{Pi}{4\pi}(e^{i\alpha} - e^{-i\alpha})\zeta,$$

$$\psi_1(\zeta) = b_0 + \frac{Pi}{2\pi}\sum_{k=1}^{\infty}\left[\frac{e^{ik\alpha} - e^{-ik\alpha}}{k} - e^{i(k+2)\alpha} + e^{-i(k+2)\alpha}\right]\zeta^k,$$

where

$$b_0 = \frac{P}{\pi}\left[\pi - \alpha - \frac{i}{2}(e^{2i\alpha} - e^{-2i\alpha})\right].$$

Since for $|x| < 1$, $\sum_{k=1}^{\infty} x^k/k = -\log(1 - x)$ and $\sum_{k=0}^{\infty} x^k = 1/(1 - x)$, the formulas for $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ can be written in closed forms:

$$\varphi_1(\zeta) = \frac{P\alpha}{\pi} + \frac{Pi}{2\pi}\left(\log\frac{e^{i\alpha} - \zeta}{e^{-i\alpha} - \zeta} - \frac{e^{i\alpha} - e^{-i\alpha}}{2}\zeta\right),$$
$$\psi_1(\zeta) = \frac{P\alpha}{\pi} + b_0 + \frac{Pi}{2\pi}\left(\log\frac{e^{i\alpha} - \zeta}{e^{-i\alpha} - \zeta} + \frac{e^{-i\alpha}}{e^{i\alpha} - \zeta} - \frac{e^{i\alpha}}{e^{-i\alpha} - \zeta}\right).$$

Since $z = R\zeta$ and $z_0 = Re^{i\alpha}$, we can write these as

$$\varphi(z) = \frac{Pi}{2\pi}\left(\log\frac{z_0 - z}{\bar{z}_0 - z} - \frac{z_0 - \bar{z}_0}{2R^2}z\right),$$
$$\psi(z) = \frac{Pi}{2\pi}\left(\log\frac{z_0 - z}{\bar{z}_0 - z} + \frac{\bar{z}_0}{z_0 - z} - \frac{z_0}{\bar{z}_0 - z}\right),$$

where we dropped the nonessential constants that do not affect the stress distribution.

The computation of stresses and displacements presents no serious difficulties.[1]

[1] These are recorded in N. I. Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953), pp. 327–328, where the functions $\varphi(z)$ and $\psi(z)$ are obtained in a different way. See also Timoshenko and Goodier, Theory of Elas-

d. Rotating Disk.[1] If a circular disk rotates with constant angular velocity ω about the axis through its center, we consider the stresses in the form

$$\tau_{\alpha\beta} = \tau_{\alpha\beta}^{(0)} + \tau_{\alpha\beta}^{(1)},$$

where the $\tau_{\alpha\beta}^{(0)}$ are given by (68.3), and the $\tau_{\alpha\beta}^{(1)}$ satisfy the homogeneous equilibrium equations

$$\tau_{\alpha\beta,\beta}^{(1)} = 0.$$

If no forces are applied to the boundary of the disk, $T_\alpha = 0$ and the boundary condition in (68.4) for $\tau_{\alpha\beta}^{(1)}$ gives

$$\begin{aligned}\tau_{\alpha\beta}^{(1)}\nu_\beta &= T_\alpha - \tau_{\alpha\beta}^{(0)}\nu_\beta \\ &= 0 - (\tau_{\alpha 1}^{(0)} \cos\theta + \tau_{\alpha 2}^{(0)} \sin\theta) = T_\alpha^{(1)}.\end{aligned}$$

Thus,

$$T_1 + iT_2 = -[\tau_{11}^{(0)} \cos\theta + \tau_{12}^{(0)} \sin\theta + i(\tau_{21}^{(0)} \cos\theta + \tau_{22}^{(0)} \sin\theta)].$$

Substituting for $\tau_{\alpha\beta}^{(0)}$ from (68.3) and noting that, on the boundary of the circle of radius R,

$$x_1 = R\cos\theta, \qquad x_2 = R\sin\theta,$$

we find,

$$T_1 + iT_2 = \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)}\, \rho\omega^2 R^2 e^{i\theta}.$$

Hence

$$f_1 + if_2 = i\int^s T_1 + iT_2\, ds = \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)}\, \rho\omega^2 R^3 e^{i\theta}.$$

It is clear from this that the problem of determining the stress distribution $\tau_{\alpha\beta}^{(1)}$ is identical with the uniform-pressure problem considered in (*a*) above, where we must set $P = -\dfrac{2\lambda + 3\mu}{4(\lambda + 2\mu)}\, \rho\omega^2 R^2$.

ticity (1951), pp. 107–111, where this problem is solved by indirect means. This problem was originally treated by H. Hertz, *Zeitschrift für Mathematik und Physik*, vol. 28 (1883), and later by J. H. Michell, *Proceedings of the London Mathematical Society*, vol. 32 (1900), pp. 35–61, vol. 34 (1902), pp. 134 142, who solved several similar problems by ingenious devices. A unified and systematic treatment of this category of problems was first given by G. V. Kolosoff and N. I. Muskhelishvili in *Izvestiya Petrograd Electrotechnical Institute*, vol. 12 (1915), pp. 39–55 (in Russian).

[1] For different solutions of this problem see Love's Treatise, Sec. 102, and Timoshenko and Goodier's Theory of Elasticity, Secs. 30 and 119. The problem of the disk rotating about an axis normal to the disk at an arbitrary point of the disk was solved by Ya. K. Il'yn, *Doklady Akademii Nauk SSSR*, vol. 67 (1949), pp. 803–806 (in Russian). A solution of the problem of rotating disk with attached concentrated masses is outlined in Sec. 80 of Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953).

It should be kept in mind that the value of λ in formulas (68.3), appropriate to the problem of a rotating disk, is given by $\bar{\lambda} = 2\lambda\mu/(\lambda + 2\mu)$, since we are dealing with the state of generalized plane stress. In the corresponding plane-deformation problem, that is, in the problem of a rotating shaft, λ is the Lamé constant given by (23.5).

78. Solution of Problems for the Infinite Region Bounded by a Circle. If we consider the region $|z| \geq R$ and map it on the unit circle in the ζ-plane by means of

$$z = \omega(\zeta) = \frac{R}{\zeta}, \tag{78.1}$$

the functions $\varphi_1(\zeta) \equiv \varphi[\omega(\zeta)]$ and $\psi_1(\zeta) \equiv \psi[\omega(\zeta)]$ assume the forms [see (76.11)],

$$\left\{\begin{aligned} \varphi_1(\zeta) &= \frac{X_1 + iX_2}{2\pi(1+\varkappa)} \log \zeta + (B + iC)\frac{R}{\zeta} + \varphi^0(\zeta), \\ \psi_1(\zeta) &= -\frac{\varkappa(X_1 - iX_2)}{2\pi(1+\varkappa)} \log \zeta + (B' + iC')\frac{R}{\zeta} + \psi^0(\zeta). \end{aligned}\right. \tag{78.2}$$

We recall that $X_1 + iX_2$ is the resultant force acting on the circular boundary and the constants B, B', C, C' are related to the stresses and rotation at infinity by formulas (72.9).

We shall assume that $C = 0$ and take

$$\left\{\begin{aligned} B &= \tfrac{1}{4}[\tau_{11}(\infty) + \tau_{22}(\infty)], \\ B' &= \tfrac{1}{2}[\tau_{22}(\infty) - \tau_{11}(\infty)], \\ C' &= \tau_{12}(\infty). \end{aligned}\right. \tag{78.3}$$

For the determination of the analytic functions $\varphi^0(\zeta)$ and $\psi^0(\zeta)$ we thus have the boundary condition

$$\varphi^0(\sigma) - \frac{1}{\sigma^3}\overline{\varphi^{0\prime}(\sigma)} + \overline{\psi^0(\sigma)} = F^0(\sigma), \tag{78.4}$$

where

$$\begin{aligned} F^0(\sigma) = F(\sigma) &- \frac{X_1 + iX_2}{2\pi} \log \sigma - \frac{BR}{\sigma} \\ &+ \frac{1}{\sigma^3}\left[\frac{X_1 - iX_2}{2\pi(1+\varkappa)}\sigma - BR\sigma^2\right] - (B' - iC')R\sigma, \end{aligned} \tag{78.5}$$

and $F(\sigma) \equiv f_1(\vartheta) + if_2(\vartheta)$, determined by the specified stress distribution on the circular boundary.

Setting

$$\varphi^0(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^k, \qquad \psi^0(\zeta) = \sum_{k=0}^{\infty} b_k\zeta^k,$$

in (78.4) and writing in the right-hand member the Fourier series representation for the single-valued function,

$$F(\sigma) - \frac{X_1 + iX_2}{2\pi} \log \sigma = \sum_{k=-\infty}^{\infty} A_k \sigma^k - \frac{X_1 + iX_2}{2\pi} \left[\pi i + \sum_{k=1}^{\infty} \frac{1}{k} (\sigma^{-k} - \sigma^k) \right],$$

we obtain

$$(78.6) \quad \sum_{k=1}^{\infty} a_k \sigma^k - \sum_{k=1}^{\infty} k \bar{a}_k \sigma^{-k-2} + \sum_{k=0}^{\infty} \bar{b}_k \sigma^{-k} = \sum_{k=-\infty}^{\infty} A_k \sigma^k - \frac{X_1 + iX_2}{2\pi} \left[\pi i + \sum_{k=1}^{\infty} \frac{1}{k} (\sigma^{-k} - \sigma^k) \right] - \frac{2BR}{\sigma} - (B' - iC')R\sigma + \frac{X_1 - iX_2}{2\pi(1 + \varkappa)} \sigma^{-2}.$$

The comparison of like powers of σ then yields:

$$(78.7) \quad \begin{cases} a_1 = A_1 + \dfrac{X_1 + iX_2}{2\pi} - (B' - iC')R, \\ a_k = A_k + \dfrac{X_1 + iX_2}{2\pi} \dfrac{1}{k}, \qquad k \geq 2, \\ \bar{b}_0 = A_0 - \dfrac{i(X_1 + iX_2)}{2}, \\ \bar{b}_1 = A_{-1} - \dfrac{X_1 + iX_2}{2\pi} - 2BR, \\ \bar{b}_2 = A_{-2} - \dfrac{X_1 + iX_2}{2\pi} \dfrac{1}{2} + \dfrac{X_1 - iX_2}{2\pi(1 + \varkappa)}. \\ \bar{b}_k = A_{-k} - (k - 2)\bar{a}_{k-2} - \dfrac{X_1 + iX_2}{2\pi} \dfrac{1}{k}, \qquad k \geq 3. \end{cases}$$

These formulas simplify considerably when $X_1 + iX_2 = 0$, that is when the stresses on the boundary are self-equilibrating.

We next specialize these results to several problems of technical interest.

a. Uniform Internal Pressure. When constant pressure P acts on the boundary of the hole, $T_1 = P \cos \theta$, $T_2 = P \sin \theta$, and[1]

$$f_1 + if_2 = +i \int^s (T_1 + iT_2)\, ds = -i \int^\theta Pe^{i\theta} R\, d\theta = -PRe^{i\theta}.$$

Thus,

$$F(\sigma) = -PR\sigma^{-1}.$$

[1] The negative sign is introduced in the integral because the positive direction of integration along the circle is clockwise, inasmuch as the normal ν is directed toward its center.

Clearly, $X_1 + iX_2 = 0$ and, if we assume that the stresses at infinity vanish, $B = B' = C' = 0$. Thus $F^0(\sigma) = F(\sigma)$, and $\varphi_1(\zeta) = \varphi^0(\zeta)$, $\psi_1(\zeta) = \psi^0(\zeta)$.

Since all Fourier coefficients A_k, with the exception

$$A_{-1} = -PR,$$

vanish in tne expansion for $F(\sigma)$, we conclude from (78.7) that

$$\begin{aligned} a_k &= 0, \qquad k \geq 1, \\ b_1 &= -PR, \qquad b_k = 0, \qquad k \neq 1. \end{aligned}$$

Hence

$$\varphi_1(\zeta) = 0, \qquad \psi_1 = -PR\zeta,$$

and, therefore,

$$\varphi(z) = 0, \qquad \psi(z) = -\frac{PR^2}{z}.$$

Using the formulas (77.6), we get,

$$u_r = \frac{PR^2}{2\mu r}, \qquad u_\theta = 0,$$

$$\tau_{rr} = -\tau_{\theta\theta} = -\frac{PR^2}{r^2}, \qquad \tau_{r\theta} = 0,$$

and the problem is completely solved.

b. Concentrated Force in the Plane. The stress distribution produced by a concentrated force applied to a point in the plane can be obtained by analyzing the effect of the constant stress distribution

$$T_1 = \frac{X_1}{2\pi R}, \qquad T_2 = \frac{X_2}{2\pi R}, \tag{78.8}$$

acting on the boundary of the circle $|z| = R$. The resultant force produced by the stress distribution $T_1 + iT_2$ is, clearly, $X_1 + iX_2$. If we assume that the stresses at infinity vanish, $F^0(\sigma)$ defined by (78.5) becomes,

$$F^0(\sigma) = F(\sigma) - \frac{X_1 + iX_2}{2\pi} \log \sigma + \frac{1}{\sigma^2} \frac{X_1 - iX_2}{2\pi(1 + \varkappa)}. \tag{78.9}$$

But

$$f_1 + if_2 = i \int^s (T_1 + iT_2)\, ds = -i \frac{X_1 + iX_2}{2\pi} \theta,$$

so that

$$F(\sigma) = \frac{i(X_1 + iX_2)}{2\pi} \vartheta \equiv \frac{X_1 + iX_2}{2\pi} \log \sigma.$$

Inserting this in (78.9), we see that the right-hand member in (78.6) reduces to the single term

$$F^0(\sigma) = \frac{X_1 - iX_2}{2\pi(1 + \varkappa)} \sigma^{-2}.$$

We thus conclude that

$$a_k = 0, \quad k > 0,$$
$$b_2 = \frac{X_1 + iX_2}{2\pi(1 + \varkappa)}, \qquad b_k = 0, \qquad k \neq 2,$$

and hence

$$\varphi^0(\zeta) = 0, \qquad \psi^0(\zeta) = \frac{X_1 + iX_2}{2\pi(1 + \varkappa)} \zeta^2.$$

Inserting these in (78.2), and recalling (78.1), we get

$$(78.10) \qquad \begin{cases} \varphi(z) = \dfrac{X_1 + iX_2}{2\pi(1 + \varkappa)} \log \dfrac{R}{z}, \\ \psi(z) = \dfrac{-\varkappa(X_1 - iX_2)}{2\pi(1 + \varkappa)} \log \dfrac{R}{z} + \dfrac{X_1 + iX_2}{2\pi(1 + \varkappa)} \dfrac{R^2}{z^2}. \end{cases}$$

The stress distribution in the region $|z| \geq R$ is determined by (77.6), if we insert

$$(78.11) \qquad \begin{cases} \varphi'(z) = -\dfrac{X_1 + iX_2}{2\pi(1 + \varkappa)} \dfrac{1}{z}, \qquad \varphi''(z) = \dfrac{X_1 + iX_2}{2\pi(1 + \varkappa)} \dfrac{1}{z^2}, \\ \psi'(z) = \dfrac{\varkappa(X_1 - iX_2)}{2\pi(1 + \varkappa)} \dfrac{1}{z} - \dfrac{X_1 + iX_2}{\pi(1 + \varkappa)} \dfrac{R^2}{z^3}. \end{cases}$$

To obtain the stress distribution produced by the concentrated force $X_1 + iX_2$ applied at $z = 0$, we let $R \to 0$ and allow T_1 and T_2 in (78.8) to increase in such a way that the resultant force is always equal to $X_1 + iX_2$. The resulting stress distribution is that given by formulas (77.6), where we use (78.11) with $R = 0$. The result of simple calculations is,

$$(78.12) \qquad \begin{cases} \tau_{rr} = -\dfrac{\varkappa + 3}{\varkappa + 1} \dfrac{X_1 \cos\theta + X_2 \sin\theta}{2\pi r}, \\ \tau_{\theta\theta} = \dfrac{\varkappa - 1}{\varkappa + 1} \dfrac{X_1 \cos\theta + X_2 \sin\theta}{2\pi r}, \\ \tau_{r\theta} = \dfrac{\varkappa - 1}{\varkappa + 1} \dfrac{X_1 \sin\theta - X_2 \cos\theta}{2\pi r}. \end{cases}$$

The solution recorded here corresponds to the state of plane strain. In dealing with the generalized plane stress, $\varkappa$ in (78.12) must be replaced by $\bar{\varkappa} = (3 - \sigma)/(1 + \sigma)$, while X_1 and X_2 are reckoned per unit thickness of the plane. That is, $X_1 = X_1^0/2h$ and $X_2 = X_2^0/2h$, where $2h$ is the thickness of the plate and $X_1^0 + iX_2^0$ is the concentrated force.

c. Concentrated Moment in the Plane. We consider next the effect of the stress distribution

$$T_1 = -\frac{M}{2\pi R^2} \sin\theta, \qquad T_2 = \frac{M}{2\pi R^2} \cos\theta,$$

applied to the boundary $|z| = R$. This distribution is produced by the constant tangential stress T of magnitude $M/2\pi R^2$.

Since

$$f_1 + if_2 = i \int^s (T_1 + iT_2)\, ds = \frac{M}{2\pi R} \int^\theta e^{i\theta}\, d\theta = -\frac{Mi}{2\pi R} e^{i\theta},$$

$$F(\sigma) = -\frac{Mi}{2\pi R} \sigma^{-1}.$$

Thus, the only nonvanishing A_k in (78.7) is $A_{-1} = -Mi/2\pi R$. If the stresses at infinity vanish, the system (78.7) yields,

$$a_k = 0, \qquad k = 1, 2, \ldots,$$
$$b_1 = \frac{Mi}{2\pi R}, \qquad b_k = 0, \qquad k \neq 1,$$

inasmuch as $X_1 + iX_2 = 0$. Hence,

$$\varphi_1(\zeta) = 0, \qquad \psi_1(\zeta) = \frac{Mi}{2\pi R} \zeta$$

and

$$\varphi(z) = 0, \qquad \psi(z) = \frac{Mi}{2\pi} \frac{1}{z}.$$

Making use of (77.6), we easily find,

$$\tau_{rr} = \tau_{\theta\theta} = 0, \qquad \tau_{r\theta} = -\frac{M}{2\pi r^2},$$

where $M = -2\pi T R^2$.

d. Uniaxial and Biaxial Tension. Pure Shear. We consider next the effect of the stress concentration in the neighborhood of the hole $|z| = R$, located in a plane subjected to the action of constant loads at a great distance from the hole.

Let us suppose first that the plane is stretched by the tensile forces acting in the x_1-direction. We take

$$\tau_{11}(\infty) = P_1, \qquad \tau_{12}(\infty) = \tau_{22}(\infty) = 0.$$

Since the hole is free of stress, $X_1 + iX_2 = 0$ and $F(\sigma) = 0$. The constants B, B', C' in (78.7) are determined by (78.3), and we find,

$$B = \tfrac{1}{4}P_1, \qquad B' = -\tfrac{1}{2}P_1, \qquad C' = 0.$$

Equations (78.7) then yield,

$$a_1 = \frac{P_1 R}{2}, \qquad a_k = 0, \qquad k > 1,$$

$$b_0 = 0, \qquad b_1 = -\frac{P_1 R}{2}, \qquad b_2 = 0, \qquad b_3 = \frac{P_1 R}{2},$$

$$b_k = 0, \qquad k > 3.$$

Thus

$$\varphi^0(\zeta) = \frac{P_1 R}{2}\,\zeta, \qquad \psi^0(\zeta) = -\frac{P_1 R}{2}\,(\zeta - \zeta^3),$$

and, from formulas (78.2),

$$\varphi_1(\zeta) = \frac{P_1 R}{2}\left(\frac{1}{2\zeta} + \zeta\right),$$
$$\psi_1(\zeta) = -\frac{P_1 R}{2}\left(\frac{1}{\zeta} + \zeta - \zeta^3\right).$$

Hence

$$\varphi(z) = \frac{P_1}{2}\left(\frac{z}{2} + \frac{R^2}{z}\right),$$
$$\psi(z) = -\frac{P_1}{2}\left[z + R^2\left(\frac{1}{z} - \frac{R^2}{z^3}\right)\right].$$

Using formulas (77.6), we find:

$$u_r = \frac{P_1}{8\mu r}\left\{(\varkappa - 1)r^2 + 2R^2 + 2\left[R^2(\varkappa + 1) + r^2 - \frac{R^4}{r^2}\right]\cos 2\theta\right\},$$
$$u_\theta = -\frac{P_1}{4\mu r}\left[r^2 + R^2(\varkappa - 1) + \frac{R^4}{r^2}\right]\sin 2\theta,$$

$$(78.13)\quad \begin{cases} \tau_{rr} = \dfrac{P_1}{2}\left[\left(1 - \dfrac{R^2}{r^2}\right) + \left(1 - \dfrac{4R^2}{r^2} + \dfrac{3R^4}{r^4}\right)\cos 2\theta\right], \\ \tau_{\theta\theta} = \dfrac{P_1}{2}\left[\left(1 + \dfrac{R^2}{r^2}\right) - \left(1 + \dfrac{3R^4}{r^4}\right)\cos 2\theta\right], \\ \tau_{r\theta} = -\dfrac{P_1}{2}\left(1 + \dfrac{2R^2}{r^2} - \dfrac{3R^4}{r^4}\right)\sin 2\theta. \end{cases}$$

For tension $\tau_{22}(\infty) = P_2$ in the x_2-direction, we have,

$$(78.14)\quad \begin{cases} \tau_{rr} = \dfrac{P_2}{2}\left[\left(1 - \dfrac{R^2}{r^2}\right) - \left(1 - \dfrac{4R^2}{r^2} + \dfrac{3R^4}{r^4}\right)\cos 2\theta\right], \\ \tau_{\theta\theta} = \dfrac{P_2}{2}\left[\left(1 + \dfrac{R^2}{r^2}\right) + \left(1 + \dfrac{3R^4}{r^4}\right)\cos 2\theta\right], \\ \tau_{r\theta} = \dfrac{P_2}{2}\left(1 + \dfrac{2R^2}{r^2} - \dfrac{3R^4}{r^4}\right)\sin 2\theta. \end{cases}$$

By superposition of (78.13) and (78.14), we find, for the uniform biaxial tension with $P_1 = P_2 = P$,

$$(78.15)\quad \tau_{rr} = P\left(1 - \frac{R^2}{r^2}\right), \qquad \tau_{\theta\theta} = P\left(1 + \frac{R^2}{r^2}\right), \qquad \tau_{r\theta} = 0.$$

On the boundary of the hole in all these cases $\tau_{rr} = \tau_{r\theta} = 0$, as it should, but for the case of the uniaxial tension the hoop stress $\tau_{\theta\theta}$ is,

$$\tau_{\theta\theta} = P_1(1 - 2\cos 2\theta).$$

This assumes a maximum value $(\tau_{\theta\theta})_{\max} = 3P_1$ at $\theta = \pi/2$ and $\theta = 3\pi/2$, which is three times the stress in a plate without the hole. For the case of biaxial tension $(\tau_{\theta\theta})_{\max} = 2P$, as is clear from (78.15).

If we set $\tau_{11}(\infty) = P$ and $\tau_{22}(\infty) = -P$ in (78.13) and (78.14), and combine the results, we shall get the solution corresponding to the plate in the state of pure shear.[1]

On the boundary of the hole, in this case,

$$\tau_{\theta\theta}\Big|_{r=R} = -4P\cos 2\theta,$$

which has an absolute maximum $|\tau_{\theta\theta}| = 4P$ at $\theta = 0, \pi/2, \pi, 3\pi/2$.

79. Infinite Region Bounded by an Ellipse. As a further illustration of the method outlined in Sec. 76, we consider the first boundary-value problem for an infinite region bounded by an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{79.1}$$

It is easy to verify that the mapping function

$$z = R\left(\frac{1}{\zeta} + m\zeta\right), \qquad R > 0, \qquad 0 \leq m \leq 1, \tag{79.2}$$

transforms the region exterior to the ellipse (79.1) into a circle $|\zeta| \leq 1$, if we take

$$R = \frac{a+b}{2}, \qquad m = \frac{a-b}{a+b}.$$

It should be noted that, as the point $\zeta = e^{i\vartheta}$ describes the circle $|\zeta| = 1$ in the positive (counterclockwise) direction, the corresponding point z traces out the ellipse (79.1) in the clockwise direction. Accordingly, the parametric equations of the ellipse must be taken in the form:

$$x_1 = R(1+m)\cos\vartheta, \qquad x_2 = -R(1-m)\sin\vartheta.$$

If $m = 0$, the ellipse becomes a circle. When $m = 1$, the point in the z-plane traces out the segment of the x_1-axis, between $x_1 = 2R$ and $x_1 = -2R$, twice, as the point ζ describes once the boundary $|\zeta| = 1$. Thus, in this case, the function (79.2) maps the z-plane, slit along the line joining the points $(2R, 0)$ and $(-2R, 0)$, onto $|\zeta| \leq 1$.

The solution of the first boundary-value problem for an infinite simply connected domain, as we saw in Sec. 76, can be reduced to the determination of two functions $\varphi^0(\zeta)$ and $\psi^0(\zeta)$, analytic in the circle $|\zeta| < 1$, which satisfy the boundary condition

$$\varphi^0(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi^{0\prime}(\sigma)} + \overline{\psi^0(\sigma)} = F^0(\sigma). \tag{79.3}$$

[1] See Sec. 19.

In our problem,

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{1 + m\sigma^2}{\sigma(m - \sigma^2)} = -\frac{1 + m\sigma^2}{\sigma^3}\left(1 + \frac{m}{\sigma^2} + \frac{m^2}{\sigma^4} + \cdots\right), \tag{79.4}$$

so that the coefficients c_n, in the expansion (76.4), vanish for all $n \geq 0$. It follows then from (76.6) that

$$a_k = A_k, \qquad k \geq 1,$$

so that

$$\varphi^0(\zeta) = \sum_{k=1}^{\infty} A_k \zeta^k, \tag{79.5}$$

where

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} F^0(\sigma)e^{-ik\vartheta}\, d\vartheta, \tag{79.6}$$

with $F^0(\sigma)$ given by (76.13).

An integral representation for $\varphi^0(\zeta)$, which is more convenient for calculation purposes than the series (79.5), can be readily deduced by substituting from (79.6) in (79.5). We have,

$$\varphi^0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} F^0(\sigma) \sum_{k=1}^{\infty} \frac{\zeta^k}{\sigma^k}\, d\vartheta,$$

since $\sigma = e^{i\vartheta}$.

Noting that $d\sigma = e^{i\vartheta} i\, d\vartheta$, we can write

$$\begin{aligned}
\varphi^0(\zeta) &= \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma} \sum_{k=1}^{\infty} \frac{\zeta^k}{\sigma^k}\, d\sigma \\
&= \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma}\left[\frac{1}{1 - (\zeta/\sigma)} - 1\right] d\sigma,
\end{aligned}$$

or

$$\varphi^0(\zeta) = \frac{\zeta}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma(\sigma - \zeta)}\, d\sigma. \tag{79.7}$$

This is the desired integral formula for $\varphi^0(\zeta)$.

Instead of the series representation of the function $\psi^0(\zeta)$, based on the calculation of the coefficients b_k, one can also deduce a useful integral representation as follows: We rewrite (79.3) in the conjugate form,

$$\overline{\varphi^0(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\, \varphi^{0\prime}(\sigma) + \psi^0(\sigma) = \overline{F^0(\sigma)}, \tag{79.8}$$

multiply both members of (79.8) by $\dfrac{1}{2\pi i}\dfrac{d\sigma}{\sigma - \zeta}$, and integrate over the

contour γ. We get,

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi^0(\sigma)}}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}}\frac{\varphi^{0\prime}(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\psi^0(\sigma)}{\sigma - \zeta}\,d\sigma = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F^0(\sigma)}}{\sigma - \zeta}\,d\sigma.$$

But,[1]

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi^0(\sigma)}}{\sigma - \zeta}\,d\sigma = \overline{\varphi^0(0)} = 0, \qquad \frac{1}{2\pi i}\int_\gamma \frac{\psi^0(\sigma)}{\sigma - \zeta}\,d\sigma = \psi^0(\zeta),$$

and we have,

$$\psi^0(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}}\frac{\varphi^{0\prime}(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{F^0(\sigma)}}{\sigma - \zeta}\,d\sigma. \tag{79.9}$$

Since in our problem

$$\frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}} = -\frac{\sigma(\sigma^2 + m)}{1 - m\sigma^2}, \qquad m < 1,$$

we see that the first integral in the right-hand member of (79.9) can be evaluated by Cauchy's Integral Formula to yield,

$$\psi^0(\zeta) = \frac{\zeta(\zeta^2 + m)}{1 - m\zeta^2}\varphi^{0\prime}(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\overline{F^0(\sigma)}}{\sigma - \zeta}\,d\sigma. \tag{79.10}$$

We proceed to illustrate the use of formulas (79.7) and (79.10) in two special problems.

a. *Uniform Pressure on Elliptical Boundary.* If the ellipse is subjected to a uniform pressure of intensity P, then

$$T_\alpha = -P\nu_\alpha, \qquad (\alpha = 1, 2),$$

where

$$\nu_1 = \cos(x_1, \nu) = \cos(x_2, s) = \frac{dx_2}{ds},$$

$$\nu_2 = \cos(x_2, \nu) = -\cos(x_1, s) = -\frac{dx_1}{ds}.$$

Thus,

$$i\int^s (T_1 + iT_2)\,ds = -\int^s P\,dz = -Pz,$$

so that

$$F^0(\sigma) = -P\omega(\sigma) = -PR(\sigma^{-1} + m\sigma),$$

if we suppose that the stresses at infinity vanish.

Substituting this in the integrals in formulas (79.7) and (79.10), we

[1] See the corresponding calculations in (42.3).

obtain, virtually without calculations,

$$\varphi(\zeta) = -PRm\zeta,$$
$$\psi(\zeta) = -PR\left[\zeta + \frac{m\zeta(\zeta^2 + m)}{1 - m\zeta^2}\right],$$

which solve the problem.

b. Stretched Plate Weakened by an Elliptical Hole. If the boundary of the opening is free of stress and the plate is deformed at infinity by the application of a uniform tensile stress of intensity P, making an angle α with the x_1-axis, the formulas in Sec. 19*b* demand that

$$\tau_{11}(\infty) = P\cos^2\alpha, \qquad \tau_{22}(\infty) = P\sin^2\alpha, \qquad \tau_{12}(\infty) = P\sin\alpha\cos\alpha.$$

Thus the constants B, B', C' in (76.13) are determined by

$$B' + iC' = -\frac{P}{2}e^{-2i\alpha}, \qquad B = \frac{P}{4},$$

and hence

$$F^0(\sigma) = -\frac{PR}{4}\left[\frac{1}{\sigma} - 2e^{2i\alpha}\sigma + \frac{\sigma(1 + m\sigma^2)}{\sigma^2 - m}\right].$$

The substitution of this in (79.7) and (79.10) yields, after simple calculations,

$$\varphi^0(\zeta) = \frac{PR\zeta}{4}(2e^{2i\alpha} - m),$$
$$\psi^0(\zeta) = \frac{PR\zeta}{2(m\zeta^2 - 1)}[m^2 - 1 - e^{2i\alpha}(\zeta^2 + m)].$$

Thus

$$\varphi(\zeta) = \frac{PR\zeta}{4}\left(2e^{2i\alpha} - m + \frac{1}{\zeta^2}\right),$$
$$\psi(\zeta) = -\frac{PR}{2}\left[e^{-2i\alpha}\frac{1}{\zeta} + \frac{e^{2i\alpha}\zeta}{m} - \frac{(1 + m^2)(e^{2i\alpha} - m)}{m}\,\frac{\zeta}{1 - m\zeta^2}\right],$$

from which the displacement and stresses can be computed without difficulty.[1]

PROBLEMS

1. Compute the displacements and stresses in the problem treated in the illustration of Sec. 79*a*, for the case when $m = 0$.

2. Solve the problem of deformation of an infinite plate with an elliptical hole, when a constant tangential force acts on the boundary of the hole.

[1] The solution of this problem was first obtained by C. E. Inglis, *Transactions of the Institute of Naval Architects*, London, vol. 55 (1913), pp. 219–230. The solution given here is due to N. I. Muskhelishvili, *Izvestiya (Bulletin) Akademii Nauk SSSR* (1919), pp. 663–686. It is also contained in Sec. 82a, pp. 337–339, of Muskhelishvili's book, Some Basic Problems of the Mathematical Theory of Elasticity (1953).

80. Problems for the Interior of an Ellipse. Theoretically the method of solution illustrated in the foregoing can be used to solve problems for simply connected domains whenever the mapping function $z = \omega(\zeta)$ is known. But the function $\omega(\zeta)$, mapping the region interior to an ellipse onto a circle, is very complicated. However, as was shown by Muskhelishvili,[1] it is possible to make an effective use of the function employed in the preceding section to solve the interior problem as well.

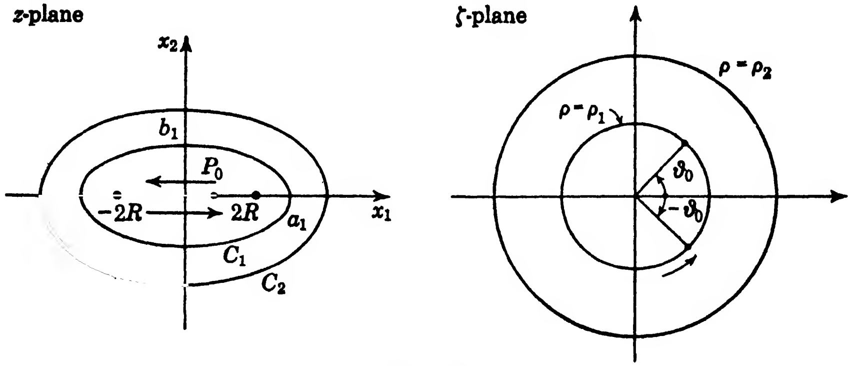

FIG. 56

We consider

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \qquad R > 0, \qquad m \geq 0, \tag{80.1}$$

and, upon setting $\zeta = \rho e^{i\vartheta}$ and $z = x_1 + ix_2$, find

$$x_1 = R\left(\rho + \frac{m}{\rho}\right)\cos\vartheta,$$

$$x_2 = R\left(\rho - \frac{m}{\rho}\right)\sin\vartheta.$$

Thus, the circle of radius $\rho = \rho_1$ in the ζ-plane corresponds to an ellipse C_1 with the semiaxes

$$a_1 = R\left(\rho_1 + \frac{m}{\rho_1}\right), \qquad b_1 = R\left(\rho_1 - \frac{m}{\rho_1}\right), \tag{80.2}$$

provided that $\rho_1^2 \geq m$.

The circle of radius $\rho = \rho_2$ corresponds to another ellipse C_2 (Fig. 56),

[1] N. I. Muskhelishvili, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 1 (1933), pp. 5–12. See also, Some Basic Problems of the Mathematical Theory of Elasticity (1953), pp. 244–250. This problem was also treated by much more complicated means by O. Tedone, *Atti della accademia delle scienze di Torino*, vol. 41 (1906), pp. 86–101, and T. Boggio, *Atti del reale instituto veneto di scienze, lettere ed vete*, vol. 60 (1901), pp. 591–609. A solution of the problem, with the aid of integral equations, was also given by D. I. Sherman, *Doklady Akademii Nauk SSSR*, vol. 31 (1941), pp. 309–310.

and the elliptical ring bounded by C_1 and C_2 is mapped conformally by (80.1) on the annulus formed by the circles $\rho = \rho_1$ and $\rho = \rho_2$. If ρ_2 is increased indefinitely, the function in (80.1) maps the region exterior to the ellipse C_1 onto the region exterior to the circle $\rho = \rho_1$. For $m = \rho_1^2$, the ellipse C_1 degenerates into a segment of the real axis. If we take $m = 1$, the mapping function

$$(80.3) \qquad z = R\left(\zeta + \frac{1}{\zeta}\right), \qquad R > 0$$

maps the entire z-plane, slit along the real axis between $x_1 = -2R$ and $x_1 = 2R$, onto the region $|\zeta| \geq 1$. As the point $\zeta = e^{i\vartheta}$ traverses the circle once, the corresponding point z traverses the slit twice, so that the points $\sigma = e^{i\vartheta_0}$ and $\sigma = e^{-i\vartheta_0}$ correspond to one and the same point P_0 on the slit. The ring bounded by the circles $\rho = \rho_0 > 1$ and $\rho = 1$ then corresponds to the interior of the ellipse C_0, cut along the real axis between the points $(-2R, 0)$ and $(2R, 0)$.

If either the displacements or the stresses are specified on the boundary C_0 of the uncut ellipse, the functions $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ are determined by the condition of the form

$$(80.4) \qquad \alpha\varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\,\overline{\varphi_1'(\zeta)} + \overline{\psi_1(\zeta)} = H(\zeta), \qquad \text{for } \zeta = \rho_0 e^{i\vartheta}.$$

Since $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ are analytic in the ring $1 < |\zeta| < \rho_0$, they can be represented in Laurent's series as

$$(80.5) \qquad \varphi_1(\zeta) = \sum_{k=-\infty}^{\infty} a_k\zeta^k, \qquad \psi_1(\zeta) = \sum_{k=-\infty}^{\infty} b_k\zeta^k.$$

Moreover, the point P_0 on the cut corresponds to the points $\zeta = e^{i\vartheta_0}$ and $\zeta = e^{-i\vartheta_0}$ on $|\zeta| = 1$, and the continuity of $\varphi(z)$ and $\psi(z)$ requires that

$$(80.6) \qquad \varphi_1(\sigma) = \varphi_1(\bar{\sigma}), \qquad \psi_1(\sigma) = \psi_1(\bar{\sigma}).$$

The condition (80.6) implies that the coefficients a_k and b_k in (80.5) are related by the formulas:

$$(80.7) \qquad a_k = a_{-k}, \qquad b_k = b_{-k}, \qquad k = 0, 1, 2, \ldots .$$

The further conditions connecting these coefficients, which enable one to determine the functions in (80.5), are obtained from the boundary condition (80.4) in the manner of Sec. 76. The reader interested in further calculational details will find them in the cited works of Muskhelishvili.

81. Basic Problems for Doubly Connected Domains. We shall see in this section that the method of solution outlined in Sec. 76 can be easily modified to yield an effective solution of the basic problems for the cir-

cular ring. Although a doubly connected domain can be mapped conformally on a circular ring, a generalization of the formulas of Sec. 76 to doubly connected domains usually leads to intractable systems of equations for the coefficients in the series representations of $\varphi(z)$ and $\psi(z)$.

The treatment of the first and second boundary-value problems for the circular ring is identical, and we confine our discussion to the first problem.

Let the ring be formed by a pair of concentric circles C_α, $\alpha = 1, 2$, of radii R_α, where $R_1 < R_2$. To simplify calculations, we shall suppose that the external stresses applied to each boundary are such that the resultant force and moment vanish for each boundary. In this event, the logarithmic terms do not appear in the representations (72.7), and the functions $\varphi(z)$ and $\psi(z)$ will be analytic in the ring $R_1 < |z| < R_2$.

Accordingly, we can write

$$\text{(81.1)} \qquad \varphi(z) = \sum_{-\infty}^{\infty} a_k z^k, \qquad \psi(z) = \sum_{-\infty}^{\infty} b_k z^k, \qquad R_1 < |z| < R_2.$$

The coefficients a_k and b_k in (81.1) must be chosen so that

$$\text{(81.2)} \quad \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = f_1^{(\alpha)}(s) + if_2^{(\alpha)}(s) + \text{const} \qquad \text{on } C_\alpha,$$

where

$$\text{(81.3)} \qquad i\int^s [T_1^{(\alpha)}(s) + iT_2^{(\alpha)}(s)]\,ds = f_1^{(\alpha)} + if_2^{(\alpha)} + \text{const}.$$

The value of the integration constant in (81.2) can be fixed arbitrarily only on one of the contours; on the other it must be determined so that the stresses and displacements in the ring are single-valued and continuous.

The arc parameter s on the circular boundaries C_α can be taken equal to $R\theta$, where θ is the polar angle; thus, the right-hand member in (81.3) can be viewed as a function of θ, say $F_\alpha(\theta)$. Assuming that $F_\alpha(\theta)$ can be represented in the complex Fourier series,

$$F_\alpha(\theta) = \sum_{-\infty}^{\infty} A_k^{(\alpha)} e^{ik\theta},$$

and, recalling (81.1), we can write the boundary conditions (81.2) in the form

$$\text{(81.4)} \quad \sum_{-\infty}^{\infty} a_k R_\alpha^k \sigma^k + R_\alpha \sigma \sum_{-\infty}^{\infty} \bar{a}_k k R_\alpha^{k-1} \sigma^{-(k-1)} + \sum_{-\infty}^{\infty} \bar{b}_k R_\alpha^k \sigma^{-k} = \sum_{-\infty}^{\infty} A_k^{(\alpha)} \sigma^k,$$

where $\sigma = e^{i\theta}$.

The system of equations for the unknown coefficients a_k and b_k is then

got by comparing the coefficients of like powers of σ in (81.4). The solution of this system presents no difficulties, and the resulting series (81.1) can be easily shown to correspond to the desired solution if the derivative of $F_\alpha(\theta)$ is of bounded variation.[1]

We limit ourselves to the study of the case in which the boundaries of the ring are subjected to constant pressures.[2]

Let the pressures acting on contours C_α be P_α. Then, on taking account of the convention for the positive direction of the normal to C_α, we have,

$$T_1^{(2)} = -P_2 \cos\theta, \qquad T_2^{(2)} = -P_2 \sin\theta,$$
$$T_1^{(1)} = P_1 \cos\theta, \qquad T_2^{(1)} = P_1 \sin\theta.$$

Thus,

$$T_1^{(1)} + iT_2^{(1)} = P_1 e^{i\theta}, \qquad T_1^{(2)} + iT_2^{(2)} = -P_2 e^{i\theta},$$

and[3]

$$(81.5)\quad \begin{cases} F_1(\theta) \equiv i\displaystyle\int^s (T_1^{(1)} + iT_2^{(1)})\,ds \\ \qquad\qquad = -i\displaystyle\int^\theta P_1 e^{i\theta} R_1\,d\theta = -P_1R_1e^{i\theta} + c_1, \\ F_2(\theta) \equiv i\displaystyle\int^s (T_1^{(2)} + iT_2^{(2)})\,ds = -i\displaystyle\int^\theta P_2 e^{i\theta} R_2\,d\theta = -P_2R_2e^{i\theta}, \end{cases}$$

where the integration constant in $F_2(\theta)$ has been set equal to zero.

Substituting in the right-hand member of (81.4) from (81.5), we get, on equating the coefficients of like powers of σ, the following systems:

$$(81.6)\quad \begin{cases} a_0 + 2R_2^2\bar{a}_2 + \bar{b}_0 = 0, \\ a_1R_2 + R_2\bar{a}_1 + \bar{b}_{-1}R_2^{-1} = -P_2R_2, \\ a_kR_2^k + (2-k)\bar{a}_{2-k}R_2^{2-k} + \bar{b}_{-k}R_2^{-k} = 0, & \text{for } k \neq 0, 1. \\ a_0 + 2R_1^2\bar{a}_2 + \bar{b}_0 = c_1, \\ a_1R_1 + R_1\bar{a}_1 + \bar{b}_{-1}R_1^{-1} = -P_1R_1, \\ a_kR_1^k + (2-k)\bar{a}_{2-k}R_1^{2-k} + \bar{b}_{-k}R_1^{-k} = 0, & \text{for } k \neq 0, 1. \end{cases}$$

In order to obtain a unique solution, we set $a_0 = 0$, $\mathscr{I}a_1 = 0$ and, after some simple algebra, find that the nonvanishing coefficients are:[4]

$$a_1 = \frac{P_2R_2^2 - P_1R_1^2}{2(R_1^2 - R_2^2)}, \qquad b_{-1} = \frac{R_2^2R_1^2(P_1 - P_2)}{R_1^2 - R_2^2}.$$

[1] An analogous system, corresponding to a somewhat different choice of functions, is discussed in detail on pp. 218–225 of N. I. Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953). There are other (more complicated) solutions available. See, for example, A. Timpe, *Zeitschrift für Mathematik und Physik*, vol. 52 (1905), pp. 348–383.

[2] This problem was first solved by G. Lamé in Leçons sur la théorie de l'élasticité (1852), with the aid of Navier's equations.

[3] The positive description of the contour C_1 is clockwise, and it is counterclockwise for C_2.

[4] The constant c_1 turns out to be zero in this problem.

Thus

$$\varphi(z) = \frac{P_2R_2^2 - P_1R_1^2}{2(R_1^2 - R_2^2)}\,z, \qquad \psi(z) = \frac{R_2^2R_1^2(P_1 - P_2)}{R_1^2 - R_2^2}\,\frac{1}{z}. \tag{81.7}$$

Using formulas (77.6) we find that,

$$\left\{\begin{aligned} \tau_{rr} &= \frac{P_2R_2^2 - P_1R_1^2}{R_1^2 - R_2^2} - \frac{P_2 - P_1}{R_1^2 - R_2^2}\,\frac{R_2^2R_1^2}{r^2}, \\ \tau_{\theta\theta} &= \frac{P_2R_2^2 - P_1R_1^2}{R_1^2 - R_2^2} + \frac{P_2 - P_1}{R_1^2 - R_2^2}\,\frac{R_2^2R_1^2}{r^2}, \\ \tau_{r\theta} &= 0. \end{aligned}\right. \tag{81.8}$$

If $P_2 = 0$,

$$\tau_{rr} = \frac{P_1R_1^2}{R_2^2 - R_1^2}\left(1 - \frac{R_2^2}{r^2}\right), \qquad \tau_{\theta\theta} = \frac{P_1R_1^2}{R_2^2 - R_1^2}\left(1 + \frac{R_2^2}{r^2}\right),$$

and we see that the radial stress τ_{rr} is compressive, while the hoop stress $\tau_{\theta\theta}$ is tensile. It is interesting to note that $(\tau_{\theta\theta})_{\max} > P_1$ regardless of how thick the ring is.

The procedure indicated above, when applied to the problem of the ring deformed by two oppositely directed concentrated forces on the exterior boundary, yields very slowly converging series (81.1) when the ring is narrow. The concentrated forces acting on the boundary C_2 give rise to the singularities in $\varphi(z)$ and $\psi(z)$, and it is advisable to modify the problem by making use of the solution of the corresponding problem for the solid disk found in Sec. 77*c*.

If the radius of the solid disk is R_2, the concentrated forces produce in it certain known shearing and normal stresses along the circle $|z| = R_1$. On subtracting the known solution of the problem of the solid disk of radius R_2, under the action of the same concentrated forces, from the desired solution of the ring problem, one is led to consider the following auxiliary problem: Find the state of stress in a ring whose exterior boundary $|z| = R_2$ is free of stress and whose interior boundary $|z| = R_1$ is subjected to continuously distributed shearing and normal stresses equal and opposite to the stresses present in the solid disk along the circle $|z| = R_1$. The superposition of the solution of this auxiliary problem on the known solution of the problem for the solid disk yields the desired solution.[1]

[1] This familiar device has been used by S. Timoshenko and J. N. Goodier [Theory of Elasticity (1951), pp. 116–123] and K. Wieghardt [*Sitzungsberichte der Akademie der Wissenschaften in Wien*, vol. 124 (1915), p. 1119] to solve the problem in Fourier series. However, the series converge slowly when R_1/R_2 is near unity. An application of the alternating method, discussed in Sec. 88, enabled M. Z. Narodetzkv [*Izvestiya Akademii Nauk SSSR*, Technical Series, No. 1 (1948), pp. 7–18] to deduce a solution that converges more rapidly. The stress distribution in a circular ring under the action of two equal and oppositely directed concentrated forces applied at the nearest

The function

$$z = \omega(\zeta) = \frac{\zeta - 1}{a\zeta - 1}, \qquad a > 1.$$

maps the region between two eccentric circles onto a circular ring. The reader may find it instructive to formulate the first boundary-value problem for the region bounded by two eccentric circles with the aid of this mapping function and deduce from the boundary conditions the appropriate systems of equations for the coefficients a_k and b_k in the expansions for $\varphi_1(\zeta)$ and $\psi_1(\zeta)$. The solution of the resulting systems presents difficulties, and it is simpler to treat the equilibrium problems for eccentric rings in bipolar coordinates.[1]

The function

$$z = R\left(\zeta + \frac{m}{\zeta}\right), \qquad R > 0, \qquad m > 0,$$

as we saw in Sec. 80, maps the region bounded by two confocal ellipses onto a circular ring of radii $\rho = \rho_\alpha$, $\alpha = 1, 2$. If the external stresses acting on the elliptical boundaries are such that the resultant force and moment acting on each boundary vanish, the functions $\varphi(z)$ and $\psi(z)$ will be single-valued and analytic in the elliptical ring. Consequently their

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points of the boundaries of the ring has been studied by D. V. Weinberg [*Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 13 (1949), pp. 151–158]. By increasing the radii of the circles, Weinberg deduces the known solution for an infinite strip subjected to the action of two oppositely directed concentrated forces. See also L. N. G. Filon's paper, entitled "The Stresses in a Circular Ring," *Institution of Civil Engineers,* London, *Selected Engineering Papers,* 12 (1924).

[1] See G. B. Jeffery, *Philosophical Transactions of the Royal Society (London) (A),* vol. 221 (1921), pp. 265–293, and Ya. S. Uflyand, Bipolar Coordinates in the Theory of Elasticity (1950), pp. 193–210 (in Russian). The equilibrium problems for a semi-infinite plane with a circular hole are also in this category. Bipolar coordinates have been used by Ya. S. Podstrigach, *Dopovidi Akademii Nauk Ukraïn'skoï RSR* (1953), pp. 456–460, to study the stress concentration in an infinite elastic plate weakened by two unequal circular holes, when the boundary of each hole is subjected to uniform pressures. The case of uniformly stretched plate weakened by two unequal circular holes is also considered in this paper.

As an illustration of the "alternating method," the equilibrium of an eccentric ring is discussed in Sec. 88.

The state of stress in a heavy semi-infinite sheet with one circular hole was investigated by R. D. Mindlin, "Stress Distribution around a Tunnel," *Proceedings of the American Society of Civil Engineers,* vol. 65 (1939), pp. 619–642.

Stress distribution in a heavy semi-infinite sheet with two circular holes was studied in detail by D. I. Sherman, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 15 (1951), pp. 297–316, 751–761.

An investigation of the stress concentration in a heavy semi-infinite sheet, near arch-shaped and trapezoidal openings stiffened by absolutely rigid rings, was made by I. S. Hara, *Dopovidi Akademii Nauk Ukraïn'skoï RSR* (1953), pp. 299–303.

transforms $\varphi_1(\zeta)$ and $\psi_1(\zeta)$, for $\rho_1 \leq |\zeta| \leq \rho_2$, have the representations,

$$(81.9) \qquad \varphi_1(\zeta) = \sum_{-\infty}^{\infty} a_k\zeta^k, \qquad \psi_1(\zeta) = \sum_{-\infty}^{\infty} b_k\zeta^k, \qquad \rho_1 \leq |\zeta| \leq \rho_2.$$

The boundary conditions in the transformed domain are:

$$(81.10) \qquad \varphi_1(\sigma_\alpha) + \frac{\omega(\sigma_\alpha)}{\overline{\omega'(\sigma_\alpha)}}\,\overline{\varphi_1'(\sigma_\alpha)} + \overline{\psi_1(\sigma_\alpha)} = f_\alpha(\sigma_\alpha) + c_\alpha, \qquad \alpha = 1, 2,$$

where $\sigma_\alpha = \rho_\alpha e^{i\vartheta}$ and the c_α are constants. The functions $f_\alpha(\sigma_\alpha)$ are completely determined from the assigned stresses on the boundaries C_α of the elliptical ring. If these functions are expanded in Fourier series and the series (81.9) are inserted in (81.10), it is possible to write down the system of equations for the determination of the a_k and b_k. Although this procedure is quite straightforward in principle, the calculations are quite involved and we shall not pursue them here.[1]

PROBLEMS

1. Use formulas (81.8) to show that

$$(\tau_{\theta\theta})_{\max} = \frac{P_1(R_1^2 + R_2^2)}{(R_1 + R_2)t},$$

where $t = R_2 - R_1$, and that for thin rings (or long pipes)

$$(\tau_{\theta\theta})_{\max} \doteq \frac{P_1R_1}{t}.$$

2. Consider the problem of the stress distribution in a hollow shaft, of inner radius R_1 and outer radius R_2, rotating with constant angular velocity ω. Take $\tau_{\alpha\beta} = \tau_{\alpha\beta}^{(0)} + \tau_{\alpha\beta}^{(1)}$, where $\tau_{\alpha\beta}^{(0)}$ is given by (68.3). Show that the formula

$$T_\alpha^{(1)} = \tau_{\alpha\beta}^{(1)}\nu_\beta,$$

[1] The solution of this problem, proposed by A. Timpe, *Mathematische Zeitschrift* vol. 52 (1923), pp. 189–205, as was noted by Muskhelishvili, is incorrect. The correct solution was given recently by A. I. Kalandiya, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 17 (1953), pp. 692–704, and a satisfactory approximate solution by M. P. Sheremetev, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 17 (1953), pp. 107–113. An outline of a solution of this problem with the aid of an integral equation whose kernel depends on Green's function for the confocal elliptical ring is contained on pp. 229–233 of S. G. Mikhlin's Integral Equations (1949). Although it is possible to deduce approximate solutions by replacing the kernel in Mikhlin's integral equation by a degenerate kernel, the necessary calculations are quite heavy. It appears that there is no royal road to the solution of the simplest elastostatic problems in multiply connected domains.

The construction of conformal maps, for the doubly connected region bounded externally by an ellipse and internally by a circle with coincident center, onto a circular ring was discussed by M. Z. Narodetzky and D. I. Sherman, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 14 (1950), pp. 209–214.

yields

$$(T_1^{(1)} + iT_2^{(1)})_{C_2} = \rho\omega^2 R_2^2 \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)} e^{i\theta} \quad \text{on the boundary } C_2,$$

and

$$(T_1^{(1)} + iT_2^{(1)})_{C_1} = -\rho\omega^2 R_1^2 \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)} e^{i\theta}, \quad \text{on the boundary } C_1.$$

Hence conclude that the solution of the equilibrium problem of rotating shaft with free lateral surface is deducible from the results of Sec. 81. Show that the maximum hoop stress is on the inner boundary.

3. Deduce the system of equations (81.6) by multiplying the boundary conditions

$$\varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)} = \begin{cases} -P_1 t + c_1, & t = R_1 e^{i\theta} \\ -P_2 t, & t = R_2 e^{i\theta} \end{cases}$$

by $\frac{1}{2\pi i}\frac{dt}{t^{n+1}}$ and integrating the result over the contours $|z| = R_1$ and $|z| = R_2$. Note that the coefficients in the Laurent series (81.1) for $\varphi(z)$ and $\psi(z)$ are given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{\varphi(t)\,dt}{t^{n+1}} \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_C \frac{\psi(t)\,dt}{t^{n+1}}.$$

82. Integrodifferential Equations for the Basic Problems. We have seen that the basic problems of plane elasticity for finite and infinite simply connected domains are reducible to the determination of two functions $\varphi(\zeta)$ and $\psi(\zeta)$, analytic in the circle $|\zeta| < 1$, which satisfy on its boundary γ a condition of the type[1]

$$\alpha\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = H(\sigma), \tag{82.1}$$

where $H(\sigma) \equiv h_1(\vartheta) + ih_2(\vartheta)$ is a single-valued function having continuous derivatives with respect to ϑ satisfying Hölder's condition.[2]

The boundary condition (82.1) can be reduced to an integrodifferential equation for the determination of $\varphi(\zeta)$ and $\psi(\zeta)$ by a technique similar to that used in deriving Schwarz's formula in Sec. 42.

If we multiply both members of (82.1) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma - \zeta}$, where $|\zeta| < 1$, and integrate over γ, we get

$$\frac{\alpha}{2\pi i}\int_\gamma \frac{\varphi(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi'(\sigma)}}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi(\sigma)}\,d\sigma}{\sigma - \zeta}\,d\sigma = A(\zeta), \tag{82.2}$$

with

$$A(\zeta) \equiv \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma - \zeta}\,d\sigma.$$

[1] We omit the subscript 1 and the superscript 0 on φ and ψ in the formulas (76.11), (76.12), (76.14) and in all expressions of this and the following three sections.

[2] See Sec. 40.

By Harnack's Theorem of Sec. 41, Eq. (82.2) is equivalent to Eq. (82.1). But for every $F(\zeta)$ continuous in $|\zeta| \leq 1$ and analytic for $|\zeta| < 1$,

$$\frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)}{\sigma - \zeta}\,d\sigma = F(\zeta), \qquad \frac{1}{2\pi i}\int_\gamma \frac{\overline{F(\sigma)}}{\sigma - \zeta}\,d\sigma = \overline{F(0)},$$

so that (82.2) yields

$$\alpha\varphi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi'(\sigma)}}{\sigma - \zeta}\,d\sigma + \overline{\psi(0)} = A(\zeta). \tag{82.3}$$

This is the desired integrodifferential equation for $\varphi(\zeta)$. It contains an unknown constant $\overline{\psi(0)}$, which can be determined by imposing the condition $\varphi(0) = 0$. Thus, if the value of $\overline{\psi(0)}$ in (82.3) is tentatively fixed in some arbitrary way and the corresponding solution for $\varphi(\zeta)$ is obtained, then the actual value of $\overline{\psi(0)}$ in (82.3) can be computed from the condition $\varphi(0) = 0$. For, if $\varphi^*(\zeta)$ is any solution of (82.3) for a given $\psi(0)$, and if $\varphi^*(0) = a_0 \neq 0$, then $\varphi^*(\zeta) - a_0$ is a solution of (82.3) with $\overline{\psi(0)}$ replaced by $\overline{\psi(0)} + a_0\alpha$.

Once a solution of (82.3) satisfying the condition $\varphi(0) = 0$ is obtained, the function $\psi(\zeta)$ can be calculated by Cauchy's Integral Formula from (82.1). The value of $\psi(\zeta)$ on γ, as determined by forming the conjugate of (82.1), is,

$$\psi(\sigma) = \overline{H(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma) - \alpha\overline{\varphi(\sigma)}.$$

If we multiply this by $\dfrac{1}{2\pi i}\dfrac{d\sigma}{\sigma - \zeta}$, integrate over γ, and note that

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi(\sigma)}}{\sigma - \zeta}\,d\sigma = \overline{\varphi(0)} = 0,$$

we get an explicit formula,

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{H(\sigma)}}{\sigma - \zeta}\,d\sigma - \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\varphi'(\sigma)}{\sigma - \zeta}\,d\sigma. \tag{82.4}$$

From considerations of the following section, where it is demonstrated that the solution of the functional equation (82.3) can be made to depend on the solution of the standard Fredholm integral equation, it follows that there exists a unique solution of Eq. (82.3), because a supplementary condition $\mathscr{I}[\varphi'(0)/\omega'(0)] = 0$ can be imposed in the first boundary-value problem for the finite domain.

We shall see in Sec. 84 how effective solutions of Eq. (82.3) can be deduced for a broad class of plane problems, without making reductions to integral equations.

83. Integral Equations for the Basic Problems. It is easy to reduce the solution of the integrodifferential equation (82.3) to the solution of

the standard Fredholm integral equation. The existence of a solution of Eq. (82.3) then would follow, almost directly, from the Fredholm theory. We outline briefly this reduction.

The equality

$$\frac{\omega(\zeta)}{2\pi i}\int_\gamma \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\frac{d\sigma}{\sigma-\zeta} = \frac{\overline{\varphi'(0)}}{\overline{\omega'(0)}}\,\omega(\zeta)$$

permits us to rewrite Eq. (82.3) in the form

$$\alpha\varphi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)-\omega(\zeta)}{\overline{\omega'(\sigma)}(\sigma-\zeta)}\overline{\varphi'(\sigma)}\,d\sigma + k\omega(\zeta) + \overline{\psi(0)} = A(\zeta), \tag{83.1}$$

where

$$k = \frac{\overline{\varphi'(0)}}{\overline{\omega'(0)}}. \tag{83.2}$$

We observe that when the domain is infinite, $k = 0$, since for such domains $\omega'(0) = \infty$. If the domain is finite, Eq. (83.1) can be reduced to the same form as for the infinite domain by setting

$$\varphi(\zeta) = -\frac{k}{\alpha}\,\omega(\zeta) + \varphi_0(\zeta), \tag{83.3}$$

where $\varphi_0(\zeta)$ is the new unknown function. On substituting (83.3) in (83.1) we readily find that

$$\alpha\varphi_0(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)-\omega(\zeta)}{\overline{\omega'(\sigma)}(\sigma-\zeta)}\overline{\varphi_0'(\sigma)}\,d\sigma + \overline{\psi(0)} = A(\zeta). \tag{83.4}$$

Differentiating both members of (83.4) with respect to ζ, and letting ζ tend to an arbitrary point t of γ, yields the Fredholm integral equation of the second kind,

$$\alpha\varphi_0'(t) + \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial t}\left[\frac{\omega(\sigma)-\omega(t)}{\sigma-t}\right]\frac{\overline{\varphi_0'(\sigma)}}{\overline{\omega'(\sigma)}}\,d\sigma = A'(t). \tag{83.5}$$

Since

$$\lim_{\sigma\to t}\frac{\omega(\sigma)-\omega(t)}{\sigma-t} = \omega'(t),$$

the kernel

$$K(\sigma, t) \equiv \frac{1}{\overline{\omega'(\sigma)}}\frac{\partial}{\partial t}\left[\frac{\omega(\sigma)-\omega(t)}{\sigma-t}\right]$$

is continuous for all σ and t in the closed circle γ (except for $\sigma = 0$, $t = 0$ in the case of the infinite domain) so long as the contour C is such that $\omega''(\zeta)$ is continuous in $|\zeta| \leq 1$.

Thus (83.5) is of the standard type.[1]

[1] By separating (83.5) into real and imaginary parts, this equation can be reduced to a pair of standard real equations, but such reduction is not necessary for our purposes

The existence of a continuous solution of (83.5) follows from the fact that the related homogeneous equation [in which $A'(t) \equiv 0$] can have no solution other than the trivial solution $\varphi_0'(t) \equiv 0$. For the homogeneous integral equation corresponds to the physical situation in which either the displacements or the stresses vanish on the boundary C, and the assumption that a nonvanishing solution exists in such cases violates the uniqueness theorem.

Let us suppose that by some means we have obtained a solution $\varphi_0'(t)$ of (83.5). Inserting it in the integral of (83.4), we obtain $\varphi_0(\zeta)$ and fix it so that $\varphi_0(0) = 0$ [see (82.3)]. We then construct $\varphi(\zeta)$, defined by (83.3), and choose k in accord with (83.2).

From (83.3),

$$\varphi'(0) = -\frac{k}{\alpha}\,\omega'(0) + \varphi_0'(0),$$

and hence

$$k = \frac{\overline{\varphi'(0)}}{\overline{\omega'(0)}} = -\frac{\bar{k}}{\alpha} + \frac{\overline{\varphi_0'(0)}}{\overline{\omega'(0)}},$$

and, therefore,

$$k + \frac{\bar{k}}{\alpha} = \frac{\overline{\varphi_0'(0)}}{\overline{\omega'(0)}}. \tag{83.6}$$

In the second boundary-value problem this equation completely determines k. In the first problem $\alpha = 1$, and (83.6) demands that

$$k + \bar{k} = \frac{\overline{\varphi_0'(0)}}{\overline{\omega'(0)}}$$

be real. But from (83.2)

$$k + \bar{k} = 2\Re\,\frac{\overline{\varphi'(0)}}{\overline{\omega'(0)}},$$

so that the first boundary-value problem will surely have a solution if $\mathcal{I}[\varphi'(0)/\omega'(0)] = 0$. This is the familiar condition we encountered previously.

When the domain is infinite, the mapping function has the structure

$$\omega(\zeta) = \frac{c}{\zeta} + \omega_0(\zeta), \tag{83.7}$$

where $\omega_0(\zeta)$ is analytic in $|\zeta| < 1$. It is easy to verify[1] that the function $\varphi^{0\prime}(\zeta)$, introduced in (76.11), satisfies the integral equation given in the problem at the end of this section. Thus the problems for the simply connected infinite domains differ in no essential particulars from the problems for finite domains. This fact has already been noted in Sec. 76.

[1] Detailed calculations will be found in N. I. Muskhelishvili's Some Basic Problems of the Mathematical Theory of Elasticity (1953), Sec. 79.

Integral equations of the type considered here have been thoroughly studied by Sherman,[1] who proved, among other things, that they can be solved by a method of successive approximations. Moreover, if the mapping function $\omega(\zeta)$ is rational, the kernel $K(\sigma, t)$ has the degenerate form

$$K(\sigma, t) = \sum_{k=1}^{n} a_k(t) b_k(\sigma),$$

and hence Eq. (83.5) is solvable in the closed form. This remarkable result, first established by Muskhelishvili, can also be deduced in the manner of Sec. 84, where two special forms of rational mapping functions are considered.

PROBLEM

Show that, if the domain is infinite and the mapping function has the form (83.7), the function $\varphi^{0\prime}(t)$ satisfies the equation

$$\alpha\varphi^{0\prime}(t) + \frac{1}{2\pi i}\int_\gamma K(\sigma, t)\varphi^{0\prime}(\sigma)\,d\sigma = A'(t),$$

where

$$K(\sigma, t) = \frac{1}{\overline{\omega'(\sigma)}}\frac{\partial}{\partial t}\left[\frac{\omega_0(\sigma) - \omega_0(t)}{\sigma - t}\right].$$

84. Solution of Integrodifferential Equations. The integration of Eq. (82.3) can be carried out in the closed form, and by quite elementary means, whenever $\omega(\zeta)$ is a rational function.[2]

We consider first the simplest case where $\omega(\zeta)$ is a polynomial

$$(84.1)\qquad \omega(\zeta) = \gamma_1\zeta + \gamma_2\zeta^2 + \cdots + \gamma_n\zeta^n, \qquad \gamma_1 \neq 0, \qquad \gamma_n \neq 0,$$

and recall[3] the notation

$$\bar{\omega}(\zeta) = \bar{\gamma}_1\zeta + \bar{\gamma}_2\zeta^2 + \cdots + \bar{\gamma}_n\zeta^n.$$

In this notation, the rational function

$$(84.2)\qquad \begin{aligned}\frac{\omega(\zeta)}{\bar{\omega}'(1/\zeta)} &= \frac{\gamma_1\zeta + \gamma_2\zeta^2 + \cdots + \gamma_n\zeta^n}{\bar{\gamma}_1 + 2\bar{\gamma}_2\zeta^{-1} + \cdots + n\bar{\gamma}_n\zeta^{-n+1}}\\ &= \zeta^n\frac{\gamma_1 + \gamma_2\zeta + \cdots + \gamma_n\zeta^{n-1}}{\bar{\gamma}_1\zeta^{n-1} + \cdots + n\bar{\gamma}_n}\end{aligned}$$

reduces to $\omega(\sigma)/\overline{\omega'(\sigma)}$ for $\zeta = \sigma$.

[1] D. I. Sherman, *Trudy Seismological Institute, Academy of Sciences of the USSR*, Nos. 82 and 83 (1938).

[2] N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity (1953), Sec. 85.

[3] Sec. 42.

Moreover, since $\omega'(\zeta) \neq 0$ for $|\zeta| \leq 1$, $\bar{\omega}'(1/\zeta) \neq 0$ for $|\zeta| \geq 1$, and thus (84.2) represents an analytic function for all $|\zeta| \geq 1$, except for $\zeta = \infty$, where it has a pole of order n. It follows, then, that for $|\zeta| \geq 1$,

$$\frac{\omega(\zeta)}{\bar{\omega}'(1/\zeta)} = c_n\zeta^n + \cdots + c_1\zeta + \sum_{k=0}^{\infty} c_{-k}\zeta^{-k}. \tag{84.3}$$

The fact that this expansion has a finite number of positive integral powers of ζ will enable us to evaluate the integral in (82.3) in finite terms.

Since

$$\varphi(\zeta) = a_1\zeta + a_2\zeta^2 + \cdots + a_n\zeta^n + \cdots, \qquad |\zeta| \leq 1,$$

$$\bar{\varphi}'\left(\frac{1}{\zeta}\right) = \bar{a}_1 + \frac{2\bar{a}_2}{\zeta} + \cdots + \frac{n\bar{a}_n}{\zeta^{n-1}} + \cdots, \qquad |\zeta| \geq 1, \tag{84.4}$$

and the product of the series (84.3) and (84.4) gives

$$\frac{\omega(\zeta)}{\bar{\omega}'(1/\zeta)}\bar{\varphi}'\left(\frac{1}{\zeta}\right) = K_n\zeta^n + K_{n-1}\zeta^{n-1} + \cdots + K_1\zeta + K_0 + \sum_{m=1}^{\infty} K_{-m}\zeta^{-m}, \tag{84.5}$$

where

$$\begin{cases} K_n = \bar{a}_1c_n, \\ K_{n-1} = \bar{a}_1c_{n-1} + 2\bar{a}_2c_n, \\ \cdots\cdots\cdots\cdots\cdots \\ K_2 = \bar{a}_1c_2 + 2\bar{a}_2c_3 + \cdots + (n-1)\bar{a}_{n-1}c_n, \\ K_1 = \bar{a}_1c_1 + 2\bar{a}_2c_2 + \cdots + n\bar{a}_nc_n. \end{cases} \tag{84.6}$$

We do not write out the expressions for K_{-m}, $m \geq 0$, because, as will be seen presently, they are not required in the calculation of $\varphi(\zeta)$.

It may be observed that Eqs. (84.6) contain only n coefficients c_i in the principal part of the Laurent expansion (84.3). The determination of the principal part calls for quite elementary algebraic computations.

If we now set $\zeta = \sigma$ in (84.5) and insert the result in (82.3), we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi'(\sigma)}}{\sigma - \zeta}\,d\sigma = \sum_{m=0}^{n} K_m\zeta^m,$$

since

$$\frac{1}{2\pi i}\int_\gamma \frac{\sum_{m=1}^{\infty} K_{-m}\sigma^{-m}}{\sigma - \zeta}\,d\sigma = 0.$$

Thus (82.3) yields an explicit formula for $\varphi(\zeta)$,

$$\alpha\varphi(\zeta) + \sum_{m=0}^{n} K_m\zeta^m + \overline{\psi'(0)} = \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma - \zeta}\,d\sigma, \tag{84.7}$$

as yet containing the unknown constants K_m. These, however, can be easily determined. Since

$$\frac{1}{\sigma - \zeta} = \frac{1}{\sigma} + \frac{\zeta}{\sigma^2} + \cdots + \frac{\zeta^n}{\sigma^{n+1}} + \cdots,$$

(84.8)
$$\frac{1}{2\pi i} \int_\gamma \frac{H(\sigma)}{\sigma - \zeta} d\sigma = \sum_{k=0}^{\infty} C_k \zeta^k,$$

where

$$C_k \equiv \frac{1}{2\pi i} \int_\gamma H(\sigma) \sigma^{-k-1} \, d\sigma, \qquad k = 0, 1, \ldots .$$

The substitution of the power series (84.8) in (84.7) and the comparison of coefficients of like powers of ζ then give,

(84.9)
$$\begin{cases} \overline{\psi(0)} + K_0 = C_0, & \\ \alpha a_m + K_m = C_m, & m = 1, 2, \ldots, n, \\ \alpha a_m = C_m, & m > n. \end{cases}$$

The first of these equations determines the value of $\overline{\psi(0)} + K_0$. If the values of the K_m from (84.6) are inserted in the n equations in the second line, one obtains the system

(84.10)
$$\begin{cases} \alpha a_1 + \bar{a}_1 c_1 + \cdots + n \bar{a}_n c_n = C_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha a_n + \bar{a}_1 c_n \qquad\qquad\qquad = C_n, \end{cases}$$

identical with the system (76.9) deduced previously. The solution of this system for the $a_1, \ldots, a_n$ completely determines the values of $K_1, \ldots, K_n$ in (84.7). The value of $\overline{\psi(0)} + K_0$, as already noted, is determined by the first equation in (84.9). Thus, $\varphi(\zeta)$ is fully determined by formula (84.7). To obtain $\psi(\zeta)$, we make use of (82.4).

Now, referring to (84.5), we can write

$$\frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)} \varphi'(\zeta) = \bar{K}_n \frac{1}{\zeta^n} + \cdots + \frac{\bar{K}_1}{\zeta} + \sum_{m=0}^{\infty} \bar{K}_{-m} \zeta^m, \qquad |\zeta| \leq 1,$$

and hence

$$\frac{1}{2\pi i} \int_\gamma \frac{\bar{\omega}(1/\sigma)}{\omega'(\sigma)} \frac{\varphi'(\sigma)}{\sigma - \zeta} d\sigma = \sum_{m=0}^{\infty} \bar{K}_m \zeta^m \equiv \frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)} \varphi'(\zeta) - \sum_{m=1}^{n} \bar{K}_{-m} \zeta^{-m}.$$

Inserting this result in (82.4), we get finally,

(84.11)
$$\psi(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{\overline{H(\sigma)}}{\sigma - \zeta} d\sigma - \frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)} \varphi'(\zeta) + \sum^{n} \bar{K}_m \zeta^{-m}.$$

If the domain is infinite and the mapping function has the form

$$\omega(\zeta) = \frac{c}{\zeta} + \sum_{k=0}^{n} \gamma_k \zeta^k, \tag{84.12}$$

we first reduce the problem to the calculation of $\varphi_0(\zeta)$ satisfying (83.4).

From the structure of $\omega(\zeta)$ we conclude that

$$\frac{\omega(\zeta)}{\bar{\omega}'(1/\zeta)} = c_{n-2}\zeta^{n-2} + \cdots + c_1\zeta + \sum_{n=0}^{\infty} c_{-n}\zeta^{-n},$$

and for $|\zeta| = 1$ we find

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\varphi_0'(\sigma)} = K_{n-2}^{(0)}\sigma^{n-2} + \cdots + K_1^{(0)}\sigma + K_0^{(0)} + \sum_{m=1}^{\infty} K_{-m}^{(0)}\sigma^{-m},$$

where

$$\begin{cases} K_{n-2}^{(0)} = \bar{a}_1 c_{n-2}, \\ K_{n-3}^{(0)} = \bar{a}_1 c_{n-3} + 2\bar{a}_2 c_{n-2}, \\ \dots\dots\dots\dots\dots\dots \\ K_1^{(0)} = \bar{a}_1 c_1 + 2\bar{a}_2 c_2 + \cdots + (n-2)\bar{a}_{n-2} c_{n-2}, \end{cases} \tag{84.13}$$

Evaluating the integral in (83.4), we get

$$\frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma) - \omega(\zeta)}{\overline{\omega'(\sigma)}(\sigma - \zeta)} \overline{\varphi_0'(\sigma)}\, d\sigma = \sum_{m=0}^{n-2} K_m^{(0)}\zeta^m.$$

so that $\varphi_0(\zeta)$ is determined by

$$\alpha\varphi_0(\zeta) + \sum_{m=0}^{n-2} K_m^{(0)}\zeta^m + \overline{\psi(0)} = \frac{1}{2\pi i}\int_\gamma \frac{F_0(\sigma)}{\sigma - \zeta}\, d\sigma. \tag{84.14}$$

As in the case of the finite domain,

$$K_0^{(0)} + \overline{\psi(0)} = \frac{1}{2\pi i}\int_\gamma \frac{F_0(\sigma)}{\sigma}\, d\sigma = \frac{1}{2\pi}\int_0^{2\pi} F_0(\sigma)\, d\vartheta$$

and the values of the remaining $K_m^{(0)}$ are determined from (84.13) upon solving for the a_m the system of $n - 2$ equations

$$\begin{cases} \alpha a_1 + \bar{a}_1 c_1 + 2\bar{a}_2 c_2 + \cdots + (n-2)\bar{a}_{n-2} c_{n-2} = A_1^0, \\ \alpha a_2 + \bar{a}_1 c_2 + 2\bar{a}_2 c_3 + \cdots + (n-3)\bar{a}_{n-3} c_{n-2} = A_2^0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha a_{n-2} + \bar{a}_1 c_{n-2} \qquad\qquad = A_{n-2}^0, \end{cases} \tag{84.15}$$

with

$$A_k^0 = \frac{1}{2\pi}\int_0^{2\pi} F_0(\sigma) e^{-ik\vartheta}\, d\vartheta, \qquad k = 1, \ldots, n-2.$$

The ideas leading to the calculation of $\varphi(\zeta)$ [or $\varphi_0(\zeta)$], in principle, are identical to those of Sec. 76, but generally formulas given in this section enable one to compute $\varphi(\zeta)$ and $\psi(\zeta)$ with less effort.

Similar results can be obtained by considering $\omega(\zeta)$ in the form of the quotient of two polynomials.

85. Illustrative Examples. For comparison purposes we apply the formulas of Sec. 84 to problems which have already been solved by the series method in Secs. 77 to 79.

When the region is a circle of radius R, the mapping function is

$$z = \omega(\zeta) = R\zeta,$$

and the expansion (84.3) reduces to

$$\frac{\omega(\zeta)}{\bar{\omega}'(1/\zeta)} = \zeta.$$

Thus all c_k, with the exception of $c_1 = 1$, vanish, and it follows from (84.6) that

$$K_1 = \bar{a}_1, \qquad K_m = 0, \qquad m = 2, \ldots, n.$$

Accordingly, formula (84.7) gives

$$\alpha\varphi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma - \zeta}\,d\sigma - K_1\zeta - [K_0 + \overline{\psi(0)}]. \tag{85.1}$$

From the first of Eqs. (84.9)

$$K_0 + \overline{\psi(0)} = C_0 \equiv \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma}\,d\sigma, \tag{85.2}$$

and from the second of Eqs. (84.9), with $m = 1$,

$$\alpha a_1 + K_1 = C_1 \equiv \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma^2}\,d\sigma. \tag{85.3}$$

The value of a_1, as follows from (84.10), is determined by

$$\alpha a_1 + \bar{a}_1 = C_1, \tag{85.4}$$

inasmuch as $c_1 = 1$, $c_k = 0$, $k > 1$.

In the first boundary-value problem $\alpha = 1$, and the imaginary part of a_1 can be set equal to zero. Equation (85.4) then yields

$$a_1 + \bar{a}_1 = 2a_1 = C_1$$

and from (85.3), with $\alpha = 1$,

$$K_1 = \frac{C_1}{2} = \frac{1}{4\pi i}\int_\gamma \frac{H(\sigma)}{\sigma^2}\,d\sigma.$$

The substitution in (85.1) then yields $\varphi(\zeta)$, for the first boundary-value problem, in the form

$$\varphi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma - \zeta}\,d\sigma - \frac{\zeta}{4\pi i}\int_\gamma \frac{H(\sigma)}{\sigma^2}\,d\sigma - \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)}{\sigma}\,d\sigma. \tag{85.5}$$

The corresponding function $\psi(\zeta)$ from formula (84.11) is determined by[1]

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{H(\sigma)}}{\sigma - \zeta}\,d\sigma - \frac{\varphi'(\zeta)}{\zeta} + \frac{1}{4\pi i\zeta}\int_\gamma \frac{H(\sigma)}{\sigma^2}\,d\sigma. \tag{85.6}$$

In the second boundary-value problem $\alpha = -\varkappa$, and a_1 is completely determined by (85.4). We leave it to the reader to write out the appropriate solutions in the form analogous to (85.5) and (85.6).

Instead of using formulas of Sec. 84, it is frequently simpler to determine the function $\varphi(\zeta)$ directly from Eq. (82.3). We illustrate this by solving the problem of Sec. 79 for the infinite region bounded by an ellipse.

Inasmuch as the boundary condition (79.3) is identical in structure with (82.1), the integrodifferential equation for $\varphi^0(\zeta)$ is

$$\varphi^0(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\varphi^{0\prime}(\sigma)}}{\sigma - \zeta}\,d\sigma + \overline{\psi^0(0)} = \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma - \zeta}\,d\sigma. \tag{85.7}$$

If we insert

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{1 + m\sigma^2}{\sigma(m - \sigma^2)}$$

from (79.4) in (85.7) we get

$$\varphi^0(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{1 + m\sigma^2}{\sigma(m - \sigma^2)}\frac{\overline{\varphi^{0\prime}(\sigma)}}{\sigma - \zeta}\,d\sigma + \overline{\psi^0(0)} = \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma - \zeta}\,d\sigma. \tag{85.8}$$

Since,

$$\frac{1 + m\sigma^2}{\sigma(m - \sigma^2)}\,\overline{\varphi^{0\prime}(\sigma)} = \sum_{n=1}^{\infty} \alpha_n\sigma^{-n}, \qquad |\sigma| \geq 1,$$

the value of the integral in (85.8) is zero.[2]

Thus,

$$\varphi^0(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma - \zeta}\,d\sigma - \overline{\psi^0(0)}.$$

But $\varphi^0(0) = 0$, so that

$$\overline{\psi^0(0)} = \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma}\,d\sigma,$$

[1] Since $a_1 = \bar{a}_1$, $K_1 = \bar{K}_1$.

[2] Note (79.4), and recall that the expansion for $\overline{\varphi^{0\prime}(\sigma)}$ contains no positive powers of σ.

and, hence,

$$\varphi^0(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma - \zeta}\,d\sigma - \frac{1}{2\pi i}\int_\gamma \frac{F^0(\sigma)}{\sigma}\,d\sigma. \tag{85.9}$$

The calculation of

$$\psi^0(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F^0(\sigma)}}{\sigma - \zeta}\,d\sigma + \frac{\zeta(\zeta^2 + m)}{1 - m\zeta^2}\,\overline{\varphi^{0\prime}(\zeta)} \tag{85.10}$$

was carried out in detail in Sec. 79.

For $m = 0$ these formulas yield the solution of the first boundary-value problem for the region exterior to the circle $|\zeta| = 1$.

The reader may find it instructive to solve these problems by determining the function $\varphi_0'(t)$ from the integral equation (83.5) and by following the argument of Sec. 83.

For either of the mapping functions considered in this section the integral in (83.5) vanishes, so that

$$\varphi_0'(t) = \frac{1}{\alpha}\,A'(t).$$

The substitution in (83.4) then yields at once

$$\alpha\varphi_0(\zeta) = A(\zeta) + \beta,$$

where β is a constant. This constant and the constant k in (83.3) can be easily determined by making use of (83.6) and recalling that $\varphi(0) = 0$.

86. Further Developments. Multiply Connected Domains. The methods of solution of plane problems considered thus far depend vitally on the knowledge of the mapping function. Since only simply connected domains can be mapped conformally on a circle in a one-to-one manner, the considerations of Secs. 82 and 83 do not apply to multiply connected domains. However, there is a simple connection between the mapping function $\omega(\zeta)$ and Green's function for the domain.[1]

Thus the integral equation (83.5) can always be written in the form whose kernel is expressed in terms of Green's function. Since Green's function can be constructed for multiply connected domains, this at once suggests a generalization of the integral equation. One such generalization has been made by Mikhlin, who reduced the basic problems of plane elasticity in multiply connected domains to the solution of certain Fredholm integral equations whose kernels depend on Green's functions.[2] Although Mikhlin's equations serve admirably to establish the existence of solutions in multiply connected domains, they possess the disadvan-

[1] If one writes the mapping function in the form $\zeta = f(z)$ and makes the point $z = z_0$ of the region R correspond to the center of the circle $|\zeta| = 1$, then Green's function $G(P, P_0) = (1/2\pi)\log(1/|f(z)|)$, with the pole P_0 at the point z_0.

[2] A connected account of this work is contained in a monograph by S. G. Mikhlin, entitled Integral Equations (1949) (in Russian).

tage of being dependent on the solution of an auxiliary Dirichlet's problem for Green's function. It is clearly desirable to formulate the relevant equations so that they depend only on the assigned boundary values. The fact that this can be done was demonstrated by Lauricella[1] in a rather involved paper concerned with the integration of the equilibrium equations for the clamped elastic plate. This particular problem, as we have already observed in Sec. 69, is closely related to the first boundary-value problem in plane elasticity. The Lauricella equations have been rediscovered by Sherman,[2] who deduced them in a very simple way and made use of them in solving the standard boundary-value problems, and certain important new types, in plane elasticity.

A detailed account of Sherman's work would consume more space than we have at our disposal, and we give only a sketch of the essential ideas. We recall that in a finite simply connected domain $\varphi(z)$ and $\psi(z)$ are analytic in the interior, and on the boundary C they satisfy the condition,

$$\alpha\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \qquad \text{on } C. \tag{86.1}$$

Sherman seeks to represent $\varphi(z)$ and $\psi(z)$ by the following integrals of Cauchy's type:[3]

$$\left\{\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int_C \frac{w(s)}{s-z}\,ds, \\ \psi(z) &= \frac{\alpha}{2\pi i}\int_C \frac{\overline{w(s)}}{s-z}\,ds - \frac{1}{2\pi i}\int_C \frac{\bar{s}w'(s)}{s-z}\,ds, \end{aligned}\right. \tag{86.2}$$

where $w(s)$ is an unknown density function whose derivative satisfies Hölder's condition on C.

The choice of $w(s)$ is restricted by the boundary condition (86.1). We proceed to determine the nature of this restriction by substituting from (86.2) in (86.1).

We first note that

$$\varphi'(z) = \frac{1}{2\pi i}\int_C \frac{w(s)}{(s-z)^2}\,ds,$$

[1] G. Lauricella, *Acta Mathematica*, vol. 32 (1909), pp. 201–256.

[2] D. I. Sherman, *Doklady Akademii Nauk SSSR*, vol. 27 (1940), pp. 911–913, vol. 28 (1940), pp. 25–32, vol. 32 (1941), pp. 314–315; *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 7 (1943), pp. 301–309, 413–420, vol. 17 (1953), pp. 685–692. Equations whose appearance is strikingly similar to the Sherman-Lauricella equations have also been deduced by N. I. Muskhelishvili, *Doklady Akademii Nauk SSSR*, vol. 3 (1934), pp. 7 and 73. However, the content of Muskhelishvili's equations is quite different, and they appear to be less susceptible of extensions to the new types of elastostatic problems.

[3] These forms are suggested by the known solution of the equilibrium problem for the semi-infinite plane. See also formula (82.4). Integrals of Cauchy's type were introduced in Sec. 40.

which is not an integral of the Cauchy type. But on integration by parts we get

$$\varphi'(z) = \frac{1}{2\pi i}\int_C \frac{w'(s)}{s - z}\,ds. \tag{86.3}$$

If we now let z in (86.2) and (86.3) approach an arbitrary point t of C, we get from Plemelj's formulas (40.7),

$$\begin{cases} \varphi(t) = \dfrac{w(t)}{2} + \dfrac{1}{2\pi i}\displaystyle\int_C \frac{w(s)}{s-t}\,ds, \\ \psi(t) = \dfrac{\alpha\overline{w(t)}}{2} + \dfrac{\alpha}{2\pi i}\displaystyle\int_C \frac{\overline{w(s)}}{s-t}\,ds - \dfrac{\bar t w'(t)}{2} - \dfrac{1}{2\pi i}\displaystyle\int_C \frac{\bar s w'(s)}{s-t}\,ds, \\ \varphi'(t) = \dfrac{w'(t)}{2} + \dfrac{1}{2\pi i}\displaystyle\int_C \frac{w'(s)}{s-t}\,ds. \end{cases} \tag{86.4}$$

The substitution from (86.4) in (86.1) leads to the integrodifferential equation,

$$\alpha w(t) + \frac{\alpha}{2\pi i}\int_C w(s)d\left(\log\frac{s-t}{\bar s - \bar t}\right) + \frac{1}{2\pi i}\int_C \overline{w'(s)}\left(\frac{s-t}{\bar s - \bar t}\right)d\bar s = f(t).$$

This equation, on integrating by parts the second integral in the left hand member, yields the desired integral equation,

$$\alpha w(t) + \frac{\alpha}{2\pi i}\int_C w(s)d\left(\log\frac{s-t}{\bar s - \bar t}\right) - \frac{1}{2\pi i}\int_C \overline{w(s)}d\,\frac{s-t}{\bar s - \bar t} = f(t). \tag{86.5}$$

If we set

$$s - t = re^{i\theta}$$

we get the equation

$$\alpha w(t) + \frac{1}{\pi}\int_C [\alpha w(s) - \overline{w(s)}e^{2i\theta}]\,d\theta = f(t). \tag{86.6}$$

It is easy to check that on writing $w(s) = p(s) + iq(s)$, where p and q are real functions, Eq. (86.6) is equivalent to two real equations,

$$\begin{cases} \alpha p(t) + \dfrac{1}{\pi}\displaystyle\int_C [p(s)(\alpha - \cos 2\theta) - q(s)\sin 2\theta]\,d\theta = f_1(t), \\ \alpha q(t) - \dfrac{1}{\pi}\displaystyle\int_C [p(s)\sin 2\theta - q(s)(\alpha + \cos 2\theta)]\,d\theta = f_2(t), \end{cases} \tag{86.7}$$

where $f_1 + if_2 = f$.

The simultaneous integral equations (86.7) are of the Fredholm type, and by a familiar device they can be reduced to a single real Fredholm's equation.

It is not necessary to write out this equation since it is essential to know only that such reduction is feasible. Equations (86.7) are sufficiently simple to permit numerical solutions.[1]

In a multiply connected domain, the boundary condition of the form (86.1) must be satisfied on the contour $C = C_1 + C_2 + \cdots + C_{m+1}$, and m unknown constants of integration will appear in the boundary conditions. If one should attempt to represent $\varphi(z)$ and $\psi(z)$ in the form[2] (86.2), the unknown integration constants would enter in the integral equation for $w(t)$. To avoid this, Sherman modifies the formulas (86.2) by adding to their right-hand members the sums $\sum_{j=1}^{m} \dfrac{b_j}{z - z_j}$, where the z_j lie within the interior contours C_j. The constants b_j are then defined so that the resulting equation for $w(t)$ is free of unknown constants.[3]

As a simple illustration of the use of Eq. (86.6) consider the determination of $\varphi(z)$ and $\psi(z)$ for the problem of the solid disk of radius R compressed by a uniform pressure P on its boundary.

[1] For example, if we take n points $s_1, s_2, \ldots, s_n$ on the boundary C and apply some formula of mechanical quadratures to the integrals in (86.7), we get a system of $2n$ algebraic equations in $2n$ unknowns,

$$\alpha p_i + \frac{1}{\pi} \sum_{j=1}^{n} [p_j(\alpha - \cos 2\theta_{ij}) - q_j \sin 2\theta_{ij}]\, \Delta\theta_{ij} = f_1(t_i) + \epsilon_1(t_i),$$

$$\alpha q_i - \frac{1}{\pi} \sum_{j=1}^{n} [p_j \sin 2\theta_{ij} - q_j(\alpha + \cos 2\theta_{ij})]\, \Delta\theta_{ij} = f_2(t_i) + \epsilon_2(t_i),$$

where the ϵ_α are the errors made in the process of replacing integrals by finite terms. The solution of this system would enable us to compute $p(t)$ and $q(t)$ approximately.

[2] The functions $\varphi(z)$ and $\psi(z)$ can be regarded as single-valued in all cases, since the multiple-valued terms in (72.7) can be incorporated in $f(t)$.

[3] An account of this is contained in the first two of Sherman's *Doklady* papers, cited on p. 314. These papers are reproduced practically without change in Muskhelishvili's book Some Basic Problems of the Mathematical Theory of Elasticity (1953), pp. 412–420.

An illustration of the use of Eq. (86.5) in the solution of the first boundary-value problem for the interior of the region bounded by an ellipse is contained in D. I. Sherman's paper in *Doklady Akademii Nauk SSSR*, vol. 31 (1941), pp. 309–310, and in S. G. Mikhlin, Integral Equations (1949), pp. 292–294. For applications of the Sherman method to doubly and triply connected domains see D. I. Sherman, "On the Stresses in a Heavy Half-plane Weakened by Two Circular Openings," *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 15 (1951), pp. 297–316, 751–762; "On the State of Stress in Some Shrink-fitted Members," *Izvestiya Akademii Nauk SSSR*, Technical Series (1948), pp. 1371–1388. See also M. F. Gur'ev, "Distribution of Stresses in a Stretched Isotropic Rectangular Plate Weakened by a Circular Hole," *Dopovidi Akademii Nauk Ukraïn'skoï RSR* (1953), pp. 133–139.

In this case, as shown in Sec. 77*a*, $f(t) = -Pt$, so that we seek the solution of

$$(86.8) \qquad w(t) + \frac{1}{\pi}\int_C [w(s) - \overline{w(s)}e^{2i\varphi}]\, d\varphi = -Pt.$$

Since $s - t = re^{i\varphi}$, we see from Fig. 57 that

$$r = 2R \sin \tfrac{1}{2}(\theta - \theta_0),$$

and hence

$$e^{i\varphi} = \frac{e^{i\theta_0}(e^{i(\theta-\theta_0)} - 1)}{2 \sin \frac{1}{2}(\theta - \theta_0)}.$$

Thus, as the point P describes the contour C, φ varies between the limits

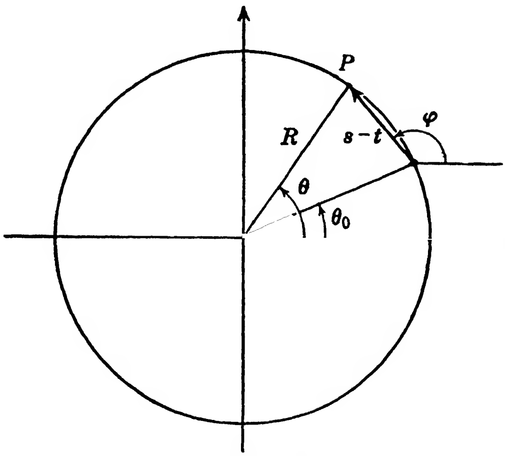

FIG. 57

$\theta_0 + \pi/2$ and $\theta_0 + 3\pi/2$. The structure of the right-hand member in (86.8) suggests that we seek the solution in the form $w(t) = at + b$, where a and b are constants.

The substitution of the assumed solution in (86.8) yields [since $w(s) = w(t + re^{i\varphi}) = at + are^{i\varphi} + b$]

$$at + b + \frac{1}{\pi}\int_{\theta_0+(\pi/2)}^{\theta_0+(3\pi/2)} (at + are^{i\varphi} + b - \bar{a}\bar{t}e^{2i\varphi} - \bar{b}e^{2i\varphi} - \bar{a}re^{i\varphi})\, d\varphi = -Pt,$$

and, on integration, we readily find that $a = -P/2$, $b = 0$. Thus

$$w(s) = -\frac{Ps}{2},$$

and substituting this in (86.2) yields at once

$$\varphi(z) = -\frac{Pz}{2}, \qquad \psi(z) = 0.$$

These agree with the values found in Sec. 77.

87. Schwarz's Alternating Method. Since elastostatic problems in multiply connected domains present serious computational difficulties, it is natural to attempt to reduce their solution to a sequence of problems in simply connected domains. This can be done by making rather obvious modifications in Schwarz's treatment of the Dirichlet problem for the overlapping domains.[1] We first sketch the essence of the method and then show how the solution of elastostatic problems for multiply connected domains can be made to hinge on the solution of the familiar problems in simply connected domains.

Consider a region (Fig. 58) formed by the overlapping domains R_1 and R_2 each of which is bounded by a simple closed contour. Let the portion of the contour C_1 bounding R_1 that lies within the region R_2 be C_1' and the part that is outside R_2 be C_1''. Then $C_1 = C_1' + C_1''$. Similarly, denote the part of the boundary C_2 of R_2 that is interior to R_1 by C_2' and the remaining part by C_2''. The region R_{12} that is common to R_1 and R_2 is thus bounded by C_1' and C_2', while the region $R_1 + R_2$ has the curve $C_1'' + C_2''$ for its boundary.

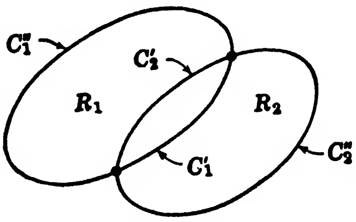

FIG. 58

We shall suppose that the values of some function φ, specified on the boundary $C_1'' + C_2''$, determine φ in the region $R_1 + R_2$ and that φ satisfies in this region the functional equation $L(\varphi) = 0$, where the operator L is linear. In the classical Dirichlet problem, L is the Laplace operator $\nabla^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$; in other problems, $L(\varphi) = 0$ might denote an integral or integrodifferential equation such as we have encountered in preceding sections.

An algorithm for the solution of the boundary-value problem

$$(87.1)\qquad \begin{cases} L(\varphi) = 0 & \text{in } R_1 + R_2, \\ \varphi = F(s) & \text{on } C_1'' + C_2'', \end{cases}$$

from the solutions of the corresponding boundary-value problems for the regions R_1 and R_2, can be constructed as follows:

Determine in the region R_1 the function u_1, which satisfies the equation $L(\varphi) = 0$ and which is such that

$$\begin{aligned} u_1 &= F(s) \qquad \text{on } C_1'', \\ &= f(s) \qquad \text{on } C_1', \end{aligned}$$

where $f(s)$ is assigned arbitrarily on C_1'. Having determined u_1, construct in the region R_2 the function v_1 which satisfies the equation

[1] H. A. Schwarz, *Gesammelte mathematische Abhandlungen*, vol. 2, pp. 133–143.

$L(\varphi) = 0$ and which assumes on C_2 the values

$$\begin{aligned} v_1 &= F(s) \qquad &&\text{on } C_2'' \\ &= u_1(s) &&\text{on } C_2'. \end{aligned}$$

Next determine the solution u_2 of $L(\varphi) = 0$ in R_1 such that

$$\begin{aligned} u_2 &= F(s) \qquad &&\text{on } C_1'', \\ &= v_1(s) &&\text{on } C_1', \end{aligned}$$

and then obtain the solution v_2 of $L(\varphi) = 0$ in R_2, subject to the condition

$$\begin{aligned} v_2 &= F(s) \qquad &&\text{on } C_2'', \\ &= u_2(s) &&\text{on } C_2'. \end{aligned}$$

The successive applications of this alternating procedure would yield two sequences of functions, $\{u_n\}$ in R_1 and $\{v_n\}$ in R_2, such that

$$\begin{aligned} u_n &= F(s) \qquad &&\text{on } C_1'', \\ &= v_{n-1}(s) &&\text{on } C_1', \end{aligned}$$

and

$$\begin{aligned} v_n &= F(s) \qquad &&\text{on } C_2'' \\ &= u_n(s) &&\text{on } C_2'. \end{aligned}$$

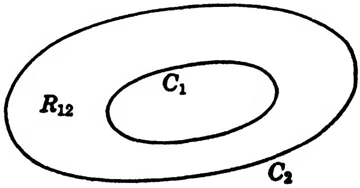

FIG. 59

If the solutions of the equation $L(\varphi) = 0$ possess suitable properties, the sequences $\{u_n\}$ and $\{v_n\}$ may converge to u and v, respectively, with $u \equiv v$ in the common region R_{12}. Also, on the boundary $C_1'' + C_2''$ of $R_1 + R_2$ the functions u and v assume specified values $F(s)$, and if they also satisfy the equation $L(\varphi) = 0$, our problem (87.1) is solved.

Whether this formal process would yield the desired solution or not clearly depends on the properties of the operator L and on the nature of assigned boundary values. If L is the Laplace operator and $F(s)$ is a continuous function defined on a sufficiently smooth boundary of the region, this process actually yields the solution of the Dirichlet problem.[1]

It was observed by Neumann that the Schwarz method can be modified to yield the solution of the Dirichlet problem for the domain R_{12} formed by the intersection of R_1 and R_2, and hence for the doubly connected domain. For the region R_{12} can be considered as the intersection of the infinite region R_1 bounded by C_1 with the finite region R_2 interior to C_2 (Fig. 59).

We indicate next how the alternating method of Schwarz can be made to yield the solution of the basic problems of elasticity for doubly connected domains.

We define the operator L by the formula

$$L(\varphi, \psi) \equiv \alpha\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}$$

[1] The proof of this is contained in many books. See, for example, É. Goursat, Cours d'analyse, 5th ed. (1942), vol. 3, pp. 207–210.

and write the boundary condition on $C_1 + C_2$ as

(87.2) $$L[\varphi(t), \psi(t)] \equiv \alpha\varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)} = f(t).$$

To obtain the first approximation $(\varphi^{(1)}, \psi^{(1)})$ to (φ, ψ), we determine the functions $\varphi^{(1)}$, $\psi^{(1)}$ in R_1 so that

$$L(\varphi^{(1)}, \psi^{(1)}) \Big|_{C_1} = f \Big|_{C_1}.$$

To get the second approximation $(\varphi^{(2)}, \psi^{(2)})$, we consider the solution in R_2 such that

$$L(\varphi^{(2)}, \psi^{(2)}) \Big|_{C_2} = f \Big|_{C_2} - L(\varphi^{(1)}, \psi^{(1)}) \Big|_{C_2}.$$

For the third approximation, we determine in R_1 the solution satisfying the condition

$$L(\varphi^{(3)}, \psi^{(3)}) = f \Big|_{C_1} - L(\varphi^{(2)}, \psi^{(2)}) \Big|_{C_1},$$

and so on.

The use of this procedure in constructing the approximate solutions of special elastostatic problems in doubly connected domains is presented in detail in Sec. 88.

The proof of convergence of the Schwarz algorithm in the solution of the second elastostatic boundary value problems for a doubly connected domain R_{12} (Fig. 59) for the case when the contours C_1 and C_2 bounding this domain are sufficiently far apart has been supplied by Mikhlin.[1] In essence Mikhlin's proof is based on Neumann's modification[2] of the Schwarz procedure for solving the Dirichlet problem in Laplace's equation for the domain $R_1 + R_2$.

A more general proof of the Schwarz algorithm for the second boundary value problem of elasticity in three dimensions was sketched out by Soboleff.[3] This proof reduces the consideration of convergence of sequences of approximate solutions for the sum $R_1 + R_2$ of the overlapping domains R_1 and R_2, and for their product domain R_{12}, to a study

[1] S. G. Mikhlin, *Trudy, Seismological Institute of the Academy of Sciences, USSR*, vol. 39 (1934), pp. 1–14.

[2] C. Neumann, *Leipziger Berichte*, vol. 22 (1870), pp. 264–321. A detailed and careful presentation of the Schwarz-Neumann method of solution of the Dirichlet problem for a class of elliptic partial differential equations in two dimensions and in solving certain systems of integral equations will be found in L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis, 4th ed. (1952), pp. 637–695. The treatment given in this book is also applicable to three-dimensional problems.

[3] S. Soboleff, "L'algorithme de Schwarz dans la théorie de l'élasticité," *Comptes Rendus (Doklady) de L'Academie des Sciences de l'URSS*, vol. IV (XIII), No. 6 (1936), pp. 243–246.

of convergence of sequences that minimize the integral for the strain energy.[1] We indicate briefly this connection because of its bearing on the methods of solution of the boundary value problems in elasticity developed in Chap. 7.

As in the beginning paragraphs of this section, we denote by C_1 the boundary of a given simply connected (two- or three-dimensional) domain R_1 and by C_2 the boundary of another such domain R_2 which intersects R_1. The part of the boundary C_1 interior to R_2 is denoted by C_1' and the part exterior to R_2 is C_1''. Similarly, the part of C_2 interior to R_1 is C_2' and the part exterior to R_1 is C_2'' (Fig. 58).

The determination of displacements u_i in the interior of $R_1 + R_2$ from specified displacements on its boundary $C_1'' + C_2''$ requires the solution of Navier's equations

$$L(u_i) \equiv \mu\nabla^2 u_i + (\lambda + \mu)u_{k,ki} = 0, \quad (i = 1,2,3), \tag{87.3}$$

subject to

$$\begin{cases} u_i \Big|_{C_1''} = F_i, \\ u_i \Big|_{C_2''} = G_i, \end{cases} \tag{87.4}$$

where the F_i and G_i are assigned on the boundary of $R_1 + R_2$.

The corresponding determination of displacements in the product domain R_{12} calls for the solution of Eq. (87.3) satisfying the conditions

$$\begin{aligned} u_i \Big|_{C_1'} &= F_i, \\ u_i \Big|_{C_2'} &= G_i. \end{aligned} \tag{87.5}$$

We shall see in Sec. 107 that the solution of the boundary value problem (87.3), (87.4) is equivalent to obtaining the vector u_i which minimizes the energy integral

$$U(u_i) = \int_{R_1+R_2} \left[(\lambda + \mu)(u_{k,k})^2 + \frac{\mu}{2}(u_{i,j} + u_{j,i}) \right] d\tau, \tag{87.6}$$

on the set of all continuously differentiable vectors u_i taking on the boundary $C_1'' + C_2''$ the values (87.4). If one is concerned with the problem (87.3), (87.5), the integral (87.6) is minimized on the set of u_i's satisfying on the boundary of R_{12} the conditions (87.5). Soboleff constructs suitable minimizing sequences $\{u_i^{(k)}\}$, $k = 0,1,2, \ldots$ for these problems (in a manner suggested at top of page 319) and shows that they converge in the mean to the desired displacements u_i. The difficult question of the rapidity of convergence of approximating sequences has

[1] See Sec. 107.

not yet been investigated. It is known, however, that if $L(u) = \nabla^2 u$, then the convergence is not slower than that of a geometric progression.[1]

88. Applications of the Alternating Method. As an illustration of the use of the Schwarz alternating method in deducing approximate solutions of the equilibrium problems in multiply connected domains, we consider two examples.

a. Eccentric Ring under Uniform Pressure. Consider the region in Fig. 60 bounded by the circles $|z| = R$ and $|z - a| = r$, where a is the distance between their centers and the circle C_1, of radius r, lies within the circle C_0 of radius R. We shall suppose that the boundary C_0 is subjected to a uniform pressure p and the interior boundary C_1 is free of stress.

Then[2] the problem reduces to the determination of two functions $\varphi(z)$ and $\psi(z)$ analytic in the ring bounded by C_0 and C_1 from the boundary conditions:

$$(88.1) \qquad \begin{aligned} \varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)} &= \text{const} && \text{on } C_1, \\ &= -pt && \text{on } C_0. \end{aligned}$$

The foregoing resumé relates to a proof of convergence to the desired solution of the minimizing sequence constructed in accordance with the Schwarz algorithm. However, it suggests no specific method for effective construction of the elements of the sequence. The construction of the set of functions $(\varphi^{(i)}, \psi^{(i)})$ can be made to depend on techniques depending on the use of integrals of Cauchy's type, or on closely related procedures involving the determination of solutions of appropriate integral equations. Exact solutions would require, of course, the determination of the limits of sequences of approximating functions, but useful approximate solutions can be got by terminating calculations after a finite number of steps. This is indicated in the following section where two particular problems for doubly connected domains are solved approximately by the Schwarz alternating method.[3]

[1] See L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis, 4th ed. (1952), p. 675 or É. Goursat, Cours d'analyse, 5th ed. (1942), vol. 3, p. 209.

[2] See Sec. 77*a*.

[3] An example of highly effective use of the alternating method in solving the Dirichlet problem for Laplace's equation for the sum of two rectangular regions forming an L-shaped polygon is given on pp. 683–695 of the L. V. Kantorovich and V. I. Krylov monograph cited in the preceding footnote. This example contains detailed calculations and tables which include estimates of errors in successive approximations. On pp. 657–682 and 679–683 of this monograph the authors discuss a reduction of the Dirichlet and Neumann problems for the second-order elliptic partial differential equation to the solution of integral equations by successive approximations. This, in fact, is equivalent to the Schwarz alternating method. See also the concluding paragraph of Sec. 88 of this book with corresponding references.

We shall seek $\varphi(z)$ and $\psi(z)$ in the forms

$$\varphi(z) = \sum_{n=0}^{\infty} \varphi^{(n)}(z), \qquad \psi(z) = \sum_{n=0}^{\infty} \psi^{(n)}(z), \tag{88.2}$$

where the functions $\varphi^{(2n)}$, $\psi^{(2n)}$ are single-valued and analytic in the finite region $|z| < R$ and $\varphi^{(2n+1)}$, $\psi^{(2n+1)}$ are single-valued and analytic in the region $|z - a| > r$, including the point at infinity.

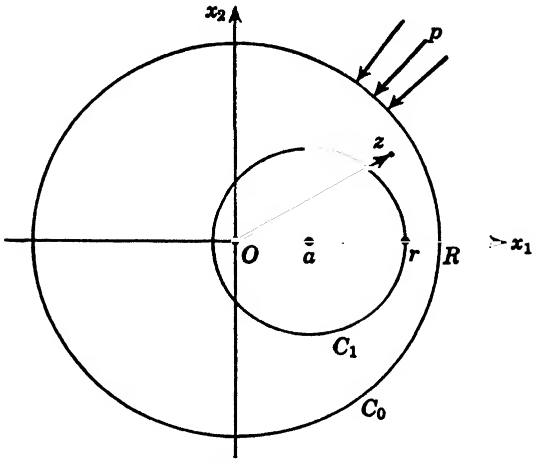

FIG. 60

The functions $\varphi^{(0)}(z)$, $\psi^{(0)}(z)$ will be determined in the region $|z| < R$, so that

$$\varphi^{(0)}(t) + t\overline{\varphi^{(0)\prime}(t)} + \overline{\psi^{(0)}(t)} = -pt \qquad \text{on } C_0.$$

These functions, clearly, will not satisfy the conditions (88.1) on the boundary C_1. We next obtain the solution $\varphi^{(1)}$, $\psi^{(1)}$ in the region $|z - a| > r$, corresponding to the zero stresses at infinity, such that

$$\varphi^{(1)}(t) + t\overline{\varphi^{(1)\prime}(t)} + \overline{\psi^{(1)}(t)} = -L[\varphi^{(0)}(t), \psi^{(0)}(t)] \qquad \text{on } C_1,$$

where[1]

$$L(\varphi, \psi) \equiv \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}. \tag{88.3}$$

Then the functions $\varphi^{(0)} + \varphi^{(1)}$, $\psi^{(0)} + \psi^{(1)}$ will be such that $L(\varphi^{(0)} + \varphi^{(1)}, \psi^{(0)} + \psi^{(1)})$ vanishes on C_1, but it does not reduce to $-pt$ on C_0.

In general, $\varphi^{(n)}(z)$, $\psi^{(n)}(z)$ will be determined from the boundary conditions,

[1] It is not difficult to show from the uniqueness theorem that, if $L[\varphi(t), \psi(t)]$ = const on C_1, then $L[\varphi(z), \psi(z)]$ = const throughout the region.

$$(88.4)\quad \begin{cases} L[\varphi^{(2n+1)}(t), \psi^{(2n+1)}(t)] = -L[\varphi^{(2n)}(t), \psi^{(2n)}(t)] \\ \qquad\qquad \text{on } C_1,\ n = 0, 1, 2, \ldots \\ L[\varphi^{(2n)}(t), \psi^{(2n)}(t)] = -L[\varphi^{(2n-1)}(t), \psi^{(2n-1)}(t)] \\ \qquad\qquad \text{on } C_0,\ n = 1, 2, \ldots . \end{cases}$$

The general solutions of the exterior and interior boundary-value problems for the circular region are known. Thus, if

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = F_0(t) \qquad \text{on } C_0,$$

then $\varphi(z)$ and $\psi(z)$ for $|z| < R$ are given by the formulas:[1]

$$(88.5)\quad \begin{cases} \varphi(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_0} \frac{F_0(t)}{t-z}\,dt - \frac{z}{4\pi i}\int_{C_0} \frac{F_0(t)}{t^2}\,dt - \frac{1}{2\pi i}\int_{C_0}\frac{F_0(t)}{t}\,dt, \\ \psi(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_0} \frac{\overline{F_0(t)}}{t-z}\,dt - \frac{R^2}{z}\varphi'(z) + \frac{R^2}{4\pi i z}\int_{C_0}\frac{F_0(t)}{t^2}\,dt, \\ \qquad\qquad t = Re^{i\theta}. \end{cases}$$

On the other hand, the functions $\varphi(z)$ and $\psi(z)$, analytic in the region $|z - a| \geq r$ and satisfying the condition

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = F_1(t) \qquad \text{on } C_1,$$

are determined by the formulas[2]

$$(88.6)\quad \begin{cases} \varphi(\mathfrak{z}) = -\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{F_1(t)}{t-\mathfrak{z}}\,dt, \\ \psi(\mathfrak{z}) = -\dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{F_1(t)}{t-\mathfrak{z}}\,dt + \frac{1}{2\pi i}\int_{C_1}\frac{\overline{F_1(t)}}{t}\,dt - \frac{r^2}{\mathfrak{z}}\varphi'(\mathfrak{z}), \end{cases}$$

where $\mathfrak{z} = z - a$, and, on C_1, $\mathfrak{z} = re^{i\varphi}$.

If we set $F_0(t) = -pt$ in (88.5), we obtain after simple calculations[3]

$$(88.7)\qquad \varphi^{(0)}(z) = -\tfrac{1}{2}pz, \qquad \psi^{(0)}(z) = 0, \qquad |z| < R.$$

Inserting from (88.7) in (88.3), we get

$$L(\varphi^{(0)}, \psi^{(0)}) = -pz,$$

and, therefore, the functions $\varphi^{(1)}(z)$, $\psi^{(1)}(z)$ are determined in the region $|z - a| > r$, from the boundary conditions,

$$\varphi^{(1)}(t) + t\overline{\varphi^{(1)\prime}(t)} + \overline{\psi^{(1)}(t)} = -L(\varphi^{(0)}, \psi^{(0)})\Big|_{C_1} = p(a + t).$$

[1] These follow from Eq. (85.5) and (85.6) upon setting $z = R\zeta$.

[2] These follow from (85.9) and (85.10) upon setting $m = 0$ and $z = r/\mathfrak{z}$, where $\mathfrak{z} = z - a$.

[3] In verifying these calculations, note that $\bar{t} = R^2/t$.

Setting $F_1 = p(a + t)$ in (88.6) and integrating, we find,

$$\varphi^{(1)}(\mathfrak{z}) = 0, \qquad \psi^{(1)}(\mathfrak{z}) = \frac{r^2 p}{\mathfrak{z}} + pa,$$

so that

$$\varphi^{(1)}(z) = 0, \qquad \psi^{(1)}(z) = \frac{r^2 p}{z - a} + pa, \qquad |z - a| > r. \tag{88.8}$$

We next form

$$L(\varphi_1^{(1)}, \psi^{(1)}) = \frac{pr^2}{\bar{z} - a} + pa,$$

and determine $\varphi^{(2)}(z)$, $\psi^{(2)}(z)$ for $|z| < R$ from the boundary condition,

$$L(\varphi^{(2)}, \psi^{(2)})\Big|_{C_0} = -L(\varphi^{(1)}, \psi^{(1)})\Big|_{C_0} = -\frac{pr^2}{\bar{t} - a} - pa.$$

Making use of the formulas (88.5) with $F_0(t)$ given by the right-hand member of the expression just found, we obtain,

$$\begin{cases} \varphi^{(2)}(z) = -\dfrac{pr^2 z(R^2 + az)}{2R^2(R^2 - az)}, \\ \psi^{(2)}(z) = -pa + \dfrac{pr^2 a}{(R^2 - az)^2}(2R^2 - az). \end{cases} \tag{88.9}$$

This process can be continued to obtain the approximating functions of higher orders. The series (88.2) constructed in this manner converge, but clearly the rapidity of convergence will depend on the magnitudes of the parameters a, r, and R. As noted earlier, this problem can be solved more simply in bipolar coordinates.[1]

b. Concentric Ring under Concentrated Forces. Let the ring bounded by concentric circles C_0 and C_1 of radii R and r, respectively, $R > r$, be acted on by the concentrated forces P at $z = \pm Ri$.

The functions $\varphi(z)$, $\psi(z)$ are determined in the region $r < |z| < R$ from the boundary conditions:

$$\begin{aligned} \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} &= \text{const} \qquad \text{on } C_1, \\ &= f(t) \qquad \text{on } C_0, \end{aligned} \tag{88.10}$$

where[2]

$$\begin{aligned} f(t) &= 0, \qquad \text{for } t = Re^{i\theta}, \qquad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ &= P, \qquad \text{for } t = Re^{i\theta}, \qquad \frac{\pi}{2} < \theta \le \frac{3}{2}\pi. \end{aligned}$$

[1] See, for example, Ya. S. Uflyand, Bipolar Coordinates in the Theory of Elasticity (1950), pp. 204–210 (in Russian).

[2] See Sec. 77c.

We again seek a solution in the form (88.2), where $\varphi^{(2n)}$, $\psi^{(2n)}$ are analytic for $|z| < R$ and $\varphi^{(2n+1)}$, $\psi^{(2n+1)}$ are analytic for $|z| > r$. As our first approximation $\varphi^{(0)}(z)$, $\psi^{(0)}(z)$ we take the known solution, deduced in Sec. 77, for the solid circle of radius R, under the action of concentrated forces. It is,

$$\varphi^{(0)}(z) = \frac{Pi}{2\pi}\left(\log\frac{z - iR}{z + iR} + \frac{z}{R}\right),$$
$$\psi^{(0)}(z) = \frac{Pi}{2\pi}\left(\log\frac{z - iR}{z + iR} + \frac{iR}{z - iR} - \frac{iR}{z + iR}\right).$$

The subsequent approximations are determined from the boundary conditions (88.4), with the aid of formulas (88.5) and (88.6).

Although the process indicated here leads to convergent series (88.2), the convergence is slow. However, because of the special character of loading, it proves possible to deduce the general expressions for $\varphi^{(2n)}$, $\psi^{(2n)}$ and sum the dominant terms in the resulting series. Narodetzky obtained in this manner an approximate solution, valid to any specified degree of accuracy.

Variants of the Schwarz method have been used by Mikhlin and Sherman to solve certain integral equations furnishing solutions of the first elastostatic boundary-value problem for the semi-infinite plate with an elliptical hole.[2]

89. Concluding Remarks. The principal object of this chapter has been to introduce the reader to certain powerful general methods of solution of the two-dimensional problems in elasticity. These methods have recently been extended to plane problems in anisotropic elastic media and modified to include the problems of transverse deflection of thin plates and several categories of contact problems[3] in elasticity. Among the more comprehensive contributions of this type are:[4]

S. G. Lekhnitzky, Anisotropic Plates (1947).

I. N. Vekua, New Methods of Solution of Elliptic Equations (1948).

I. Ya. Shtaerman, The Contact Problem of Elasticity (1949).

S. G. Lekhnitzky, Theory of Elasticity of an Anisotropic Elastic Body (1950).

[1] M. Z. Narodetzky, *Izvestiya Akademii Nauk SSSR, Technical Series*, No. 1 (1948), pp. 7–18 (in Russian).

[2] S. G. Mikhlin, *Trudy Seismological Institute, Academy of Science of the USSR*, No. 391 (1934) (in Russian).

D. I. Sherman, *Trudy Seismological Institute, Academy of Science of the USSR*, Nos. 53 and 54 (1935) (in Russian).

[3] The contact problems are treated in Chap. 13 of N. I. Muskhelishvili's Singular Integral Equations (1953), as well as in his monograph Some Basic Problems of the Mathematical Theory of Elasticity (1953).

[4] With the exception of the book by Green and Zerna all these monographs are in the Russian language.

G. N. Savin, Concentration of Stresses around Openings (1951).
A. E. Green and W. Zerna, Theoretical Elasticity (1954).

Savin's book contains solutions of numerous special problems on the stress concentration near openings in stretched isotropic elastic plates.[1]

A survey of the recent work on the theory of plates, published in the USSR, is contained in a paper by G. Dzhanelidze, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 12 (1948), pp. 109–128. An English translation of this paper, prepared by the American Mathematical Society, *Translation* 6 (1950), is available. References contained in this translation should be supplemented by the following papers dealing with the deflection of thin elastic plates whose boundaries are simply supported, clamped, or partly clamped and partly simply supported. All these papers[2] appeared in vols. 14 to 17 of the Russian journal *Applied Mathematics and Mechanics* (*Prikl. Mat. Mekh., Akademiya Nauk SSSR*)*:* Z. I. Havilov, vol. 14 (1950), pp. 405–414; M. M. Friedman, vol. 14 (1950), pp. 429–432, vol. 15 (1951), pp. 258–260, vol. 16 (1952), pp. 429–436; G. F. Mandzhavidze, vol. 15 (1951), pp. 279–296; V. K. Prokopov, vol. 14 (1950), pp. 527–536, vol. 16 (1952), pp. 45–56; A. I. Kalandiya, vol. 16 (1952), pp. 271–282, vol. 17 (1953), pp. 293–310, 692–704; G. A. Greenberg, N. N. Lebedev, and Y. S. Uflyand, vol. 17 (1953), pp. 73–86; G. A. Greenberg, vol. 17 (1953), pp. 211–228.

[1] These may be supplemented by J. R. M. Radok's paper concerned with the problems of plane elasticity for reinforced boundaries, *Journal of Applied Mechanics*, vol. 22 (1955), and by Eugene Levin's doctoral dissertation entitled "Reinforced Openings in Plane Structural Members," University of California, Los Angeles (1955). See also I. S. Hara's paper cited in Sec. 81, and I. G. Abramovich, *Doklady Akademii Nauk SSSR* (NS), vol. 104 (1955), pp. 372–375.

[2] The following papers on the deflection of thin elastic plates were published while this book was in press:

V. A. Likhachev, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 19 (1955), pp. 255–256; O. M. Sapondzhyan, *Izvestiya Akademii Nauk Armyanskoĭ SSR, Phys. Mat. Nauki*, No. 5 (1954), pp. 19–43, No. 6 (1955), pp. 27–34; D. I. Sherman, *Doklady Akademii Nauk SSSR*, vol. 10 (1955), pp. 623–626.

CHAPTER 6

THREE-DIMENSIONAL PROBLEMS

90. General Solutions. The key to effective treatment of the two-dimensional boundary-value problems, discussed in Chap. 5, is in the special representation of solutions of appropriate field equations with the aid of certain arbitrary functions. Although several attempts have been made to construct analogous "general solutions" of the three-dimensional field equations of elasticity, such solutions have not been exploited in a systematic way. The so-called general solutions are but particular forms of solutions of the field equations involving arbitrary functions of special types. Thus one can construct a solution of Navier's equations, containing arbitrary harmonic functions that enter in particular combinations with certain known functions. The choice of known functions and the form of solution are determined, in part, by the differential equations and, in part, by the topology of the region. Another "general solution" of Navier's equations can be constructed with the aid of the biharmonic functions, and there is no a priori reason why one form of general solution should be readily transformable into another. The criterion of the generality of a given form of solution lies in the possibility of determining the arbitrary functions so that the boundary conditions are fulfilled.

Thus, in dealing with the two-dimensional elastostatic problems in simply connected domains, the general solution of the homogeneous Navier's equations was obtained in the form[1]

$$2\mu(u_1 + iu_2) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{90.1}$$

where $\varphi(z)$ and $\psi(z)$ are single-valued analytic functions. This solution is general in the sense that the unknown functions φ and ψ can be determined, essentially uniquely, when suitable boundary conditions are imposed. If one relaxes restrictions on the connectivity of the region, or on the behavior of displacements on the boundary, the representation (90.1) may cease to be valid.

An equivalent form of the general solution involving four arbitrary plane harmonic functions can be deduced from (90.1) by setting,

[1] Sec. 71.

$$\varphi(z) = \varphi_1(x_1, x_2) + i\varphi_2(x_1, x_2),$$
$$\psi(z) = \psi_1(x_1, x_2) + i\psi_2(x_1, x_2).$$

We readily find

$$(90.2) \qquad 2\mu u_\alpha = \varkappa\varphi_\alpha - x_\beta\varphi_{\beta,\alpha} - \psi_\alpha, \qquad (\alpha, \beta = 1, 2),$$

which is the general solution of the two-dimensional Navier equations in a simply connected domain involving four harmonic functions φ_α, ψ_α. However, only two of these are independent in the sense that the specification of φ_1 and ψ_1 enables one to calculate the conjugate harmonics φ_2, ψ_2 to within nonessential constants of integration. The determination of these functions, whenever the displacements or tractions on the boundary are specified, is clearly possible, since the problem is equivalent to the calculation of $\varphi(z)$ and $\psi(z)$.

Inasmuch as the apparatus of the complex variable theory is not readily available for the treatment of the three-dimensional Navier equations,

$$(90.3) \qquad \begin{cases} \mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} = 0, & (i = 1, 2, 3), \\ \vartheta \equiv u_{i,i}, \end{cases}$$

it is natural to seek a general solution of these equations[1] in terms of space harmonic functions. To avoid the introduction of multiple-valued harmonic functions, we confine our considerations to simply connected domains τ bounded by smooth surfaces.

It is well known that the divergence and curl of the displacement vector can be specified independently.[2] It follows from this that the displacement vector $\mathbf{u}$ can be represented as the sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ of class C^2 such that div $\mathbf{v} = 0$ and curl $\mathbf{w} = 0$. But a necessary and sufficient condition for this is the existence of the vector $\mathbf{A}$ and the scalar Ψ such that $\mathbf{v} = \text{curl } \mathbf{A}$ and $\mathbf{w} = \nabla\Psi$. It is easy to show that, for a given $\mathbf{u}$, $\mathbf{A}$ and Ψ can be determined from the solution of Poisson's equations whenever div $\mathbf{A} = 0$. This justifies us in seeking a solution of the system (90.3) in the form

$$\mathbf{u} = \frac{1}{\lambda + \mu}\nabla\Psi + \frac{1}{\mu}\text{curl } \mathbf{A},$$

[1] Only homogeneous systems (90.3) need be considered in the problem of general integration since body forces can always be eliminated in the manner explained in Sec. 68.

[2] If the region τ is finite, the system of equations

$$\text{curl } \mathbf{u} = \mathbf{f}(x_1, x_2, x_3),$$
$$\text{div } \mathbf{u} = g(x_1, x_2, x_3),$$

is known to have a solution whenever $\mathbf{u}$ is of class C^2 in τ. If τ is infinite, we further require that the specified functions $\mathbf{f}$ and g vanish at infinity as $1/r^2$. See, for example, M. Mason and W. Weaver, The Electromagnetic Field (1932), pp. 352–365. See also A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Sec. 16.

or

(90.4) $$u_i = \frac{1}{\lambda + \mu}\Psi_{,i} + \frac{1}{\mu}\,\mathrm{curl}_i\,\mathbf{A}.$$

On calculating the divergence of u_i in (90.4), we get

(90.5) $$u_{i,i} = \frac{1}{\lambda + \mu}\nabla^2\Psi \equiv \vartheta,$$

so that (90.3) can be written in the form

$$\nabla^2(\mu u_i + \Psi_{,i}) = 0.$$

Hence

(90.6) $$\mu u_i + \Psi_{,i} = \Phi_i,$$

where Φ_i is an arbitrary harmonic vector. It follows from (90.6) that

$$\mu u_{i,i} + \nabla^2\Psi = \Phi_{i,i}$$

and, on noting (90.5), we get

(90.7) $$\nabla^2\Psi = \frac{\lambda + \mu}{\lambda + 2\mu}\Phi_{i,i}.$$

A particular integral of this equation is $\frac{1}{2}\frac{\lambda + \mu}{\lambda + 2\mu}x_j\Phi_j$, and hence[1] the general solution can be written in the form

(90.8) $$\Psi = \Phi_0 + \frac{1}{2}\frac{\lambda + \mu}{\lambda + 2\mu}x_j\Phi_j,$$

where Φ_0 is an arbitrary harmonic function. Referring to (90.6), we see that the displacement vector u_i can be represented in the form

(90.9) $$\mu u_i = \Phi_i - \Phi_{0,i} - \frac{1}{2}\frac{\lambda + \mu}{\lambda + 2\mu}(x_j\Phi_j)_{,i},$$

involving four arbitrary harmonic functions.

This formula can be cast in the form whose structure is identical with the representation (90.2) of the displacements in plane elasticity. On carrying out the indicated differentiation in (90.9) and simplifying, we find,

$$\mu u_i = \frac{\lambda + 3\mu}{2(\lambda + 2\mu)}\Phi_i - \frac{1}{2}\frac{\lambda + \mu}{\lambda + 2\mu}x_j\Phi_{j,i} - \Phi_{0,i}.$$

But

$$\frac{\lambda + 3\mu}{2(\lambda + 2\mu)} = \frac{3 - 4\sigma}{4(1 - \sigma)}, \qquad \frac{\lambda + 2\mu}{\lambda + \mu} = 2(1 - \sigma)$$

and hence

$$\mu u_i = \frac{3 - 4\sigma}{4(1 - \sigma)}\Phi_i - \frac{1}{4(1 - \sigma)}x_j\Phi_{j,i} - \Phi_{0,i},$$

[1] Note that $\nabla^2(x_i\Phi_i) = 2\Phi_{i,i}$, since Φ_i is harmonic.

and if we define,

$$\varphi_i \equiv \Phi_i/[2(1-\sigma)], \qquad \varphi_0 \equiv 2\Phi_0,$$

and recall from (71.8) that $\varkappa = 3 - 4\sigma$, we get

$$2\mu u_i = \varkappa\varphi_i - x_j\varphi_{j,i} - \varphi_{0,i}. \tag{90.10}$$

The formula (90.10) involving four arbitrary harmonic functions φ_i $(i = 0, 1, 2, 3)$ is identical in structure[1] with (90.2). Aside from the mode of derivation and notational differences the formula (90.10) is that deduced independently by Papkovich and Neuber.[2]

We remarked, in connection with the two-dimensional problems, that the general solution (90.2) contains, in effect, only two independent harmonic functions. This suggests the likelihood of eliminating one of the space harmonics in the representation (90.10), so that the general solution of the three-dimensional Navier equations involves only three independent harmonic functions. Unsupported statements to this effect are common.[3] Thus, it is frequently asserted that any one of the functions in (90.10) may be set equal to zero without affecting the generality of the solution. Neuber in his book, Theory of Notch Stresses, indicates that the substitution

$$\begin{cases} \varphi_0 = (\varkappa + 1)\bar{\varphi}_0 - x_j\bar{\varphi}_{0,j}, \\ \varphi_j = \bar{\varphi}_j + \bar{\varphi}_{0,j}, \end{cases} \tag{90.11}$$

in (90.10) yields

$$2\mu u_i = \varkappa\bar{\varphi}_i - x_j\bar{\varphi}_{j,i}, \tag{90.12}$$

which involves only $\bar{\varphi}_1$, $\bar{\varphi}_2$, and $\bar{\varphi}_3$. Likewise a substitution

$$\begin{cases} \varphi_\alpha = \bar{\varphi}_\alpha + \bar{\varphi}_{3,\alpha}, & (\alpha = 1, 2), \\ \varphi_3 = \bar{\varphi}_{3,3}, & \\ \varphi_0 = (\varkappa + 1)\bar{\varphi}_3 - x_i\bar{\varphi}_{3,i} + \bar{\varphi}_0, & (i = 1, 2, 3), \end{cases} \tag{90.13}$$

results in the expression of the form (90.10) involving only $\bar{\varphi}_0$, $\bar{\varphi}_1$, and $\bar{\varphi}_2$. To establish the validity of the assertion, it is necessary to show[4] that the systems of Eqs. (90.11) and (90.13) possess harmonic solutions $\bar{\varphi}$ for the arbitrarily specified harmonic functions φ. If we suppose that the same set of displacements u_i can be represented in either of the forms

[1] To make the formal analogy complete, set $\varphi_{0,1} = \psi_2$, $\varphi_{0,2} = \psi_1$.

[2] P. F. Papkovich, *Comptes rendus hebdomadaires des séances de l'académie des sciences*, Paris, vol. 195 (1932), pp. 513–515, 754–756; *Izvestiya Akademii Nauk SSSR*, Physics-Mathematics Series (1932), pp. 1425–1435.

H. Neuber, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 14 (1934), p. 203, or his book Theory of Notch Stresses (1946), pp. 21–25.

[3] These usually stem from misconceptions about the meaning of the term "general solution" and from inadequate recognition of the fact that the form of such solutions depends on the topology of the domain.

[4] The proof of this in Neuber's book is lacking. Indeed, as we shall see presently, the statement is not always true.

(90.10) or (90.12), wherein the functions φ are harmonic, we get on subtraction

$$\varkappa\psi_i - x_j\psi_{j,i} = \varphi_{0,i}, \tag{90.14}$$

where

$$\psi_i \equiv \varphi_i - \bar{\varphi}_i.$$

We rewrite (90.14) in the form

$$(\varkappa + 1)\psi_i - (x_j\psi_j)_{,i} = \varphi_{0,i} \tag{90.15}$$

and find, on differentiation with respect to x_k, that

$$(\varkappa + 1)\psi_{i,k} - (x_j\psi_j)_{,ik} = \varphi_{0,ik}. \tag{90.16}$$

Interchanging the indices i and k and subtracting the result from (90.16) yields

$$\psi_{i,k} = \psi_{k,i},$$

which implies the existence of a scalar function F such that

$$\psi_i = F_{,i}. \tag{90.17}$$

On the other hand, if we set $k = i$ in (90.16) and take cognizance of the fact that φ_0 and the ψ_i by hypothesis are harmonic functions, we find that

$$\psi_{i,i} = 0. \tag{90.18}$$

It follows from (90.17) and (90.18) that F is a harmonic function. Now if it is possible to construct F, for an arbitrary harmonic function φ_0, then the functions ψ_i, and hence the $\bar{\varphi}_i$, will be determined for the preassigned harmonic functions φ_i. The substitution (90.11) will then yield displacements in the form (90.12).

The differential equation satisfied by F can be got by substituting (90.17) in (90.15). We have

$$(\varkappa + 1)F_{,i} - (x_jF_{,j})_{,i} = \varphi_{0,i},$$

so that

$$(\varkappa + 1)F - x_jF_{,j} = \varphi_0 + c \tag{90.19}$$

where c is the integration constant.

But if one assumes that every harmonic function φ_0 defined in the finite, simply connected closed region can be represented[1] in the series of solid integral harmonics as

$$\varphi_0 = \sum_{n=0}^{\infty} r^n Y_n(\theta, \varphi),$$

[1] A proof of this for the general *three-dimensional* domains is lacking. A summary of the basic facts about integral harmonics is given in Sec. 95.

then Eq. (90.19) can be written in spherical coordinates as[1]

$$(\varkappa + 1)F - r\frac{\partial F}{\partial r} = \sum_{n=0}^{\infty} r^n Y_n + c,$$

and one can, clearly, take

$$F = \frac{c}{\varkappa + 1} + \sum_{n=0}^{\infty} \frac{1}{\varkappa + 1 - n} r^n Y_n.$$

This solution is valid so long as $\varkappa + 1 - n \neq 0$. The exceptional case arises when $\varkappa = 2$ and $n = 3$, and since $\varkappa = 3 - 4\sigma$, we see that the representation in the form (90.12), in general, is impossible when $\sigma = \frac{1}{4}$.

If the domain under consideration is an infinite domain exterior to some closed surface τ containing the origin, and if φ_0 in (90.19) can be represented in the form

$$\varphi_0 = \sum_{n=0}^{\infty} r^{-(n+1)} Y_n,$$

we find, as above, that F can be taken as

$$F = \frac{c}{\varkappa + 1} + \sum_{n=0}^{\infty} \frac{1}{\varkappa + n + 2} r^{-(n+1)} Y_n.$$

Since $\varkappa > 0$, this solution is valid and hence a representation of the form (90.12) may prove possible in an infinite simply connected domain.

The possibility of representing every solution of Navier's equations in the form (90.10), wherein one of the functions φ_i is set equal to zero, say $\varphi_3 = 0$, hinges on the construction of the harmonic function $\bar{\varphi}_3$ from the specified values of its derivatives. It is clear from (90.13) that the harmonic functions $\bar{\varphi}_1$, $\bar{\varphi}_2$, and $\bar{\varphi}_0$ will be uniquely determined once the harmonic function $\bar{\varphi}_3$ is obtained from the equation

(90.20) $$\bar{\varphi}_{3,3} = \varphi_3.$$

If the region is a sphere, the function φ_3 can be represented in the series of spherical harmonics as

$$\begin{aligned}\varphi_3 &= \sum_{n=0}^{\infty} r^n Y_n \\ &= \sum_{n=0}^{\infty} r^n \left[A_n P_n(\cos\theta) + \sum_{m=1}^{n} (A_n^m \cos m\varphi + B_n^m \sin m\varphi) P_n^m(\cos\theta)\right].\end{aligned}$$

[1] The scalar product $x_i F_{,i}$ of the vector $\mathbf{r}$ with the gradient ∇F of F is clearly equal to $r\dfrac{\partial F}{\partial r}$.

It is easy to check[1] that the solution of (90.20) can be taken in the form

$$\varphi_3 = \sum_{n=0}^{\infty} r^{n+1} \left[\frac{A_n}{n+1} P_{n+1}(\cos\theta) + \sum_{m=1}^{n} \left(\frac{A_n^m}{n+m+1} \cos m\varphi + \frac{B_n^m}{n+m+1} \sin m\varphi \right) P_{n+1}^m(\cos\theta) \right].$$

It was argued[2] that the harmonic solution of (90.20) in an infinite simply connected region can be obtained only when certain terms in the representation of φ_3 in the series of spherical harmonics do not appear in the expansion.

The formulas for the components of the stress tensor associated with the representation (90.10) can be easily written down with the aid of the stress-strain relations.[3]

Another interesting form of solution,

$$\mu u_i = 2(1-\sigma)\nabla^2 F_i - F_{j,ji}, \tag{90.21}$$

where the F_i are biharmonic functions, was obtained by Galerkin.[4] This solution is closely related to the Neuber-Papkovich solution (90.10). Indeed, if we set

$$\nabla^2 F_i = \frac{1}{1-\sigma} \Phi_i, \qquad F_{j,j} = \Psi,$$

(90.21) becomes

$$\mu u_i = \Phi_i - \Psi_{,i},$$

which is precisely the formula (90.6). This connection was first noted apparently by Mindlin.[5] We shall see that in a finite simply connected domain every biharmonic function can be expressed in terms of two harmonic functions. It follows from this, and from the representation (90.10), that at least two of the six harmonic functions entering in the Galerkin solution are not independent.

[1] In verifying it is advisable to use the integral representation of solid harmonics such as is recorded in Sec. 18.31 of Whittaker and Watson's Modern Analysis.

[2] M. G. Slobodyanski, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 18 (1954), pp. 54–78.

[3] Such formulas have been recorded by several authors: G. S. Shapiro, *Comptes rendus (Doklady) de l'académie des sciences de l'URSS*, vol. 55 (1947), pp. 693–695; W. Freiberger, *Australian Journal of Scientific Research (A)*, vol. 2 (1949), pp. 483–492; G. Yu. Dzhanelidze, *Doklady Akademii Nauk SSSR*, New Series, vol. 88 (1953), pp. 423–425; M. Brdička, *Czechoslovak Journal of Physics*, vol. 3 (1953), pp. 36–52.

[4] B. G. Galerkin, *Comptes rendus hebdomadaires des séances de l'académie des sciences*, Paris, vol. 190 (1930), p. 1047; *Comptes rendus (Doklady) de l'académie des sciences de l'URSS*, ser. *A*, vol. 14 (1930), p. 353, vol. 10 (1931), p. 281; *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 6 (1942), p. 487.

[5] R. D. Mindlin, *Bulletin of the American Mathematical Society*, vol. 42 (1936), pp. 373–376.

The function Ψ, as is clear from (90.5), is biharmonic, and calculations leading to (90.8) show that every biharmonic function is expressible in terms of four harmonic functions. One can thus represent every biharmonic function F in the form

$$F = \Phi_0 + x_i\Phi_i, \qquad (i = 1, 2, 3), \tag{90.22}$$

where the Φ's are harmonic.[1] Any two of the functions Φ_i can be set equal to zero without loss of generality, so that every biharmonic function (in a finite simply connected domain) is expressible in terms of two harmonic functions in one of the forms:

$$F = \Phi_0 + x_1\Phi, \qquad F = \Phi_0 + x_2\Phi, \qquad F = \Phi_0 + x_3\Phi.$$

It would suffice to consider F in the form

$$F = \Phi_0 + x_1\Phi. \tag{90.23}$$

Let F be an arbitrary biharmonic function. The functions Φ_0 and Φ can then be constructed as follows: On forming the Laplacian of (90.23), we get

$$\nabla^2 F = 2\Phi_{,1}, \tag{90.24}$$

and since $\nabla^2 F$ is known, we can construct the harmonic function Φ satisfying this equation. Having determined Φ, we insert it in (90.23) and get

$$\Phi_0 = F - x_1\Phi,$$

which is harmonic by virtue of (90.24).

We conclude this section with a brief mention of the sets of solutions of Cauchy's equilibrium equations,

$$\tau_{ij,j} = 0, \tag{90.25}$$

deduced by Maxwell and Morera. It is easy to verify that Eqs. (90.25) are formally satisfied if one assumes that

$$\left\{\begin{aligned} \tau_{11} &= \varphi_{22,33} + \varphi_{33,22} - 2\varphi_{23,23} \\ \tau_{22} &= \varphi_{33,11} + \varphi_{11,33} - 2\varphi_{31,31} \\ \tau_{33} &= \varphi_{11,22} + \varphi_{22,11} - 2\varphi_{12,12} \\ \tau_{23} &= \varphi_{31,12} + \varphi_{12,13} - \varphi_{11,23} - \varphi_{23,11} \\ \tau_{31} &= \varphi_{12,23} + \varphi_{23,21} - \varphi_{22,31} - \varphi_{31,22} \\ \tau_{12} &= \varphi_{23,31} + \varphi_{31,32} - \varphi_{33,12} - \varphi_{12,33}, \end{aligned}\right. \tag{90.26}$$

where the $\varphi_{ij} = \varphi_{ji}$ are of class C^3. On setting $\varphi_{12} = \varphi_{23} = \varphi_{31} = 0$, we obtain solutions proposed by Maxwell, and on taking $\varphi_{11} = \varphi_{22} = \varphi_{33} = 0$,

[1] This is identical in form with the Goursat representation of the plane biharmonic function $U(x_1, x_2) = \varphi_0 + x_\alpha\varphi_\alpha$ deduced in Sec. 70.

we get solutions due to Morera.[1] The functions φ_{ij} are further restricted by Beltrami's compatibility equations. These restrictions have been formulated implicitly (in tensor form) by Schaefer,[2] who also indicates a connection of relations (90.26) with the formulas for stresses deduced from the Neuber-Papkovich solution (90.10).

91. Concentrated Forces. The general solution of the nonhomogeneous Navier's equations,

$$\mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} = -F_i \qquad \text{in } \tau, \tag{91.1}$$

can be got by adding a particular integral of (91.1) to one of the general solutions deduced in Sec. 90. We record one useful form of the particular integral due to Lord Kelvin.[3] It is,

$$u_i(x) = A \int_\tau \left[B \frac{F_i(\xi)}{r} - \left(\frac{1}{r}\right)_{,i} (x_j - \xi_j) F_j(\xi) \right] d\tau, \tag{91.2}$$

where

$$A \equiv \frac{\lambda + u}{8\pi\mu(\lambda + 2\mu)}, \qquad B \equiv \frac{\lambda + 3\mu}{\lambda + \mu}, \qquad \left(\frac{1}{r}\right)_{,i} = \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right),$$

and $r = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{1/2}$ is the distance from the field point (x_1, x_2, x_3) to the variable point (ξ_1, ξ_2, ξ_3) in τ. The functions $F_i(\xi)$ are the components of the body force F_i expressed in terms of the variables of integration ξ_i.

The fact that (91.2) is indeed an integral of (91.1) can be verified by direct substitution.[4]

A solution of Eqs. (91.1), appropriate to the deformation of an elastic body by the concentrated force F_i^0 applied at some point ξ_i, can be easily deduced from (91.2). We suppose that the body forces F_i are distributed over some subregion τ_1 of τ, including the point ξ_i, and vanish over the rest of the region. The resultant of the body forces acting on τ_1 is

$$F_i^0 = \int_{\tau_1} F_i \, d\tau.$$

[1] J. Maxwell, *Transactions of the Royal Society of Edinburgh*, vol. 26 (1870), p. 27, or Collected Papers, vol. 2, pp. 161–207; G. Morera, *Atti della reale accademia dei Lincei*, Rome, ser. 5, vol. 1 (1892), pp. 137–141, 233–234.

[2] H. Schaefer, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 33 (1953), pp. 356–362. See also a paper by R. V. Southwell in Timoshenko Anniversary Volume, pp. 211–216 and a paper by W. Ornstein, "Stress Functions of Maxwell and Morera," *Quarterly of Applied Mathematics*, vol. 12 (1954), p. 198.

[3] Sir William Thomson, *Cambridge and Dublin Mathematical Journal* (1848), or Mathematical and Physical Papers, vol. 1, p. 97. See also A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity (1927), pp. 183–185.

[4] For the field points in the region τ, the integral is improper, and care must be used in differentiating under the integral sign. See analogous calculations in M. Mason and W. Weaver, The Electromagnetic Field, pp. 93–96.

If we now let F_i increase in such a way that this integral has a finite limit F_i^0 as $\tau_1 \to 0$, we arrive at the notion of the concentrated force F_i^0 acting at the point ξ_i.

The displacements $u_i(x)$ produced at the point $x_i \neq \xi_i$ by the force F_i^0 applied at ξ_i, as follows from (91.2), are

$$u_i(x) = \frac{\lambda + 3\mu}{8\pi\mu(\lambda + 2\mu)} \frac{F_i^0}{r} + \frac{\lambda + \mu}{8\pi\mu(\lambda + 2\mu)} \frac{(x_i - \xi_i)(x_j - \xi_j)}{r^3} F_j^0. \tag{91.3}$$

These expressions satisfy the homogeneous Navier equations at all points of the region except at the point of application of the force. They cease having meaning at the singular point $x_i = \xi_i$; but if this point is deleted from the region by enclosing it in a sphere S of small radius a, the solutions (91.3) in the remaining region correspond to the deformation present in a body τ with a cavity S subjected to the action of forces with the resultant F_i^0.

If we choose the coordinate axes so that F_i^0 acts at the origin $\xi_i = 0$ and take $F_1^0 = F_2^0 = 0$, $F_3^0 = P$, the formulas (91.3) yield

$$\begin{cases} u_\alpha = C \dfrac{x_\alpha x_3}{r^3}, & (\alpha = 1, 2), \\ u_3 = C \left(\dfrac{\lambda + 3\mu}{\lambda + \mu} \dfrac{1}{r} + \dfrac{x_3^2}{r^3} \right), \end{cases} \tag{91.4}$$

where

$$C = \frac{(\lambda + \mu)P}{8\pi\mu(\lambda + 2\mu)} \quad \text{and} \quad r^2 = x_i x_i. \tag{91.5}$$

Using the stress-strain relations,

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}),$$

we find,

$$\begin{cases} \tau_{11} = -\dfrac{2\mu C x_3}{r^3}\left[3\left(\dfrac{x_1}{r}\right)^2 - \dfrac{\mu}{\lambda + \mu}\right], \\ \tau_{22} = -\dfrac{2\mu C x_3}{r^3}\left[3\left(\dfrac{x_2}{r}\right)^2 - \dfrac{\mu}{\lambda + \mu}\right], \\ \tau_{33} = -\dfrac{2\mu C x_3}{r^3}\left[3\left(\dfrac{x_3}{r}\right)^2 + \dfrac{\mu}{\lambda + \mu}\right], \\ \tau_{23} = -\dfrac{2\mu C x_2}{r^3}\left[3\left(\dfrac{x_3}{r}\right)^2 + \dfrac{\mu}{\lambda + \mu}\right], \\ \tau_{13} = -\dfrac{2\mu C x_1}{r^3}\left[3\left(\dfrac{x_3}{r}\right)^2 + \dfrac{\mu}{\lambda + \mu}\right], \\ \tau_{12} = -\dfrac{6\mu C x_1 x_2 x_3}{r^5}. \end{cases} \tag{91.6}$$

The tractions T_i, produced by these stresses over the sphere S of radius $r = a$, are determined from

$$T_i = \tau_{ij}\nu_j,$$

with $\nu_j = x_j/a$. We get,

$$(91.7) \qquad T_\alpha = -\frac{6\mu C x_\alpha x_3}{a^4}, \qquad T_3 = -\frac{6\mu C x_3^2}{a^4} - \frac{2\mu^2 C}{(\lambda + \mu)a^2}$$

and, on integrating over the surface of the sphere $r = a$, we find,

$$(91.8) \qquad \int_S T_\alpha \, d\sigma = 0, \qquad \int_S T_3 \, d\sigma = -\frac{8\pi\mu C(\lambda + 2\mu)}{\lambda + \mu}.$$

These are the components of the resultant force exerted on S by matter exterior to S. On noting the value of C in (91.5), we see that the component in the x_3-direction is $-P$, as it should be, since τ is in equilibrium.

To solve the problem of deformation of the elastic half space bounded by a plane subjected to the action of a concentrated force, Boussinesq combined solutions (91.4) with certain other singular solutions of Navier's equations, which we give next.

It is easy to verify that the displacements

$$(91.9) \qquad u_\alpha = \frac{D x_\alpha}{r(r + x_3)}, \qquad u_3 = \frac{D}{r}, \qquad (\alpha = 1, 2),$$

with $r^2 = x_i x_i$ and $D = \text{const}$, represent the dilatationless[1] solution of Navier's equations so long as $r \neq 0$. The corresponding stresses are

$$(91.10) \qquad \begin{cases} \tau_{11} = 2\mu D \left[\dfrac{x_2^2 + x_3^2}{r^3(r + x_3)} - \dfrac{x_1^2}{r^2(r + x_3)^2} \right], \\ \tau_{22} = 2\mu D \left[\dfrac{x_1^2 + x_3^2}{r^3(r + x_3)} - \dfrac{x_2^2}{r^2(r + x_3)^2} \right], \\ \tau_{33} = -2\mu D \dfrac{x_3}{r^3}, \qquad \tau_{13} = -2\mu D \dfrac{x_1}{r^3}, \qquad \tau_{23} = -2\mu D \dfrac{x_2}{r^3}, \\ \tau_{12} = -2\mu D \dfrac{x_1 x_2 (x_3 + 2r)}{r^3(r + x_3)^2}. \end{cases}$$

As in the preceding example we calculate the tractions T_i over the sphere S of radius a and find

$$(91.11) \qquad T_\alpha = -2\mu D \frac{x_\alpha}{r^2(r + x_3)}, \qquad T_3 = -2\mu D \frac{1}{r^2}, \qquad r = a.$$

The corresponding components of the resultant force R_i exerted on S by the matter exterior to the sphere of radius a are

$$(91.12) \qquad R_\alpha = \int_S T_\alpha \, d\sigma = 0, \qquad R_3 = \int_S T_3 \, d\sigma = -8\pi u D.$$

[1] We note that, when $\vartheta = 0$, the solution of Navier's equations reduces to the familiar problem in potential theory.

We shall see in the following section that a superposition of the elementary solutions (91.4) and (91.9) can be made to yield the state of deformation present in an elastic half space whose plane boundary is under the action of the concentrated normal force.[1]

92. Deformation of Elastic Half Space by Normal Loads. Let the semi-infinite region $x_3 \geq 0$ be occupied by an elastic medium, and assume that the concentrated force P, applied at the origin, acts in the positive direction of the x_3-axis. Since the point of application of the load is a singular point in the solution of Navier's equations, we delete it from the region $x_3 \geq 0$ by describing a hemisphere of small radius a and confine our attention to the semi-infinite region bounded by the hemisphere and the x_1x_2-plane.

We shall construct a solution such that the resultant of all external stresses acting on the hemisphere is P, and

(92.1) $$\tau_{13} = \tau_{23} = \tau_{33} = 0$$

over the rest of the boundary. To this end we form the sum of displacements in (91.4) and (91.9) and get

(92.2) $$u_\alpha = C\frac{x_\alpha x_3}{r^3} + \frac{Dx_\alpha}{r(r + x_3)}, \qquad (\alpha = 1, 2),$$

(92.3) $$u_3 = C\left(\frac{\lambda + 3\mu}{\lambda + \mu}\frac{1}{r} + \frac{x_3^2}{r^3}\right) + \frac{D}{r},$$

The distribution of tractions over the surface of the hemisphere, corresponding to the displacements (92.2) and (92.3), can be got by adding the tractions in (91.7) and (91.11). From computations leading to formulas (91.8) and (91.12), it is obvious that the resultant force on the surface of the hemisphere acts in the x_3-direction and has the magnitude[2]

$$R = -\frac{4\pi\mu C(\lambda + 2\mu)}{\lambda + \mu} - 4\pi\mu D.$$

Since this represents the action on the hemisphere from the side of the medium, we equate R to $-P$ and get the equation

(92.4) $$P = \frac{4\pi\mu C(\lambda + 2\mu)}{\lambda + \mu} + 4\pi\mu D,$$

involving two unknown constants C and D. Another equation involving these constants is got by imposing the conditions (92.1).

[1] Several problems in this category have been worked out by J. Boussinesq, Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques (1885).

[2] This is one-half the sum of the values given by (91.8) and (91.12), wherein the integration was performed over the entire sphere.

Forming the sum of appropriate stress components in (91.6) and (91.10) with $x_3 = 0$, we get,

$$\tau_{\alpha 3} = -\frac{2\mu C x_\alpha}{r^3}\frac{\mu}{\lambda+\mu} - 2\mu D\frac{x_\alpha}{r^3}, \qquad (\alpha = 1, 2),$$
$$\tau_{33} = 0,$$

so that

(92.5) $$\frac{2\mu^2 C}{r^3(\lambda+\mu)} + \frac{2\mu D}{r^3} = 0.$$

The solution of (92.5) and (92.4) yields

$$C = \frac{P}{4\pi\mu}, \qquad D = -\frac{P}{4\pi(\lambda+\mu)},$$

and the substitution of these values in (92.3) gives,

(92.6) $$\begin{cases} u_\alpha = \dfrac{P}{4\pi\mu}\dfrac{x_3 x_\alpha}{r^3} - \dfrac{P}{4\pi(\lambda+\mu)}\dfrac{x_\alpha}{r(x_3+r)}, & (\alpha = 1, 2), \\ u_3 = \dfrac{P}{4\pi\mu}\dfrac{x_3^2}{r^3} + \dfrac{P(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\dfrac{1}{r}. \end{cases}$$

It is worth noting[1] that at a great distance from the origin the displacements vanish as $1/r$, and hence the stresses vanish as $1/r^2$. In this connection it should also be observed that the concept of the concentrated load is a mathematical abstraction resulting from specific assumptions concerning the behavior of continuous distributions of loads when a definite limiting process is followed. It is not surprising, therefore, that different limiting processes might yield singular solutions different from (92.6). A decision about the practical validity of any given singular solution should rest on physical rather than mathematical grounds. The definition of the concentrated load in the instance of curved surfaces obviously involves an even greater degree of arbitrariness. Because of the usefulness of the solution in the form (92.6) it is natural to use it as a criterion for an acceptable definition of the concentrated load acting on a curved surface.[2]

The solutions (92.6) can be generalized, in an obvious way, to yield the displacements produced in an infinite region $x_3 \geq 0$ by suitably restricted continuous distributions of normal loads.

[1] See remarks in Sec. 74 regarding the behavior of displacements and stresses in the two-dimensional case and their bearing on the uniqueness of solution.

[2] See in this connection:

E. Sternberg and F. Rosenthal, "The Elastic Sphere under Concentrated Loads," *Journal of Applied Mechanics*, vol. 19, No. 4 (1952), pp. 413–421.

E. Sternberg and R. A. Eubanks, "On the Singularity at a Concentrated Load Applied to a Curved Surface," A Technical Report to ONR, Department of Mechanics, Illinois Institute of Technology (1953).

A. Huber, "The Elastic Sphere under Concentrated Torques," *Quarterly of Applied Mathematics*, vol. 13 (1955), pp. 98–102.

If we let $p(\xi, \eta)$ be the distributed normal load acting at the point (ξ, η) of the x_1x_2-plane, the resultant force P on an element of area $d\sigma$ is $P = p(\xi, \eta)\, d\sigma$. Inserting this in (92.6) and integrating over the x_1x_2-plane, we get,

$$
(92.7) \qquad \begin{cases} u_\alpha = \dfrac{x_3 x_\alpha}{4\pi\mu} \displaystyle\iint_{-\infty}^{\infty} \frac{p(\xi, \eta)\, d\xi\, d\eta}{r^3} - \frac{x_\alpha}{4\pi(\lambda + \mu)} \iint_{-\infty}^{\infty} \frac{p(\xi, \eta)\, d\xi\, d\eta}{r(r + x_3)}, \\ u_3 = \dfrac{x_3^2}{4\pi\mu} \displaystyle\iint_{-\infty}^{\infty} \frac{p(\xi, \eta)\, d\xi\, d\eta}{r^3} + \frac{\lambda + 2\mu}{4\pi\mu(\lambda + \mu)} \iint_{-\infty}^{\infty} \frac{p(\xi, \eta)\, d\xi\, d\eta}{r}, \end{cases}
$$

where $r^2 = (x_1 - \xi)^2 + (x_2 - \eta)^2 + x_3^2$.

The evaluation of the double integrals in (92.7) presents serious computational difficulties except in those cases where simplifying assumptions are made about the nature of the load $p(\xi, \eta)$ and the shape of the region over which the load is distributed. If the load is axially symmetric about the x_3-axis, it is possible to deduce tractable expressions for the displacements by the method of Hankel transforms.[1]

A solution of the problem of deformation of the semi-infinite elastic half space by the concentrated force acting in the interior of the solid was given by Mindlin.[2] Mindlin's solution specializes to that of Boussinesq when the force is assumed to act on the boundary of the solid.

93. The Problem of Boussinesq. As an illustration of the use of general integrals of Navier's equations

$$
(93.1) \qquad \mu\nabla^2 u_j + (\lambda + \mu)\vartheta_{,j} = 0,
$$

we construct a solution of the second boundary-value problem for the semi-infinite region $x_3 > 0$ bounded by the plane $x_3 = 0$.[3]

[1] I. M. Sneddon, Fourier Transforms (1951), pp. 468–486. This book contains a treatment of several problems concerned with the deformation of semi-infinite elastic media.

The torsion of an elastic half-space by shearing forces distributed over a circular area was considered by N. A. Rostovcev, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 19 (1955), pp. 55–60.

[2] R. D. Mindlin, *Comptes rendus hebdomadaires des séances de l'académie des sciences*, Paris, vol. 201 (1935), pp. 536–537; *Physics*, vol. 7 (1936), pp. 195–202. An exposition of Mindlin's work is contained in H. M. Westergaard's monograph Theory of Elasticity and Plasticity (1952), pp. 142–148.

[3] The first, second, and certain types of mixed boundary-value problems of elasticity for the semi-infinite region bounded by a plane are associated with the names of J. Boussinesq and V. Cerruti. These authors solved a number of special problems with the aid of potential theory. A résumé of earlier work is contained in Chap. 10 of Love's Treatise. Love [*Philosophical Transactions of the Royal Society* (*London*) (*A*), vol. **228** (1929), p. 377] applied the Boussinesq method to study the deformation of the semi-infinite space by pressures distributed over a circle and a rectangle. An account of recent developments in related problems utilizing the Fourier and related transforms, is contained in I. N. Sneddon's book Fourier Transforms (1951), pp. 450–

Since the displacements u_j are biharmonic functions, they can be represented in the form

$$u_j = \varphi_j + x_3\psi_{,j}, \tag{93.2}$$

where the φ_j and ψ are harmonic functions. These functions, as noted in Sec. 90, are not independent since the u_j satisfy Navier's equations. Indeed, on substituting (93.2) in (93.1) we easily find that

$$[(\lambda + 3\mu)\psi_{,3} + (\lambda + \mu)\varphi_{k,k}]_{,j} = 0.$$

Hence, on disregarding the nonessential constant, we get

$$\psi_{,3} = -\frac{\lambda + \mu}{\lambda + 3\mu}\varphi_{k,k}. \tag{93.3}$$

The functions φ_j and ψ must be chosen so that, on the boundary $x_3 = 0$, the displacements u_j assume specified values $f_j(x_1, x_2)$ and vanish at infinity in a suitable manner. Setting $x_3 = 0$ in (93.2), we see that the harmonic functions φ_i are required to satisfy the boundary conditions,

$$\varphi_j(x_1, x_2, 0) = f_j(x_1, x_2). \tag{93.4}$$

The determination of the φ_j has thus been reduced to the familiar problem in potential theory, and there are several methods available for constructing these functions. Perhaps the simplest of these is a method based on the Fourier integral representation of harmonic functions.

If we suppose that

$$\varphi_j(x_1, x_2, x_3) = \iint_{-\infty}^{\infty} g_j(\alpha, \beta)e^{\gamma x_3 + i(\alpha x_1 + \beta x_2)}\, d\alpha\, d\beta, \tag{93.5}$$

where $i^2 = -1$, and require that (93.5) represent harmonic functions vanishing for $x_3 = \infty$, we find that $\gamma = -\sqrt{\alpha^2 + \beta^2}$. Moreover, since the φ_j satisfy conditions (93.4),

$$f_j(x_1, x_2) = \iint_{-\infty}^{\infty} g_j(\alpha, \beta)e^{i(\alpha x_1 + \beta x_2)}\, d\alpha\, d\beta. \tag{93.6}$$

But it is well known[1] that properly restricted functions $f_j(x_1, x_2)$ can be represented by the Fourier integral, in the form (93.6), where

$$g_j(\alpha, \beta) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} f_j(\xi, \eta)e^{-i(\alpha\xi + \beta\eta)}\, d\xi\, d\eta. \tag{93.7}$$

510, and a comprehensive treatment of the contact problems of elasticity (including the study of deformation produced by a rigid stamp, as a special case) is presented in I. Ya. Shtaerman's monograph The Contact Problem of Elasticity (1949) (in Russian).

[1] See, for example, Courant-Hilbert, Methoden der mathematischen Physik, vol. 1, Chap. II, or I. N. Sneddon, Fourier Transforms, Chap. 1.

The substitution from (93.7) in (93.5) then yields the desired functions φ_j.

To determine the displacements, we also need the functions $\psi_{,j}$. But once the φ_j are determined, we find, on integrating (93.3) with respect to x_3,

$$(93.8) \quad \psi = -\frac{\lambda+\mu}{\lambda+3\mu}\iint\limits_{-\infty}^{\infty} \frac{\gamma g_3 + i(\alpha g_1 + \beta g_2)}{\gamma} e^{\gamma x_3 + i(\alpha x_1 + \beta x_2)}\, d\alpha\, d\beta.$$

The substitution from (93.5) and (93.8) in (93.2) then gives,

$$u_1 = \iint\limits_{-\infty}^{\infty} \left\{ g_1(\alpha, \beta) - \frac{\lambda+\mu}{\lambda+3\mu}\frac{i\alpha x_3}{\gamma}[\gamma g_3 + i(\alpha g_1 + \beta g_2)]\right\} e^{\gamma x_3 + i(\alpha x_1 + \beta x_2)}\, d\alpha\, d\beta,$$

$$u_2 = \iint\limits_{-\infty}^{\infty} \left\{ g_2(\alpha, \beta) - \frac{\lambda+\mu}{\lambda+3\mu}\frac{i\beta x_3}{\gamma}[\gamma g_3 + i(\alpha g_1 + \beta g_2)]\right\} e^{\gamma x_3 + i(\alpha x_1 + \beta x_2)}\, d\alpha\, d\beta,$$

$$u_3 = \iint\limits_{-\infty}^{\infty} \left\{ g_3(\alpha, \beta) - \frac{\lambda+\mu}{\lambda+3\mu} x_3[\gamma g_3 + i(\alpha g_1 + \beta g_2)]\right\} e^{\gamma x_3 + i(\alpha x_1 + \beta x_2)}\, d\alpha\, d\beta.$$

The evaluation of these double integrals is a formidable problem. If the stress distribution in the region $x_3 > 0$ is axially symmetric, the problem can be treated more effectively by the integral equations and Hankel transform methods. Problems of the indentation of a semi-infinite space by a rigid punch of circular and elliptical cross sections are in this category.[1]

The method of Fourier integrals can also be applied to solve the corresponding first boundary-value problem, but since such calculations present no points of novelty, we do not include them here.[2]

The possibility of reducing the elastostatic problem for the semi-infinite space to the simpler problem in potential theory hinged on the special form (93.2) of the general solution of Eqs. (93.1). In applying this method to problems involving spherical boundaries, it is natural to take solutions in the form

$$u_i = \varphi_i + (r^2 - a^2)\psi_{,i}.$$

If the displacements are specified on the surface of the sphere $r = a$, one is led to the Dirichlet problem for the sphere. However, instead of selecting this mode of attack, we solve the problem of elastic equilibrium of the sphere with the aid of certain orthogonal functions.

94. Spherical Shell under External and Internal Pressures. In rare instances an interesting problem in elasticity can be solved by quite ele-

[1] See I. N. Sneddon, Fourier Transforms (1951), Chap. 10; I. Ya. Shtaerman and A. I. Lourje, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 5 (1941); I. Ya. Shtaerman, The Contact Problem of Elasticity (1949), pp. 191–196, 205–210.

[2] See E. Trefftz, *Handbuch der Physik* (1928), vol. 6.

mentary means. Such is the problem of deformation of a spherical shell by uniform internal and external pressures.

Let the internal and external radii of the shell be a_1 and a_2, respectively, and let the interior pressure be p_1 and the exterior pressure p_2. We take the center of the shell to be at the origin and consider the system

$$\mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} = 0, \qquad (i = 1, 2, 3), \tag{94.1}$$

with appropriate boundary conditions.

Since the deformation of the shell is symmetric with respect to the origin, we take displacements in the form

$$u_i = \varphi(r)x_i, \tag{94.2}$$

where $r^2 = x_ix_i$ and φ depends only on r. The substitution from (94.2) in (94.1) yields the equation

$$\frac{d^2\varphi}{dr^2} + \frac{4}{r}\frac{d\varphi}{dr} = 0,$$

whose general solution is

$$\varphi(r) = A_1 + \frac{A_2}{r^3}. \tag{94.3}$$

The stress-strain relations

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}),$$

upon using (94.2), give

$$\tau_{ij} = \lambda\vartheta\delta_{ij} + 2\mu\left[\varphi\delta_{ij} + \frac{1}{r}\varphi'(r)x_ix_j\right], \tag{94.4}$$

where $\vartheta = 3A_1$. The stress T_r in the radial direction $\nu_i = x_i/r$ is given by

$$T_r = \tau_{ij}\nu_i\nu_j,$$

and, on inserting in this formula from (94.4) and (94.3), we find

$$T_r = (3\lambda + 2\mu)A_1 - \frac{4\mu A_2}{r^3}. \tag{94.5}$$

Also, the stress T_θ acting on the planar element with the unit normal n_i orthogonal to ν_i can be easily computed from

$$\begin{aligned} T_\theta &= \tau_{ij}n_in_j, \\ &= (\lambda\vartheta + 2\mu\varphi)n_in_i + \frac{2\mu}{r}\varphi'(r)(n_ix_i)^2. \end{aligned}$$

But $n_in_i = 1$, and $n_ix_i = 0$, so that

$$\begin{aligned} T_\theta &= \lambda\vartheta + 2\mu\varphi \\ &= (3\lambda + 2\mu)A_1 + \frac{2\mu A_2}{r^3}. \end{aligned} \tag{94.6}$$

For the determination of the A_i we have the boundary conditions:

$$T_r = -p_1, \qquad \text{for } r = a_1,$$
$$T_r = -p_2, \qquad \text{for } r = a_2.$$

Inserting these in (94.5), we find that

$$A_1 = \frac{p_1 a_1^3 - p_2 a_2^3}{(3\lambda + 2\mu)(a_2^3 - a_1^3)},$$
$$A_2 = \frac{a_1^3 a_2^3 (p_1 - p_2)}{4\mu(a_2^3 - a_1^3)},$$

and hence (94.5) and (94.6) become:

$$(94.7) \qquad \begin{cases} T_r = \dfrac{p_1 a_1^3 - p_2 a_2^3}{a_2^3 - a_1^3} - \dfrac{a_1^3 a_2^3}{r^3} \dfrac{p_1 - p_2}{a_2^3 - a_1^3}, \\ T_\theta = \dfrac{p_1 a_1^3 - p_2 a_2^3}{a_2^3 - a_1^3} + \dfrac{a_1^3 a_2^3}{2r^3} \dfrac{p_1 - p_2}{a_2^3 - a_1^3}. \end{cases}$$

If the external pressure $p_2 = 0$, (94.7) yield,

$$T_r = \frac{p_1 a_1^3}{a_2^3 - a_1^3}\left(1 - \frac{a_2^3}{r^3}\right) \leq 0,$$
$$T_\theta = \frac{p_1 a_1^3}{a_2^3 - a_1^3}\left(1 + \frac{a_2^3}{2r^3}\right) > 0.$$

Thus, the maximum tension $(T_\theta)_{max}$ is at the inner surface of the shell. We have,

$$(T_\theta)_{max} = \frac{p_1}{2} \frac{2a_1^3 + a_2^3}{a_2^3 - a_1^3},$$

and if the shell is of small thickness $t = a - b$, we get an approximate formula

$$(T_\theta)_{max} \doteq \frac{p_1 a_2}{2t}.$$

The maximum extension $e_{\theta\theta}$ obviously will occur on the inner surface of the shell, so that the yielding will begin on the inner surface.

Most of the results recorded above have been deduced[1] by Lamé.

95. Spherical Harmonics. The considerations of Sec. 90 indicate the great usefulness of harmonic functions in elasticity. In solving the problems of equilibrium of an elastic sphere, one special class of harmonic functions, known as *spherical harmonics*, is particularly useful. The essential facts about these functions are summarized in this section.

We first determine a class of particular solutions of Laplace's equation

$$(95.1) \qquad \Phi_{,ii} = 0, \qquad (i = 1, 2, 3),$$

[1] G. Lamé, Leçons sur la théorie de l'élasticité (1852).

in the form of homogeneous polynomials of degree n. That is, the solutions we seek have the form

$$\Phi_n = \sum_{p+q+r=n} a_{pqr} x_1^p x_2^q x_3^r, \tag{95.2}$$

where the sum is extended over all positive integral values p, q, r such that $p + q + r = n$.

It is obvious that the polynomial of degree 0, satisfying (95.1), is $\Phi_0 = a_0$, where a_0 is a constant. The polynomial

$$\Phi_1 = a_i x_i, \qquad (i = 1, 2, 3),$$

clearly satisfies (95.1) for an arbitrary choice of the constants a_i. In this case we have three linearly independent solutions of (95.1), namely, x_1, x_2, x_3. The linear combination of these solutions is the most general solution of Laplace's equation in the form of the homogeneous polynomial of degree 1.

If we take the homogeneous polynomial of degree 2, namely,

$$\Phi_2 = a_{ij} x_i x_j, \qquad (i, j = 1, 2, 3), \tag{95.3}$$

and substitute it in (95.1), we obtain one relation

$$a_{11} + a_{22} + a_{33} = 0$$

connecting six distinct constants in (95.3). Hence there are five linearly independent homogeneous polynomials of degree 2 that satisfy (95.1). These polynomials can be determined explicitly by setting

$$a_{33} = -(a_{11} + a_{22})$$

in (95.3). We thus get,

$$\Phi_2 = a_{11}(x_1^2 - x_3^2) + a_{22}(x_2^2 - x_3^2) + a_{12} x_1 x_2 + a_{23} x_2 x_3 + a_{31} x_3 x_1.$$

Hence the desired polynomials are

$$x_1^2 - x_3^2, \qquad x_2^2 - x_3^2, \qquad x_1 x_2, \qquad x_2 x_3, \qquad x_3 x_1,$$

and every solution of (95.1) in the form of the homogeneous polynomial of degree 2 is a linear combination of these five linearly independent solutions.

By taking a homogeneous polynomial,

$$\Phi_3 = a_{ijk} x_i x_j x_k, \tag{95.4}$$

of degree 3, substituting it in (95.1), and equating in the resulting expression the coefficients of the x_i to zero, we obtain three relations among ten a_{ijk} in (95.4). Accordingly there are seven linearly independent homogeneous polynomials of degree 3 that satisfy Laplace's equation.

It is not difficult to prove[1] that, in general, there are $2n + 1$ linearly independent homogeneous polynomials of degree n satisfying (95.1). The polynomials are called the *integral harmonics of degree n*. It is worth remarking that the corresponding polynomials in the two-dimensional case can be got by separating into real and imaginary parts $(x_1 + ix_2)^n$.

If we consider an integral harmonic Φ_n of degree n, its derivatives $\Phi_{n,i}$, clearly, satisfy Laplace's equation, and since the $\Phi_{n,i}$ are homogeneous polynomials of degree $n - 1$, they are integral harmonics of degree $n - 1$.

From integral harmonics Φ_n one can deduce an important class of solutions of Laplace's equation in spherical coordinates known as *spherical harmonics*.

We introduce the transformation

$$\begin{cases} x_1 = r \sin\theta \cos\varphi, \\ x_2 = r \sin\theta \sin\varphi, \\ x_3 = r \cos\theta, \end{cases} \tag{95.5}$$

and, on substituting from (95.5) in the integral harmonic $\Phi_n(x_1, x_2, x_3)$, we get

$$\Phi_n(x_1, x_2, x_3) \equiv r^n Y_n(\theta, \varphi), \tag{95.6}$$

where $Y_n(\theta, \varphi)$ is a polynomial in $\sin\theta$, $\cos\theta$, $\sin\varphi$, and $\cos\varphi$. The function $Y_n(\theta, \varphi)$ is termed *surface*, or *zonal*, harmonic and $r^n Y_n(\theta, \varphi)$ is the *solid spherical harmonic*. Inasmuch as the number of linearly independent integral harmonics of degree n is $2n + 1$, the number of linearly independent surface harmonics $Y_n(\theta, \varphi)$ is also $2n + 1$.

We deduce next an explicit representation of the surface harmonic $Y_n(\theta, \varphi)$, by investigating the solutions of Laplace's equation in spherical coordinates in the form

$$\Phi = f(r) Y(\theta, \varphi). \tag{95.7}$$

In spherical coordinates Eq. (95.1) reads:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0. \tag{95.8}$$

Substituting (95.7) in (95.8) and separating variables yields,

$$\frac{\dfrac{d}{dr}[r^2 f'(r)]}{f(r)} = -\frac{\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\varphi^2}}{Y},$$

[1] See, for example, Courant-Hilbert, Methoden der mathematischen Physik, vol. 1, Chap. 7.

which implies that $f(r)$ and $Y(\theta, \varphi)$ satisfy the equations:

$$r^2f''(r) + 2rf'(r) - kf(r) = 0, \tag{95.9}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + kY = 0, \tag{95.10}$$

where k is a constant.

We are interested only in continuous solutions of Eq. (95.10) since, as observed above, the surface harmonics are trigonometric polynomials. Equation (95.10) will have such (nontrivial) solutions only for certain values of the parameter k, and our problem is to determine these characteristic values and construct the corresponding functions $Y(\theta, \varphi)$. On comparing (95.7) with (95.6) we see that $f(r)$ satisfying Eq. (95.9) is r^n, and on inserting this in (95.9) we get an infinite number of the values of k, namely

$$k = n(n + 1), \qquad n = 0, 1, 2, \dots .$$

The substitution of this value of k in (95.10) then yields the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\varphi^2} + n(n + 1)Y_n = 0 \tag{95.11}$$

for the surface harmonic $Y_n(\theta, \varphi)$. The considerations pertaining to the number of linearly independent spherical harmonics leads us to expect that, corresponding to each characteristic number $k = n(n + 1)$, there will be $2n + 1$ linearly independent solutions $Y_n(\theta, \varphi)$ of (95.11). We can obtain these solutions by taking

$$Y_n(\theta, \varphi) = Q_n(\theta)R_n(\varphi), \tag{95.12}$$

for $Y_n(\theta, \varphi)$ is known to be a trigonometric polynomial in $\cos\theta$, $\sin\theta$, $\cos\varphi$, and $\sin\varphi$.

The substitution of (95.12) in (95.11) leads, by familiar argument, to the pair of equations:

$$\frac{R_n''(\varphi)}{R_n(\varphi)} = -\frac{(Q_n'\sin\theta)'\sin\theta}{Q_n} - n(n + 1)\sin^2\theta = -m^2, \tag{95.13}$$

if we recall that $R_n(\varphi)$ in (95.12) is a trigonometric function. From (95.13) we see at once that suitable linearly independent solutions for R_n are

$$R_n(\varphi) = \begin{cases} \sin m\varphi, \\ \cos m\varphi, \end{cases}$$

where $m = 0, 1, 2, \dots, n$.

The equation for $Q_n(\theta)$ can be cast in the standard form by introducing a new independent variable $x = \cos\theta$. On making this change

and writing $Q_n(\theta) = P_n^{(m)}(x)$, we find that $P_n^{(m)}(x)$ satisfies Legendre's equation,

$$\frac{d}{dx}\left[(1 - x^2)\frac{dP_n^{(m)}(x)}{dx}\right] + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]P_n^{(m)}(x) = 0, \quad (m = 0, 1, 2, \ldots, n). \tag{95.14}$$

There are two linearly independent solutions of this equation, only one of which is continuous in the interval $|x| \le 1$, that is, for $0 \le \theta \le \pi$. This solution is

$$P_n^{(m)}(x) = (1 - x^2)^{m/2}\frac{d^m P_n(x)}{dx^m},$$

where $P_n(x)$ are the Legendre polynomials defined by the formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2 - 1)^n}{dx^n}.$$

We are now in a position to write out an explicit expression for the surface harmonic $Y_n(\theta, \varphi)$. It is,

$$Y_n(\theta, \varphi) = a_0 P_0(\cos\theta) + \sum_{m=1}^{n}(a_m \cos m\varphi + b_m \sin m\varphi)P_n^{(m)}(\cos\theta), \tag{95.15}$$

where a_0, a_m, and b_m are arbitrary constants.

The solution (95.6) of Laplace's equation, with $Y_n(\theta, \varphi)$ defined by (95.15), is continuous throughout all space. Another solution, which becomes infinite for $r = 0$, and which is no longer an integral harmonic, can be got by noting that the second solution of (95.9) is $f(r) = r^{-n-1}$. It is

$$\Phi_n = \frac{Y_n(\theta, \varphi)}{r^{n+1}}. \tag{95.16}$$

The functions $Y_n(\theta, \varphi)$ can be viewed as being defined on the surface of the unit sphere. They possess a number of remarkable properties, which we state without proofs.[1]

The set of functions $\{Y_n(\theta, \varphi)\}$ is orthogonal on the surface Σ of the unit sphere, so that

$$\int_\Sigma Y_i(\theta, \varphi)Y_j(\theta, \varphi)\, d\sigma = 0, \qquad \text{if } i \ne j.$$

Moreover, this set is complete in the sense that every function $f(\theta, \varphi)$, specified on the unit sphere Σ, whose square is integrable over Σ, can be represented in the series of spherical harmonics which converges in the

[1] See, E. W. Hobson, Theory of Spherical and Ellipsoidal Harmonics (1931).

mean to $f(\theta, \varphi)$. If $f(\theta, \varphi)$ is of class C^2 on Σ, the series converges to $f(\theta, \varphi)$ uniformly. In fact, we have the following representation:

$$f(\theta, \varphi) = a_0^{(0)} + \sum_{n=1}^{\infty} \left\{a_0^{(n)} P_n(\cos\theta) + \sum_{m=1}^{n} (a_m^{(n)} \cos m\varphi + b_m^{(n)} \sin m\varphi) P_n^{(m)}(\cos\theta)\right\},$$

where

$$a_m^{(n)} = \frac{(2n+1)(n-m)!}{2\pi\delta_m(n+m)!} \int_\Sigma f(\theta, \varphi) P_n^{(m)}(\cos\theta) \cos m\varphi \, d\sigma,$$

$$b_m^{(n)} = \frac{(2n+1)(n-m)!}{2\pi\delta_m(n+m)!} \int_\Sigma f(\theta, \varphi) P_n^{(m)}(\cos\theta) \sin m\varphi \, d\sigma,$$

with $\delta_m = 2$ if $m = 0$, $\delta_m = 1$ if $m > 0$, and $P_n^{(0)}(\cos\theta) \equiv P_n(\cos\theta)$.

This representation permits us to solve the problem of Dirichlet for the sphere of radius a in the series of solid harmonics. We first represent the function $f(\theta, \varphi)$, specified on the surface of the sphere in the series

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi),$$

and then form the series

$$\Phi(r, \theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) \left(\frac{r}{a}\right)^n.$$

This gives the formal solution for the region $r < a$. The solution of the corresponding exterior problem is

$$\Phi(r, \theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) \left(\frac{a}{r}\right)^{n+1}, \qquad r > a.$$

96. Elastic Equilibrium of a Sphere and Other Problems. We have just seen that the problem of Dirichlet for the sphere can be solved in the series of spherical harmonics. It is natural to make an attempt to solve in an analogous way the problems of elastic equilibrium of the sphere. To do this, it is necessary to construct the family of particular solutions of Navier's equations

$$\mu\nabla^2 u_i + (\lambda + \mu) u_{k,ki} = 0, \tag{96.1}$$

which are such that on the surface of the sphere of radius a they reduce to a complete set of surface harmonics $Y_n(\theta, \varphi)$.

Following Lord Kelvin,[1] we seek a set of particular solutions of (96.1) in the form (cf. Sec. 90),

$$u_i = \varphi_i(x_1, x_2, x_3) + cr^2\varphi_{k,ki}, \qquad (i, k = 1, 2, 3) \tag{96.2}$$

where the φ_i are integral harmonics of degree n and c is a constant.

We note that the second term in (96.2) is not an integral harmonic because of the presence of the factor r^2, but $\varphi_{k,ki}$ being the sum of the second derivatives of φ_i is an integral harmonic of degree $n - 2$. Nevertheless, on the surface of the sphere $r = a$ both terms in (96.2) reduce to the surface harmonics, the first term being the surface harmonic of the type $Y_n(\theta, \varphi)$ and the second $Y_{n-2}(\theta, \varphi)$. This fact will permit us to combine the particular solutions (96.2) in such a way that on the surface of the sphere they reduce to the specified displacements. The value of the constant c is readily determined by substitution of (96.2) in (96.1). Making use of the Euler formula $x_i\varphi_{,i} = n\varphi$ for homogeneous functions of degree n and of the assumption that the φ_i are harmonic, we easily find

$$c = -\frac{\lambda + \mu}{2[\lambda(n - 1) + (2n - 3)\mu]}. \tag{96.3}$$

Since c depends on the degree n of the integral harmonic φ_i, we denote it by $c^{(n)}$ and write the formal solution of Navier's equations in the form

$$u_i = \sum_{n=0}^{\infty} (\varphi_i^{(n)} + c^{(n)}r^2\varphi_{k,ki}^{(n)}),$$

But $\varphi_{i,jk}^{(0)} = \varphi_{i,jk}^{(1)} = 0$, and hence

$$u_i = \sum_{n=0}^{\infty} (\varphi_i^{(n)} + c^{(n+2)}r^2\varphi_{k,ki}^{(n+2)}). \tag{96.4}$$

If we go over into spherical coordinates with the aid of (95.5) and set $r = a$, each term in the series (96.4) becomes a surface harmonic of degree n.

Now if the displacements $u_i = f_i(\theta, \varphi)$, specified on the surface of the sphere $r = a$, are represented in the series of surface harmonics as

$$u_i = \sum_{n=0}^{\infty} A_i^{(n)}(\theta, \varphi), \tag{96.5}$$

it should prove possible to determine the solid harmonics $\varphi_i^{(n)}$ in (96.4) so that the series (96.4) reduces to (96.5) when r is set equal to a.

[1] Sir William Thomson, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 153 (1863); Mathematical and Physical Papers, vol. 3 (1890), p. 351.

We next form the series of solid harmonics,

$$\sum_{n=0}^{\infty} A_i^{(n)} \left(\frac{r}{a}\right)^n \tag{96.6}$$

and note that the series of solid harmonics,

$$\sum_{n=0}^{\infty} (\varphi_i^{(n)} + c^{(n+2)} a^2 \varphi_{k,ki}^{(n+2)}) \tag{96.7}$$

must converge for $r = a$ to the same function $f_i(\theta, \varphi)$ as the series (96.6).

Making use of the theorem on uniqueness of representation in the series of solid harmonics, we can write

$$\sum_{n=0}^{\infty} A_i^{(n)} \left(\frac{r}{a}\right)^n = \sum_{n=0}^{\infty} (\varphi_i^{(n)} + c^{(n+2)} a^2 \varphi_{k,ki}^{(n+2)}), \qquad \text{for } r = a,$$

and deduce

$$A_i^{(n)} \left(\frac{r}{a}\right)^n = \varphi_i^{(n)} + c^{(n+2)} a^2 \varphi_{k,ki}^{(n+2)}. \tag{96.8}$$

Using (96.8), we compute

$$\left[A_i^{(n)} \left(\frac{r}{a}\right)^n\right]_{,i} = \varphi_{i,i}^{(n)} + c^{(n+2)} a^2 \varphi_{k,kii}^{(n+2)},$$

and since $\varphi_{k,k}^{(n+2)}$ is harmonic, we get

$$\varphi_{i,i}^{(n)} = \left[A_i^{(n)} \left(\frac{r}{a}\right)^n\right]_{,i}. \tag{96.9}$$

Thus $\varphi_{i,i}^{(n)}$ is completely determined since the $A_i^{(n)}$ are known functions.

The substitution from (96.9) in (96.8) determines the $\varphi_i^{(n)}$ in the form,

$$\varphi_i^{(n)} = A_i^{(n)} \left(\frac{r}{a}\right)^n - c^{(n+2)} a^2 \left[A_k^{(n+2)} \left(\frac{r}{a}\right)^{n+2}\right]_{,ki}. \tag{96.10}$$

Hence the solution (96.4) can be written entirely in terms of the $A_i^{(n)}$ determined in (96.5). It is

$$u_i = \sum_{n=0}^{\infty} \left\{A_i^{(n)} \left(\frac{r}{a}\right)^n + c^{(n+2)} (r^2 - a^2) \left[A_k^{(n+2)} \left(\frac{r}{a}\right)^{n+2}\right]_{,ki}\right\}. \tag{96.11}$$

Despite its formal simplicity the solution (96.11) is difficult to apply to specific problems because it can be effectively carried out only for very simple distributions of assigned displacements. Kelvin also applied the

method described here to solve the first boundary-value problem for the sphere. Although, conceptually, this problem is not any more difficult than the second boundary-value problem, the necessary calculations are considerably heavier.[1]

An obvious generalization of the method, making use of particular solutions in the form $\varphi^{(n)}r^{-2n-1}$, where the $\varphi^{(n)}$ are surface harmonics, has enabled Kelvin to treat the problems of equilibrium of an elastic shell. These problems have recently been reconsidered by Lourje, who, in following the Kelvin mode of attack, uses the Neuber-Papkovich expressions for displacements, involving four harmonic functions instead of three used by Kelvin. This results in some simplifications, enabling Lourje to carry out the computations more fully.[2]

When the radius of the inner shell is small, the solutions indicate the nature of stress concentration in a large body near a spherical cavity. A brief survey of the technically important problems on stress concentration is contained[3] in Timoshenko and Goodier's Theory of Elasticity.

Among the three-dimensional problems for which explicit solutions are available are several problems in the category of contact problems of elasticity. The problem of deformation of an elastic half space by a rigid circular punch, first treated by Boussinesq, was developed in some detail by Shtaerman and Lourje and, more recently, by Leonov.[4] A systematic treatment of this and related contact problems of elasticity will be found in I. Ya. Shtaerman's monograph entitled The Contact Problem of the Theory of Elasticity (1949).

Since exact solutions of the three-dimensional problems pose serious mathematical difficulties,[5] recourse is made to approximate solutions

[1] See, for example, A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity (1929), pp. 267–270.

[2] A. I. Lourje, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 17 (1953), pp. 311–332.

[3] See also H. M. Westergaard, Theory of Elasticity and Plasticity (1952), pp. 154–157 and R. A. Eubanks, "Stress Concentration Due to a Hemispherical Pit at a Free Surface," *Journal of Applied Mechanics,* vol. 21 (1954), pp. 57–62.

[4] I. Ya. Shtaerman and A. I. Lourje, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 5 (1941); M. Ya. Leonov, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 17 (1953), pp. 87–98. See also N. A. Rostovcev, "On the Problem of Torsion of an Elastic Half-space," *Prikl. Mat. Mekh, Akademiya Nauk SSSR,* vol. 19 (1955), pp. 55–60.

[5] Interesting recent contributions to exact solutions of the axially symmetric problems of elasticity are contained in two papers by E. Sternberg, R. A. Eubanks, and M. A. Sadowsky, *Journal of Applied Physics,* vol. 22 (1951), p. 1121, *Proceedings of the First United States National Congress of Applied Mechanics* (1952), and in a brief paper by G. S. Shapiro, *Doklady Akademii Nauk SSSR,* vol. 58 (1947), pp. 1309–1312, in which the equilibrium of an ellipsoid of revolution is considered.

The equilibrium of an elastic parallelepiped was considered by M. M. Filonenko-Borodich, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 15 (1951), pp. 136–148, 562–574, with the aid of Papkovich-Neuber stress functions.

based on the variational or similar techniques. The problem of elastic equilibrium of a parallelepiped was thus treated recently.[1]

97. Betti's Method of Integration. In view of the close connection of the general solutions of Navier's equations with harmonic functions, it is natural to attempt to reduce the fundamental problems of elasticity to the basic problems in potential theory. We shall see that this can be done provided that the dilatation $\vartheta = u_{i,i}$ and the rotation tensor

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i})$$

can be determined from the displacements or tractions specified on the surface of the body.

Since ϑ is a harmonic function,

$$u_i = -\frac{1}{2}\frac{\lambda + \mu}{\mu} x_i \vartheta \tag{97.1}$$

is a particular integral of Navier's equations

$$\nabla^2 u_i = -\frac{\lambda + \mu}{\mu} \vartheta_{,i}. \tag{97.2}$$

Thus, if the displacements are assigned on the surface, and if ϑ can be computed from them throughout the body, the second boundary-value problem in elasticity reduces essentially to the Dirichlet problem.

On the other hand, when the surface tractions T_i are known, we can write

$$\begin{aligned} T_i = \tau_{ij}\nu_j &= (\lambda\vartheta\delta_{ij} + 2\mu e_{ij})\nu_j \\ &= \lambda\vartheta\nu_i + 2\mu u_{i,j}\nu_j + \mu(u_{j,i} - u_{i,j})\nu_j \\ &= \lambda\vartheta\nu_i + 2\mu\frac{\partial u_i}{\partial \nu} + 2\mu\omega_{ji}\nu_j. \end{aligned}$$

Thus, on the surface Σ of the body τ,

$$\frac{du_i}{d\nu} = \frac{1}{2\mu}(T_i - \lambda\vartheta\nu_i) - \omega_{ji}\nu_j. \tag{97.3}$$

Accordingly, if ϑ and ω_{ij} can be determined from specified tractions, the first boundary-value problem reduces to the problem of Neumann.

A mode of computing these functions, devised by Betti,[2] hinges on the construction of certain auxiliary functions analogous to Green's functions in Potential Theory. To derive the described formulas, we need a special form of the Reciprocal Theorem (established in Sec. 109) stating that when an elastic body, in the absence of body forces, is subjected to the

[1] M. M. Filonenko-Borodich, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 17 (1953), pp. 464–468. See also two earlier papers by this author in vol. 15 (1951) of the same journal.

[2] E. Betti, *Il Nuovo cimento*, ser. 2, vols. 6–10 (1872ff.).

action of two systems of surface tractions T_i and T'_i producing the displacements u_i and u'_i, respectively, then

$$\int_\Sigma T_i u'_i \, d\sigma = \int_\Sigma T'_i u_i \, d\sigma. \tag{97.4}$$

In deriving this theorem it is assumed that the functions u_i, u'_i and their derivatives are continuous throughout the region τ.

If we now consider a solution of Navier's equations in the form

$$u_i^0 = \frac{\partial r^{-1}}{\partial x_i}, \tag{97.5}$$

which has a singularity at the origin,[1] and denote the corresponding tractions by T_i^0, we can apply formula (97.4) with $u'_i = u_i^0$ and $T'_i = T_i^0$ to a region bounded externally by Σ and internally by a sphere S of small radius R centered at the origin. We thus get

$$\int_\Sigma T_i u_i^0 \, d\sigma + \int_S T_i u_i^0 \, d\sigma = \int_\Sigma T_i^0 u_i \, d\sigma + \int_S T_i^0 u_i \, d\sigma. \tag{97.6}$$

But.

$$\begin{aligned}
\int_S T_i u_i^0 \, d\sigma &= \int_S \tau_{ij} \nu_j u_i^0 \, d\sigma \\
&= \int_S (\lambda\vartheta\delta_{ij} + 2\mu e_{ij}) \frac{x_j}{R} \frac{\partial r^{-1}}{\partial x_i} \, d\sigma \\
&= \int_S (\lambda\vartheta\delta_{ij} + 2\mu e_{ij}) \frac{x_i x_j}{R^4} \, d\sigma \\
&= \int_S \left(\frac{\lambda\vartheta}{R^2} + \frac{2\mu}{R^4} x_i x_j e_{ij} \right) d\sigma.
\end{aligned}$$

Since ϑ and e_{ij} are continuous at the origin and

$$\int_S x_i x_j \, d\sigma = \tfrac{4}{3}\pi R^4 \delta_{ij}, \qquad \int_S d\sigma = 4\pi R^2,$$

it follows that, as $R \to 0$,

$$\begin{aligned}
\int_S T_i u_i^0 \, d\sigma &\to 4\pi\lambda\vartheta(0) + 2\mu e_{ij}(0)\tfrac{4}{3}\pi\delta_{ij} \\
&= 4\pi(\lambda + \tfrac{2}{3}\mu)\vartheta(0).
\end{aligned} \tag{97.7}$$

We find similarly that

$$\lim_{R\to 0} \int_S T_i^0 u_i \, d\sigma = -\tfrac{16}{3}\pi\mu\vartheta(0). \tag{97.8}$$

Hence, on substituting from (97.7) and (97.8) in (97.6), we get

$$4\pi(\lambda + 2\mu)\vartheta(0) = \int_\Sigma (T_i^0 u_i - T_i u_i^0) \, d\sigma. \tag{97.9}$$

[1] We used such singular solutions in Sec. 91. The displacements (97.5) correspond to a center of uniform compression at the origin.

This formula enables one to compute the dilatation at any point (which we have chosen to be the origin) whenever *both* the displacement u_i and traction T_i are known over Σ. If only the u_i are known, we can eliminate the T_i by solving the following auxiliary problem: *Find a solution u_i' of Navier's equations in the region τ such that $u_i' = u_i^0$ on Σ.*

For if such u_i' are known, then, by the theorem (97.4),

$$\int_\Sigma T_i' u_i \, d\sigma = \int_\Sigma T_i u_i' \, d\sigma = \int_\Sigma T_i u_i^0 \, d\sigma,$$

and hence the formula (97.9) assumes the form

$$4\pi(\lambda + 2\mu)\vartheta(0) = \int_\Sigma (T_i^0 - T_i') u_i \, d\sigma, \tag{97.10}$$

where both the T_i^0 and T_i' can be computed since the corresponding displacements u_i^0 and u_i' are known.

We note that the determination of the u_i' is equivalent to finding a set of functions

$$v_i = u_i^0 - u_i'$$

satisfying the following conditions:

1. v_i satisfy Navier's equations, except at the origin.
2. $v_i = 0$ on Σ.
3. v_i become infinite at the origin in the manner (97.5).

It is thus clear that the v_i are analogous to Green's functions.

If the surface tractions T_i are specified, we can compute ϑ by finding the solutions u_i'' of Navier's equation corresponding to the tractions $T_i'' = T_i^0$ on Σ. Then, from the Reciprocal Theorem,

$$\int_\Sigma T_i^0 u_i \, d\sigma = \int_\Sigma T_i u_i'' \, d\sigma,$$

and this time we have the formula,

$$4\pi(\lambda + 2\mu)\vartheta(0) = \int_\Sigma T_i(u_i'' - u_i^0) \, d\sigma.$$

The calculation of the ω_{ij} is similar. We confine our discussion to the computation of the x_1-component of the rotation vector $\boldsymbol{\omega}$, namely,

$$\omega_1 = \omega_{32}.$$

We introduce a singular solution $\mathbf{u}^0$ with components,

$$\left(0, \frac{\partial r^{-1}}{\partial x_3}, -\frac{\partial r^{-1}}{\partial x_2}\right),$$

which corresponds to a center of rotation, at the origin, about the x_1-axis. The tractions associated with the u_i^0 are T_i^0. Using the Recip-

rocal Theorem and integrating, as before, over Σ and S, we find,

$$\begin{aligned} \lim_{R\to 0} \int_S T_i u_i^0 \, d\sigma &= 0, \\ \lim_{R\to 0} \int_S T_i^0 u_i \, d\sigma &= 4\pi\mu(u_{3,2} - u_{2,3})_0 \\ &= 8\pi\mu\omega_1(0). \end{aligned}$$

Hence,

$$8\pi\mu\omega_1(0) = \int_\Sigma (T_i u_i^0 - T_i^0 u_i)\, d\sigma. \tag{97.11}$$

In order to express this formula in terms of the surface displacements alone, we consider a regular solution u_i' of Navier's equations such that $u_i' = u_i^0$ on Σ and find, as we did for the dilatation,

$$8\pi\mu\omega_1(0) = \int_\Sigma (T_i' - T_i^0) u_i \, d\sigma,$$

where the T_i' are tractions corresponding to the displacements u_i'.

The elimination of the u_i from (97.11) is slightly more involved this time because the tractions T_i^0 associated with the singular solution u_i^0 are not self-equilibrating. Hence no solutions of the equilibrium equations analogous to u_i'' above can be found directly. To put the body in equilibrium, we introduce a second center of rotation at a point $\bar{O}$, so selected that the couple at $\bar{O}$ is equal and opposite to that at the origin O. This can be done by considering another singular solution $\bar{\mathbf{u}}^0$ with components

$$\left(0, \frac{\partial \bar{r}^{-1}}{\partial x_3}, -\frac{\partial \bar{r}^{-1}}{\partial x_2}\right),$$

where the origin of $\bar{r}$ is at $\bar{O}$.

Let the tractions corresponding to $u_i^0 - \bar{u}_i^0$ be T_i'', and let u_i'' be the regular solution of Navier's equations in τ such that on Σ it yields the tractions T_i''. Then the Reciprocal Theorem and (97.11) yield

$$\begin{aligned} 8\pi\mu[\omega_1(O) - \omega_1(\bar{O})] &= \int_\Sigma [T_i(u_i^0 - \bar{u}_i^0) - T_i'' u_i]\, d\sigma \\ &= \int_\Sigma T_i(u_i^0 - \bar{u}_i^0 - u_i'')\, d\sigma. \end{aligned}$$

The function $u_i^0 - \bar{u}_i^0 - u_i''$ is analogous to the second Green's function.

The obvious difficulty in the application of the Betti method to specific problems is in the construction of the auxiliary functions. They have been deduced for the semi-infinite region bounded by a plane and used by Cerutti[1] to solve the Boussinesq problem.

[1] V. Cerutti, *Atti della accademia dei nazinale Lincei, Rendiconti, Classe di scienze fisiche, matematiche e naturali*, Rome (1882); vol. 4 (1888).

An exposition of Cerutti's work is contained in Chap. 10 of Love's Treatise.

The integral equations for ϑ and ω have recently been derived by Arzhanykh.[1]

98. Existence of Solutions. We saw in the preceding chapter that the existence of solutions of the fundamental two-dimensional problems follows directly from the existence of solutions of certain well-known integral equations. The demonstration of existence of solutions of the three-dimensional problems can also be made to depend on the existence of the solution of integral equations of the Fredholm type or, alternatively, on the construction of Betti's auxiliary functions. We shall not pursue this subject here and shall merely remark that the matter of existence of solutions has been satisfactorily resolved for domains of great generality by Fredholm, Lauricella, Korn, Weyl, Lichtenstein, and Sherman.[2]

The caliber of mathematicians who have concerned themselves with the problem is indicative of its complexity.

An extension of the uniqueness theorems in the linear theory of elasticity to problems involving concentrated loads is provided in a report by E. Sternberg and R. A. Eubank.[3]

99. Thermoelastic Problems. We have assumed in preceding chapters that the elastic bodies undergoing deformations were maintained at constant temperatures. Thermal changes in a body are accompanied by shifts in the relative positions of particles composing the body. Such shifts, in general, cannot proceed freely, and the body becomes stressed. Under free thermal expansion of isotropic bodies a volume element in the shape of a rectangular parallelepiped with edges l_i^0 parallel to coordinate axes deforms into a similar parallelepiped with edges l_i. For small temperature changes $T(x_1, x_2, x_3)$ the relationship between l_i^0 and l_i has the

[1] *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 15 (1951), pp. 387–391.

[2] For proofs relating to the first boundary-value problem see:

A. Korn, *Annales de la faculté des sciences de Toulouse*, ser. 2, vol. 10 (1908), pp. 165–269;

H. Weyl, *Rendiconti del circolo matematico di Palermo*, vol. 39 (1915), pp. 1–49

For the second:

I. Fredholm, *Arkiv för Matematik, Astronomi och Fysik*, vol. 2 (1906), pp. 3–8.

G. Lauricella, *Atti della reale accademia nazionale dei Lincei*, ser. 5, vol. 15 (1906), pp. 426–432, vol. 16 (1907), p. 373; *Il Nuovo cimento*, ser. 5, vol. 13 (1907), pp. 104–118, 155–174, 237–262, 501–518.

A. Korn, *Annales de l'école normale supérieure*, ser. 3, vol. 24 (1907), pp. 9–75; *Rendiconti del circolo matematico di Palermo*, vol. 30 (1910), pp. 138, 336; *Mathematische Annalen*, vol. 75 (1914), pp. 497–544;

L. Lichtenstein, *Mathematische Zeitschrift*, vol. 20 (1924), pp. 21–28; vol. 24 (1925), p. 640;

D. I. Sherman, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 7 (1943), pp. 341–360.

[3] A Technical Report to the Office of Naval Research, Department of the Navy, from the Department of Mechanics, Illinois Institute of Technology, June 15 (1954).

form

$$l_i = l_i^0(1 + \alpha T),$$

where α is the coefficient of linear expansion.

Thus, the strain components e'_{ij} due to the free thermal expansion are,

$$e'_{ij} = \alpha T \delta_{ij}. \tag{99.1}$$

Since the expansion of volume elements cannot ordinarily proceed freely, the total strain e_{ij} can be thought to consist of the sum of the thermal strain e'_{ij} and the elastic strain e''_{ij} produced by the resistance of the medium to thermal expansion. Thus,

$$e_{ij} = e'_{ij} + e''_{ij}, \tag{99.2}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

In 1841, Franz Neumann[1] proposed a hypothesis that the components e''_{ij} of the elastic deformation are related to the τ_{ij} by the usual stress-strain relations, so that

$$e''_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\Theta\delta_{ij}, \tag{99.3}$$

where $\Theta = \tau_{ii}$. It is implied, of course, that the temperature changes T are so small that the elastic moduli remain sensibly constant. On taking account of (99.1) to (99.3), we get,

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \left(\frac{\sigma}{E}\Theta - \alpha T\right)\delta_{ij}, \tag{99.4}$$

and, solving these for the τ_{ij}, we obtain,

$$\tau_{ij} = \lambda\vartheta\delta_{ij} + 2\mu e_{ij} - \alpha(3\lambda + 2\mu)T\delta_{ij}, \tag{99.5}$$

where $\vartheta = e_{ii} = u_{i,i}$.

The law (99.5) is called the *Duhamel-Neumann law*, because it was also deduced in 1838 by Duhamel,[2] who proceeded from a different hypothesis, based on the conception of an elastic body as a system of material points under molecular interactions.

The substitution from (99.5) in the dynamical equations

$$\tau_{ij,j} + F_i = \rho\ddot{u}_i,$$

yields

$$\mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} + F_i - \beta T_{,i} = \rho\ddot{u}_i,$$

[1] *Abhandlungen der deutschen Akademie der Wissenschaften Berlin*, Part 2, (1841), pp. 1–254. See also his textbook Vorlesgungen über die Theorie der Elasticität der festen Körper (1885), pp. 107–120.

[2] J. M. C. Duhamel, *Mémoires par divers savants*, Paris, vol. 5 (1838), pp. 440–498.

where

$$\beta \equiv (3\lambda + 2\mu)\alpha,$$

or, in the static case,

$$\mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} = -(F_i - \beta T_{,i}). \tag{99.6}$$

The system of differential equations (99.6) must be solved subject to the specified displacements u_i or tractions T_i on the surface of the body. The tractions T_i can be expressed, of course, in terms of the displacement derivatives on substituting from (99.5) in

$$T_i = \tau_{ij}\nu_j. \tag{99.7}$$

The temperature function T is assumed to be known, and ordinarily it is determined from the solution of the Fourier heat equation.

It is clear from Eqs. (99.6), (99.7), and (99.5) that the effect of the temperature change T is equivalent to replacing the body forces F_i in Navier's equations by $F_i - \beta T_{,i}$ and to substituting $T_i + \beta T \nu_i$ for the surface tractions T_i in the boundary conditions. The additional term, $\beta T \nu_i$, is equivalent to a hydrostatic pressure Thus, formally, the elastostatic problem with assigned body forces and the thermoelastic problem are identical. As we have noted already, by introducing suitable particular integrals the problem involving body forces can always be reduced to the solution of the homogeneous Navier's equations. It is not difficult to write down a particular integral for (99.6) when the body forces F_i have potentials.[1] For let us assume a solution of (99.6) in the form

$$u_i = \varphi_{,i}, \tag{99.8}$$

where φ is a suitably differentiable scalar function. Then

$$\vartheta = u_{k,k} = \varphi_{,kk}$$

and

$$\nabla^2 u_i = \varphi_{,ikk}.$$

On substituting in (99.6), we get

$$\mu\varphi_{,ikk} + (\lambda + \mu)\varphi_{,kki} = -F_i + \beta T_{,i},$$

and if there exists a potential Φ such that $F_i = -\Phi_{,i}$, we can write,

$$(\lambda + 2\mu)\varphi_{,kki} = (\Phi + \beta T)_{,i}.$$

The integration of these equations with respect to x_i yields

$$\varphi_{,kk} = \frac{1}{\lambda + 2\mu}(\Phi + \beta T) + \text{const.}$$

[1] We recall that this is always the case with the gravitational and centrifugal forces.

Since we are concerned only with the determination of particular integrals, it suffices to seek an integral of the Poisson equation

$$\varphi_{,kk} = \frac{1}{\lambda + 2\mu} (\Phi + \beta T).$$

Such an integral can be taken in the form of the gravitational potential

$$\varphi(x) = -\frac{1}{4\pi} \int_\tau \frac{\rho(x')\, d\tau(x')}{r(x, x')} \tag{99.9}$$

due to a distribution of matter of density

$$\rho = \frac{1}{\lambda + 2\mu} (\Phi + \beta T).$$

In the solution (99.9), $r(x, x')$ is the distance from the point (x) with coordinates x_i to the point (x') with coordinates x'_i, and the integration is performed with respect to the primed variables. Once φ is determined from (99.9), the desired particular integral is given by the formulas (99.8). We note that, when the body forces vanish, the function ρ in (99.9) is simply

$$\rho = \frac{\beta}{\lambda + 2\mu} T.$$

Borchardt[1] has made use of integrals of the form (99.9) in the general discussion of the thermoelastic problems and in solving certain special problems for spheres and circular plates subjected to asymmetric temperature distributions. A method of integration of the thermoelastic equations, with the aid of integrals similar to those of Betti and Somigliana, was outlined by Rosenblatt.[2] Goodier, Mindlin, Cheng, and Mykelstad used integrals of the type (99.9) to study the effect of special temperature distributions in the infinite and semi-infinite elastic solids.[3]

Instead of dealing with Eqs. (99.6) we can start with Cauchy's equations,

$$\tau_{ij,j} + F_i = 0,$$

where the τ_{ij} satisfy appropriate compatibility conditions. The latter can be written down at once from (24.14) by replacing the F_i in (24.14) by the "effective body force components," $F_i - \beta T_{,i}$. Another way of

[1] C. W. Borchardt, *Monatsberichte der Akademie der Wissenschaft*, Berlin (1873), pp. 9–56.

[2] A. Rosenblatt, *Rendiconti del circolo matematico di Palermo*, vol. 29 (1910), pp. 324–328. See also W. Nowacki, *Arch. Mech. Stos.*, vol. 6 (1954), pp. 481–492 (in Polish).

[3] J. N. Goodier, *Philosophical Magazine*, vol. 23 (1927), pp. 1017–1032; R. D. Mindlin and D. H. Cheng, *Journal of Applied Physics*, vol. 21 (1950), pp. 926, 931; N. O. Myklestad, *Journal of Applied Mechanics* (1942), p. A-131.

deducing such equations is to insert from (99.4) in the Saint-Venant compatibility equations (10.9).

The thermoelastic problem is further complicated by the fact that in many instances it proves necessary to determine first the temperature T from the Fourier heat-conduction equation. The available exact solutions of the heat-conduction problems are limited to spheres and cylinders and to a few problems involving plates and rods subjected to special temperature distributions.[1] We shall consider some of these in the following sections.

100. Thermal Stresses in Spherical Bodies. The deformation of a spherical shell subjected to a centrally symmetric distribution of temperature can be determined[2] in the manner of Sec. 94.

We take the temperature function in the form $T(r)$, where r is measured from the center of the sphere, and seek a solution of the system (99.6) with $F_i = 0$ in the form

$$u_i = x_i\varphi(r), \qquad r^2 = x_ix_i. \tag{100.1}$$

On substitution from (100.1) in (99.6) we get the equation

$$(\lambda + 2\mu)\left(\varphi'' + \frac{4}{r}\varphi'\right) - \frac{\beta}{r}T' = 0, \tag{100.2}$$

where primes denote the derivatives with respect to r. The general solution of this equation has the form[3]

$$\varphi(r) = A_1 + \frac{A_2}{r^3} + \frac{\beta}{\lambda + 2\mu}\varphi_0(r), \tag{100.3}$$

where

$$\varphi_0(r) = \frac{1}{r^3}\int_{r_1}^{r} T(r)r^2\,dr. \tag{100.4}$$

The lower limit r_1 in the particular integral (100.4) can be chosen in any convenient, but definite, manner.

On noting (100.1), we get

$$u_{i,j} = \delta_{ij}\varphi + x_i\varphi'(r)\frac{x_j}{r},$$

[1] Several approximate solutions of the engineering problems concerned with thermal stresses in plates and rods are discussed in Chap. 14 of Timoshenko and Goodier's Theory of Elasticity (1951).

[2] This problem and the corresponding problem for the circular cylinder were first solved by Duhamel in the memoir cited in Sec. 99. An independent solution was also given by F. Neumann in 1841. There are numerous papers on these problems rediscovering the Duhamel-Neumann solution; some of these contain elaborate calculations.

[3] Cf. (94.3).

and, on making use of this formula in (99.5), we find

$$(100.5)\qquad \begin{cases} \tau_{ij} = \lambda\vartheta\delta_{ij} + 2\mu\left(\varphi\delta_{ij} + \dfrac{x_ix_j}{r}\varphi'\right) - \beta T\delta_{ij}, \\ \vartheta = 3\varphi + r\varphi'. \end{cases}$$

Hence, the stress $T_r = \tau_{ij}\nu_i\nu_j$ in the radial direction $\nu_i = x_i/r$ is

$$(100.6)\qquad T_r = \lambda\vartheta + 2\mu[\varphi + r\varphi'(r)] - \beta T,$$

and the "hoop stress" T_θ, in the tangential direction, is[1]

$$(100.7)\qquad T_\theta = (3\lambda + 2\mu)\varphi + \lambda r\varphi' - \beta T.$$

The substitution from (100.3) in (100.6) gives

$$(100.8)\qquad T_r = (3\lambda + 2\mu)A_1 - \frac{4\mu}{r^3}A_2 - \frac{4\mu\beta}{\lambda + 2\mu}\varphi_0(r).$$

If the surface of the shell is free of external loads,

$$T_r = 0 \qquad \text{for } r = a_1,\ r = a_2, \qquad a_2 > a_1,$$

and, on solving these equations, we get

$$(3\lambda + 2\mu)A_1 = \frac{4\mu\beta[a_2^3\varphi_0(a_2) - a_1^3\varphi_0(a_1)]}{(\lambda + 2\mu)(a_2^3 - a_1^3)},$$
$$A_2 = \frac{\beta a_1^3a_2^3[\varphi_0(a_2) - \varphi_0(a_1)]}{(\lambda + 2\mu)(a_2^3 - a_1^3)}.$$

On setting $a_1 = 0$, we get the solution for the solid sphere. In this case it is convenient to take the lower limit of the integral (100.4) as zero.

If the flow of heat in the shell is produced by maintaining the outer surface of the shell at zero temperature and the inner surface at a constant temperature T_0, then, for steady heat flow,

$$T = \frac{T_0a_1}{a_2 - a_1}\left(\frac{a_2}{r} - 1\right).$$

Hence,

$$\varphi_0 = \frac{1}{r^3}\int_{a_1}^{r}\frac{T_0a_1}{a_2 - a_1}\left(\frac{a_2}{r} - 1\right)r^2\,dr$$
$$= \frac{T_0a_1}{a_2 - a_1}\frac{1}{r^3}\left[\frac{a_2(r^2 - a_1^2)}{2} - \frac{r^3 - a_1^3}{3}\right].$$

On performing elementary calculations, we find,

$$T_r = \frac{\alpha ET_0}{1 - \sigma}\frac{a_1a_2}{a_2^3 - a_1^3}\left[a_1 + a_2 - \frac{1}{r}(a_2^2 + a_1a_2 + a_1^2) + \frac{a_1^2a_2^2}{r^3}\right],$$
$$T_\theta = \frac{\alpha ET_0}{1 - \sigma}\frac{a_1a_2}{a_2^3 - a_1^3}\left[a_1 + a_2 - \frac{1}{2r}(a_2^2 + a_1a_2 + a_1^2) - \frac{a_1^2a_2^2}{2r^3}\right].$$

[1] We omit details of simple calculations quite identical with those performed in Sec. 94.

Since $T'_\theta > 0$ for $T_0 > 0$, the hoop stress is a monotone increasing function, so that the largest value of T_θ is on the outer surface of the shell. The extreme value of T_r occurs when[1] $r^2 = 3a_1^2a_2^2/(a_1^2 + a_1a_2 + a_2^2)$.

101. Two-dimensional Thermoelastic Problems. The state of stress induced in a long cylindrical body by the distribution of temperature, which does not vary along the length of the cylinder, can be determined by solving the familiar problem in plane elasticity. For if we take the x_3-axis along the length of the cylinder and assume that the temperature $T(x_1, x_2)$ is independent of the x_3-coordinate, then the stress components τ_{3i} will not depend on x_3. These stress components can be balanced by the application of suitable longitudinal forces and bending couples applied to the ends of the cylinder, so that its cross sections remain plane. If, now, the solution of the plane-deformation problem is superimposed on the solution of the simple problems of tension and pure bending, the result will represent a valid solution of the original problem not too near the ends of the cylinder.

We thus need consider only the plane-deformation problem wherein

$$u_\alpha = u_\alpha(x_1, x_2), \qquad u_3 = 0, \qquad (\alpha = 1, 2).$$

Since

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad (i, j = 1, 2, 3),$$

$e_{13} = e_{23} = e_{33} = 0$, and we see from (99.5) that[2]

$$\text{(101.1)} \qquad \begin{cases} \tau_{13} = \tau_{23} = 0, \\ \tau_{\alpha\beta} = \lambda u_{\gamma,\gamma}\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha}) - kT\delta_{\alpha\beta}, \\ \tau_{33} = \lambda u_{\gamma,\gamma} - kT. \end{cases}$$

From the second of these equations we find that

$$u_{\gamma,\gamma} = \frac{\tau_{\alpha\alpha} + 2kT}{2(\lambda + \mu)},$$

so that

$$\text{(101.2)} \qquad \tau_{33} = \frac{\lambda}{2(\lambda + \mu)}\tau_{\alpha\alpha} - \frac{k\mu}{\lambda + \mu}T.$$

Thus, the longitudinal stress component τ_{33} is completely determined by $\tau_{\alpha\alpha} = \tau_{11} + \tau_{22}$ and T.

For the determination of the $\tau_{\alpha\beta}$ we have the equilibrium equations,

$$\text{(101.3)} \qquad \tau_{\alpha\beta,\beta} = 0,$$

which are identically satisfied if we take

$$\text{(101.4)} \qquad \tau_{11} = W_{,22}, \qquad \tau_{12} = -W_{,12}, \qquad \tau_{22} = W_{,11},$$

where $W(x_1, x_2)$ is the stress function.

[1] A discussion of numerical results of engineering interest and further references to such results are contained in Chap. 14 of Timoshenko and Goodier's Elasticity.

[2] Hereafter we denote the constant $\beta = (3\lambda + 2\mu)\alpha$, introduced in Sec. 99, by k to avoid possible confusion with Greek indices having the values 1 and 2.

On recalling the compatibility equation

$$e_{11,22} + e_{22,11} = 2e_{12,12}$$

and making use of (99.4) and (101.4), we find that W satisfies the equation

$$\nabla^2\nabla^2 W + c\nabla^2 T = 0, \tag{101.5}$$

where[1]

$$c = \frac{2\mu k}{\lambda + 2\mu}.$$

If we set

$$W \equiv U - V, \tag{101.6}$$

where V is a solution of the Poisson equation

$$\nabla^2 V = cT, \tag{101.7}$$

we find from (101.5) that U is biharmonic, so that

$$\nabla^4 U = 0. \tag{101.8}$$

Thus the problem can be phrased entirely in terms of the biharmonic function U and some particular integral[2] of (101.7). We can take such an integral in the form

$$V(x_1, x_2) = \frac{c}{2\pi} \int_R T(x_1', x_2') \log r \, dx_1' \, dx_2', \tag{101.9}$$

where $r^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2$ and R is the cross section of the cylinder.

When the tractions T_α are specified on the boundary C of the cross section R, we have

$$\tau_{\alpha\beta}\nu_\beta = T_\alpha, \tag{101.10}$$

and since, from (101.4) and (101.6),

$$\tau_{22} = U_{,11} - V_{,11}, \qquad \tau_{11} = U_{,22} - V_{,22}, \qquad \tau_{12} = V_{,12} - U_{,12},$$

we find that[3]

$$\begin{cases} \dfrac{dU_{,2}}{ds} = T_1(s) + \dfrac{dV_{,2}}{ds}, \\ \dfrac{dU_{,1}}{ds} = -T_2(s) + \dfrac{dV_{,1}}{ds}, \end{cases} \tag{101.11}$$

where s is the arc parameter measured along C.

[1] In terms of E, σ, and the coefficient of linear expansion α, $c = E\alpha/(1 - \sigma)$.

[2] The harmonic function entering in the general solution of (101.7) can be absorbed in the general representation of W inasmuch as the general solution (70.4) of the biharmonic equation contains an arbitrary harmonic function.

[3] Cf. (69.8).

Accordingly, the boundary conditions for the biharmonic function U can be written in the form

$$(101.12)\qquad \begin{aligned} U_{,1} + iU_{,2} &= i\int_{s_0}^{s} (T_1 + iT_2)\, ds + V_{,1} + iV_{,2} + \text{const} \\ &\equiv f_1(s) + if_2(s) + \text{const} \qquad \text{on } C. \end{aligned}$$

This is precisely the boundary-value problem we have considered in detail in Chap. 5.

When the flow of heat is steady,

$$\nabla^2 T = 0,$$

and it follows from (101.5) that W is biharmonic. In this case we can take $V \equiv 0$, and we conclude from the foregoing that *the state of stress induced in a cylinder by the steady heat flow is identical with that present in the same cylinder at constant temperature* (*that is, with* $T = 0$) *under the same surface loading.* Reference here is made only to the stress components $\tau_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$. The stress τ_{33} necessary to maintain the state of plane deformation is given by the formula (101.2). The strains and, of course, the displacements do depend on $T(x_1, x_2)$, and they can be computed from (101.1) once the $\tau_{\alpha\beta}$ are determined.

As a corollary to the italicized statement just above, we can state that, when the cross section of the cylinder is simply connected and the cylinder is free of external loads, the steady heat flow produces no stresses $\tau_{\alpha\beta}$. These remarkable results, pertaining to the steady heat flow in cylinders, were established by Muskhelishvili,[1] who was also responsible for an interesting physical interpretation of the discontinuous, or multiple-valued, displacements that arise in the study of the thermoelastic problems in multiply connected domains.

A comprehensive treatment of the two-dimensional thermoelastic problems, based on methods developed in Chap. 5, is contained in a dissertation "Thermal Stresses in Long Cylindrical Bodies," University of Wisconsin (1939), by Gatewood.[2] As an illustration Gatewood considers the deformation of a composite circular cylinder with a concentric circular core when the temperature $T(r)$ is a function of the radius r. He also solves the problem for a composite circular cylinder with an eccentric circular core when the temperature T is constant. It is easy to show that,

[1] N. I. Muskhelishvili, *Bulletin of the Electrotechnical Institute*, Petrograd, vol. 13 (1916), pp. 23–37; *Atti della accademia nazionale dei Lincei, Rendiconti, Classe di scienze fisiche, matematiche e naturali*, Rome, ser. 5, vol. 31 (1922), pp. 548–551. A detailed discussion is also contained in Muskhelishvili's monograph Some Basic Problems of the Mathematical Theory of Elasticity (1953), pp. 157–165.

[2] See also B. E. Gatewood, *Philosophical Magazine*, ser. 7, vol. 32 (1941), pp 282 301.

when the cylinder is of radius a and the temperature $T(r)$ is a function of the radius alone, then[1]

$$\begin{aligned}
\tau_{rr} &= \frac{c}{a^2}\int_0^a rT(r)\,dr - \frac{c}{r^2}\int_0^r T(r)\,dr,\\
\tau_{\theta\theta} &= \frac{c}{a^2}\int_0^a rT(r)\,dr + \frac{c}{r^2}\int_0^r rT(r)\,dr,\\
\tau_{r\theta} &= 0.
\end{aligned}$$

Thermal stresses in a circular ring when the temperature T is a function of both the radius and polar angle were calculated by Lebedev.[2]

The contents of this section can be modified in obvious ways to apply to the two-dimensional problems involving thin plates. The transverse deflections of thin elastic plates, under fairly general distributions of temperature, have been considered by[3] Galerkin, Nadai, Marguerre, Sokolnikoff and Sokolnikoff, and Pell.

102. Vibration of Elastic Solids. We have formulated the basic dynamical problems of elasticity and discussed the existence and uniqueness of their solutions in Chap. 3. Analytical difficulties attending the determination of explicit solutions of such problems are so great that the available explicit solutions are concerned with special types of vibration in spheres and cylindrical rods, and with a few types of propagation of elastic waves in unbounded media.

In this section we indicate one mode of attack on the problem of vibration of bounded elastic media, and in the remaining sections of this chapter we discuss some important aspects of wave propagation in the infinite and semi-infinite solids.

In the study of free small vibrations of coupled dynamical systems with a finite number of degrees of freedom, it is shown that the most general motion about the equilibrium configuration is compounded of a finite number of certain special modes of vibration, known as the *normal modes.* The number of such modes is equal to the number of degrees of freedom. Each particle in the system executing a given mode moves with simple harmonic motion, the period and the phase of which are the

[1] This problem, and the corresponding problem for the hollow cylinder, can also be solved by an elementary method of Sec. 100. See also Timoshenko and Goodier's Theory of Elasticity (1951), pp. 408–416.

[2] N. Lebedev, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 3 (1936), pp. 76–84.

[3] B. G. Galerkin, *Ingenieurbauten und Baumechanik,* Leningrad (1924), pp. 131–148.

A. Nádai, Elastische Platten (1925), pp. 264–268.

K. Marguerre, *Zeitschrift für angewandte Mathematik und Mechanik,* vol. 15 (1935), pp. 369–372; *Ingenieur Archiv,* vol. 8 (1937), pp. 216–228.

I. S. Sokolnikoff and E. S. Sokolnikoff, *Transactions of the American Mathematical Society,* vol. 45 (1939), pp. 235–255.

W. H. Pell, *Quarterly of Applied Mathematics,* vol. 4 (1946), pp. 27–44.

same for each particle. Thus, the general motion of the system of n degrees of freedom can be represented by a linear combination of n simple harmonic motions with n distinct frequencies. These frequencies are determined by solving the secular equation which is completely determined by the quadratic forms representing kinetic and potential energies of the system.[1]

When the system is continuous, the corresponding secular equation has infinitely many real roots and hence infinitely many *characteristic functions* representing normal modes of vibration. These characteristic functions are solutions of the differential equations of motion with appropriate boundary conditions. Thus, in dealing with small free vibrations of an elastic solid, the characteristic functions satisfy the equations

$$\mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} = \rho\ddot{u}_i \qquad \text{in } \tau, \tag{102.1}$$

and the homogeneous boundary conditions

$$\tau_{ij}\nu_j = 0 \qquad \text{on } \Sigma, \qquad \text{for all } t, \tag{102.2}$$

that correspond to the absence of external surface forces. Since normal oscillations are simple harmonic, it is natural to seek particular solutions of this system in the form

$$u_i(x, t) = u_i'(x_1, x_2, x_3) \cos(\omega t + \epsilon). \tag{102.3}$$

On inserting this in (102.1), we find that the functions u_i' satisfy the equations

$$\mu\nabla^2 u_i' + (\lambda + \mu)\vartheta'_{,i} + \rho\omega^2 u_i' = 0, \tag{102.4}$$

with $\vartheta' = u'_{i,i}$. The solution of Eqs. (102.4) satisfying the boundary conditions (102.2) is known to exist only for a denumerable set ω_k ($k = 1, 2, \ldots$) of values of ω, all of which are real. Thus, the characteristic functions are

$$u_i^{(k)}(x, t) = u_i'^{(k)} \cos(\omega_k t + \epsilon), \qquad (k = 1, 2, \ldots), \tag{102.5}$$

where the $u_i'^{(k)}$ are the appropriate solutions of (102.4). One then concludes that every oscillation of the body can be represented in the series

$$u_i = \sum_{k=1}^{\infty} A_k u_i^{(k)},$$

where the A_k are suitable constants, whenever the $u_i^{(k)}$ are orthogonal with respect to the region under consideration.[2]

[1] See, for example, E. T. Whittaker, Analytical Dynamics, Chap. 7, or H. Goldstein, Classical Mechanics, Chap. 10.

[2] The mode of solution described here is precisely that used in solving the problem of small transverse vibrations of an elastic string by the Fourier method. The A_k are determined from conditions characterizing the initial disturbance.

It is tolerably clear that the determination of characteristic functions, even for such simple regions as spheres and cylinders, is accompanied by very laborious computations.

As a simple example we consider the determination of these functions for the problem of oscillation of a sphere, every particle of which executes a vibration in the radial direction. We take

$$u_i = x_i f(r) \cos(\omega t + \epsilon) \tag{102.6}$$

where $r^2 = x_i x_i$. Then

$$u_i' = x_i f(r). \tag{102.7}$$

On substitution from (102.7) in (102.4), we find that the function $f(r)$ is required to satisfy the equation,

$$f'' + \frac{4}{r} f' + k^2 f = 0, \tag{102.8}$$

where

$$k^2 = \frac{\rho\omega^2}{\lambda + 2\mu}. \tag{102.9}$$

The solution of (102.8), which does not become infinite for $r = 0$, is

$$f = \frac{C}{r^3}(kr \cos kr - \sin kr), \tag{102.10}$$

where C is an arbitrary constant.

The component of displacement u_r in the radial direction is,

$$\begin{aligned} u_r &= u_i \frac{x_i}{r} \\ &= rf(r) \cos(\omega t + \epsilon), \end{aligned}$$

where we have recalled (102.6). The component τ_{rr} of stress in the radial direction is

$$\tau_{rr} = \lambda\vartheta + 2\mu \frac{\partial u_r}{\partial r}.$$

But from (102.6)

$$u_{i,i} = (3f + rf') \cos(\omega t + \epsilon),$$

so that

$$\tau_{rr} = [(3\lambda + 2\mu)f + (\lambda + 2\mu)rf'] \cos(\omega t + \epsilon). \tag{102.11}$$

Since the condition (102.2), in our case, is $\tau_{rr} = 0$ for $r = a$, where a is the radius of the sphere, we find, on setting $r = a$ and on using (102.10) in (102.11), that

$$(\lambda + 2\mu)[(2 - k^2a^2) \sin ka - 2ka \cos ka] + 2\lambda(ka \cos ka - \sin ka) = 0.$$

This is our frequency equation in which k is related to ω by (102.9). The

values of the six lowest roots of this transcendental equation for $\lambda = \mu$, which corresponds to $\sigma = \frac{1}{4}$, are recorded[1] in Sec. 196, p. 285, of Love's Treatise.

It is clear from this simple example that the determination of characteristic functions in the more general case is extremely difficult.[2]

103. Wave Propagation in Infinite Regions. If a region is so large that the effects of the boundaries can be disregarded, it is possible to represent the disturbance as a sum of two waves propagated with velocities that depend only on the density and elastic constants of the medium. Indeed, the displacement vector **u** can be represented as a sum of two vectors, one of which is solenoidal and the other irrotational. This leads to a consideration of two special types of disturbance for one of which div **u** = 0 and for the other curl **u** = 0.

Now if div **u** = 0, $u_{i,i} = \vartheta = 0$ and the general equations

$$\mu\nabla^2 u_i + (\lambda + \mu)\vartheta_{,i} = \rho \ddot{u}_i, \tag{103.1}$$

reduce to

$$\mu\nabla^2 u_i = \rho\ddot{u}_i. \tag{103.2}$$

These are familiar wave equations of the form

$$\frac{\partial^2 u_i}{\partial t^2} = c^2\nabla^2 u_i, \tag{103.3}$$

where the velocity c of propagated waves is

$$c = \sqrt{\frac{\mu}{\rho}} \tag{103.4}$$

This is the case of an *equivoluminal* wave propagation, since div **u** = 0 for waves moving with this velocity.

On the other hand, when the motion is irrotational, curl **u** = 0 and the vector identity

$$\text{curl curl } \mathbf{u} = \nabla \text{ div } \mathbf{u} - \nabla^2\mathbf{u},$$

enables one to write (103.1) in the form

$$(\lambda + 2\mu)\nabla^2 u_i = \rho\ddot{u}_i. \tag{103.5}$$

These equations show that the velocity of propagation of irrotational waves is

$$\sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{103.6}$$

[1] These results are due to H. Lamb, *Proceedings of the London Mathematical Society*, vol. 13 (1882), but the problem of radial vibrations of the solid sphere was first discussed by Poisson in 1828. A solution of the corresponding problem for the hollow sphere is given in Sec. 198 of Love's Treatise.

[2] The reader may find it instructive to consult a monograph by H. Kolsky, Stress Waves in Solids (1953), which contains a summary of recent contributions to problems of longitudinal, torsional, and flexural vibration of cylinders.

We thus see that the disturbance in an infinite medium can be described with the aid of two special types of waves; one of these travels with the *equivoluminal velocity* $c_1 = \sqrt{\mu/\rho}$, and the other with the *irrotational velocity* $c_2 = \sqrt{(\lambda + 2\mu)/\rho}$. Clearly, $c_2 > c_1$.

When the initial disturbance is symmetric about the origin, the displacement is a function of r and t only and Eq. (103.3) can be written as

$$\frac{\partial^2(ru)}{\partial t^2} = c^2 \frac{\partial^2}{\partial r^2}(r^2 u).$$

On recalling the D'Alembert solution of the wave equation, we have

$$u = \frac{1}{r^2} F(r - ct) + \frac{1}{r^2} G(r + ct),$$

where F and G are arbitrary functions. This represents two trains of spherical waves, one diverging from the source $r = 0$ with the velocity c and the other moving toward the source with the same velocity. At a great distance from the source, spherical waves become nearly plane. This suggests that in an infinite medium plane waves can travel with one or the other of the velocities found above. A direct verification of this is simple. When the waves are plane,

$$u_i = F_i(x_j \nu_j - ct), \tag{103.7}$$

where the F_i are arbitrary functions and the ν_i are the direction cosines of the normal to the plane of the wave. If we insert (103.7) in (103.1) and eliminate the F_i from the resulting equations, we get the equation for c in the form

$$(\rho c^2 - \mu)(\rho c^2 - \lambda - 2\mu) = 0,$$

which shows that plane waves travel with the equivoluminal and irrotational velocities.

PROBLEMS

1. Show that, when the displacement vector $\mathbf{u}$ is written in the form

$$\mathbf{u} = \nabla\varphi + \operatorname{curl} \boldsymbol{\psi},$$

then Eqs. (103.1) are satisfied if

$$\frac{\partial^2\varphi}{\partial t^2} = c_2^2\nabla^2\varphi, \qquad c_2 = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

and

$$\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = c_1^2\nabla^2\boldsymbol{\psi}, \qquad c_1 = \sqrt{\frac{\mu}{\rho}}.$$

2. Referring to Prob. 1, show that a class of particular solutions of (103.1) can be generated by taking

$$\varphi = Ae^{-ax_2 + i(cx_1 - pt)},$$
$$\psi = Be^{-bx_2 + i(cx_1 - pt)},$$

when $u_3 = 0$ and u_1 and u_2 are independent of x_3.

3. Referring to Prob. 1, show that when the gradient and curl are expressed in cylindrical coordinates (r, θ, z), then, in axially symmetric problems,

$$u_r = \frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial z}, \qquad u_z = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{1}{r}\psi, \qquad u_\theta = 0,$$

and

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{c_2^2}\frac{\partial^2 \varphi}{\partial t^2},$$
$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\psi + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c_1^2}\frac{\partial^2 \psi}{\partial t^2}.$$

Deduce two sets of particular solutions of these equations by taking

$$\varphi = F(r)e^{i(pt+cz)}, \qquad \psi = G(r)e^{i(pt+cz)}$$

and show that F and G satisfy certain Bessel's equations.

104. Surface Waves. Let a semi-infinite solid occupying the region $x_2 \geq 0$, be set in motion by forces applied at some distance from the free boundary $(x_2 = 0)$ of the solid. We shall suppose that the resulting waves are propagated in the positive direction of the x_1-axis and that the deformation is plane with $u_3 = 0$. We are then led to consider solutions of the two-dimensional equations

$$\text{(104.1)} \qquad \mu\nabla^2 u_\alpha + (\lambda + \mu)\vartheta_{,\alpha} = \rho\ddot{u}_\alpha, \qquad (\alpha = 1, 2),$$

in the region $x_2 > 0$, such that

$$\text{(104.2)} \qquad \tau_{\alpha\beta}\nu_\beta = 0 \qquad \text{on } x_2 = 0$$

Since the nature of disturbing forces is not specified, the system (104.1), (104.2) has infinitely many solutions. It occurred to Lord Rayleigh to investigate one type of characteristic solutions in which the amplitude of the waves dies off exponentially as one recedes from the free surface of the solid. It was anticipated by Rayleigh[1] that solutions of this type might approximate the behavior of seismic waves observed during earthquakes.

The considerations in the preceding section suggest that the solutions of (104.1) be expressed in the form

$$\text{(104.3)} \qquad u_\alpha = v_\alpha + w_\alpha,$$

where the vectors v_α and w_α satisfy the equations,

$$\text{(104.4)} \qquad \mu\nabla^2 v_\alpha = \rho\ddot{v}_\alpha,$$
$$\text{(104.5)} \qquad (\lambda + 2\mu)\nabla^2 w_\alpha = \rho\ddot{w}_\alpha.$$

[1] *Proceedings of the London Mathematical Society*, vol. 17 (1887), or Scientific Papers vol. 2, p. 441.

Thus the v_α are associated with an equivoluminal wave moving with the velocity

$$c_1 = \sqrt{\frac{\mu}{\rho}}, \tag{104.6}$$

while the w_α are propagated with the velocity

$$c_2 = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{104.7}$$

Since the amplitudes of these waves are to decrease exponentially, we take solutions in the form[1]

$$\begin{cases} v_\alpha = A_\alpha e^{-ax_2+i(sx_1-pt)}, & a > 0, \\ w_\alpha = B_\alpha e^{-bx_2+i(sx_1-pt)}, & b > 0. \end{cases} \tag{104.8}$$

The period of the waves in (104.8) is $2\pi/p$, and the wave length is $2\pi/s$; hence the velocity of propagation is

$$c_3 = \frac{p}{s}. \tag{104.9}$$

To determine the constants in (104.8), we insert from (104.8) in (104.4) and (104.5) and find that

$$\begin{cases} a^2 - s^2 + k^2 = 0, \\ b^2 - s^2 + h^2 = 0, \end{cases} \tag{104.10}$$

where

$$k^2 \equiv \frac{\rho p^2}{\mu}, \qquad h^2 \equiv \frac{\rho p^2}{\lambda + 2\mu}. \tag{104.11}$$

Further, since v_α and w_α represent equivoluminal and irrotational waves,

$$v_{\alpha,\alpha} = 0, \qquad w_{\alpha,\beta} = w_{\beta,\alpha}.$$

Hence

$$A_2 ai = -A_1 s, \qquad B_2 is = -B_1 b,$$

and we can take the vectors in (104.3) to be

$$\begin{aligned} v_1 &= Aae^{-ax_2} \sin (sx_1 - pt), \\ v_2 &= Ase^{-ax_2} \cos (sx_1 - pt), \\ w_1 &= Bse^{-bx_2} \sin (sx_1 - pt), \\ w_2 &= Bbe^{-bx_2} \cos (sx_1 - pt), \end{aligned} \tag{104.12}$$

where A and B are real constants.

Further restrictions on the choice of constants in (104.12) are provided by the boundary conditions (104.2). Since $\nu_1 = 0$, $\nu_2 = -1$, Eqs.

[1] Cf. Prob. 2, Sec. 103. In accordance with the usual practice, the desired vectors v_α and w_α are determined by the real parts of expressions in (104.8).

(104.2) demand that $\tau_{22} = \tau_{12} = 0$ on $x_2 = 0$ and thus, from Hooke's law,

$$(104.13) \qquad \begin{cases} \lambda u_{\alpha,\alpha} + 2\mu u_{2,2} = 0, \\ u_{1,2} + u_{2,1} = 0, \qquad \text{for } x_2 = 0. \end{cases}$$

The substitution from (104.12) in these equations then yields two linear equations

$$(104.14) \qquad \begin{cases} 2\mu a s A + [2\mu b^2 + \lambda(b^2 - s^2)]B = 0, \\ (a^2 + s^2)A + 2sbB = 0, \end{cases}$$

which have nonvanishing solutions for A and B if, and only if,

$$(104.15) \qquad [\lambda(b^2 - s^2) + 2\mu b^2](a^2 + s^2) - 4\mu abs^2 = 0.$$

On eliminating a and b from (104.15) with the aid of (104.10), we deduce the equation,

$$(104.16) \qquad \left(2 - \frac{k^2}{s^2}\right)^4 = 16\left(1 - \frac{h^2}{s^2}\right)\left(1 - \frac{k^2}{s^2}\right).$$

But

$$\frac{k^2}{s^2} = \frac{c_3^2}{c_1^2}, \qquad \frac{h^2}{s^2} = \frac{c_3^2}{c_2^2},$$

as follows from (104.11), (104.9), (104.7), and (104.6). Hence Eq. (104.16) can be cast in the form,

$$(104.17) \qquad (2 - \omega)^4 = 16(1 - \varkappa\omega)(1 - \omega),$$

where

$$(104.18) \qquad \omega \equiv \frac{c_3^2}{c_1^2}, \qquad \varkappa \equiv \frac{c_1^2}{c_2^2} = \frac{\mu}{\lambda + 2\mu}.$$

On simplifying we get,

$$(104.19) \qquad \omega^3 - 8\omega^2 + (24 - 16\varkappa)\omega - 16(1 - \varkappa) = 0.$$

If we take[1] $\lambda = \mu$, then $\varkappa = 1/3$ and (104.19) becomes

$$3\omega^3 - 24\omega^2 + 56\omega - 32 = 0,$$

or

$$(\omega - 4)(3\omega^2 - 12\omega + 8) = 0.$$

The roots of this equation are

$$\omega_1 = 4, \qquad \omega_2 = 2 + 2/\sqrt{3}, \qquad \omega_3 = 2 - 2/\sqrt{3}.$$

It is easy to check that the first two of these do not give positive values to a and b in (104.10). Thus the only suitable root is $\omega = 2 - 2\sqrt{3}$, and hence, from (104.18),

$$c_3 = c_1\sqrt{\omega} = 0.9194\sqrt{\frac{\mu}{\rho}}.$$

[1] This corresponds to $\sigma = 1/4$.

In the limiting case of incompressible body ($\vartheta = 0$, $\sigma = \frac{1}{2}$, $\kappa = 0$), Eq. (104.19) assumes the form

$$\omega^3 - 8\omega^2 + 24\omega - 16 = 0,$$

and one finds, as above, that the velocity of the surface wave is

$$c_3 = 0.9553\sqrt{\frac{\mu}{\rho}}.$$

In either case, c_3 is slightly less than the velocity of the equivoluminal wave.

Having determined c_3, one can compute a and b with the aid of Eqs. (104.10) in terms of s and write out the corresponding expressions for the u_α. The rate of attenuation with depth depends on a and b, and it is easy to see that the waves of higher frequency are attenuated more rapidly than those of low frequency. Since c_3 is independent of wave-length, there is no dispersion.

Waves roughly similar in appearance to Rayleigh's waves are often recorded by seismographs. However, seismographic records of distant earthquakes indicate a dispersion, which is to be expected since the earth is not a homogeneous medium.

If we consider forced vibrations of the semi-infinite solid by normal forces $P \cos(sx_1 - pt)$ distributed along the x_3-axis, we must replace the boundary conditions (104.13) by

$$\begin{aligned} \tau_{22} &= \lambda u_{\alpha,\alpha} + 2\mu u_{2,2} = P\cos(sx_1 - pt), \\ \tau_{12} &= \mu(u_{1,2} + u_{2,1}) = 0. \end{aligned}$$

Equations corresponding to (104.14) in this case are

$$\begin{aligned} 2\mu asA + [2\mu b^2 + \lambda(b^2 - s^2)]B &= P, \\ (a^2 + s^2)A + 2sbB &= 0. \end{aligned}$$

These can be solved for A and B. The corresponding expressions for u_α, with $x_2 = 0$, give the displacements on the free surface of the solid under the action of forces $P\cos(sx - pt)$ distributed along the x_3-axis. These can be generalized[1] in a familiar way with the aid of Fourier integrals to provide formulas for the displacements on the free surface due to the forced vibrations of a more general sort. The analysis of resulting formulas shows that the steady-state disturbance of the free surface consists of three waves one of which moves with velocity c_1, the other with velocity c_2, and the third with the Rayleigh velocity c_3.

A similar analysis has been carried out by Lamb for the semi-infinite

[1] H. Lamb, *Philosophical Transactions of the Royal Society (London) (A)*, vol. 203 (1904); *Proceedings of the Royal Society (London) (A)*, vol. 93 (1917), p. 114.

See also S. Timoshenko, *Philosophical Magazine*, vol. 43 (1922), p. 125, and J. H. Jeans, *Proceedings of the Royal Society (London) (A)*, vol. 102 (1923), p. 554.

half space a portion of whose boundary is subjected to the axially symmetric *forced vibrations*. *This problem was treated by a different* method by Smirnoff, Soboleff, and Narychkina.[1]

[1] V. Smirnoff and S. Soboleff, "Sur une méthode nouvelle dans le problème plan des vibrations élastiques," *Trudy, Seismological Institute of the Academy of Sciences of the USSR*, No. 24 (1932); No. 29 (1933).

S. Soboleff, *Matematicheski Sbornik*, vol. 40 (1933).

E. Narychkina, "Sur les vibrations d'un demi-espace aux conditions initiales arbitraires," *Trudy, Seismological Institute of the Academy of Sciences of the USSR*, No. 45 (1934); No. 90 (1940).

See also closely related papers by:

Ya. A. Mindlin, *Doklady Akademii Nauk SSSR*, vol. 15 (1937), vol. 24 (1939), vol. 26 (1940), vol. 27 (1940), vol. 42 (1944); *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 10 (1946), pp. 229–240.

CHAPTER 7

VARIATIONAL METHODS

105. Introduction. The determination of the state of stress in the preceding chapters was made to depend on a solution of certain boundary-value problems involving partial differential equations. A different approach, exploiting certain broad minimum principles that characterize the equilibrium states of elastic bodies, is developed in this chapter. This approach is based on the use of direct methods in the calculus of variations, first proposed by Lord Rayleigh and W. Ritz and extended by R. Courant, K. Friedrichs, B. G. Galerkin, L. V. Kantorovich, S. G. Mikhlin, E. Trefftz, and others.

We shall see that it is possible to construct certain integrals, related to the work performed on an elastic body in the process of deformation, and to show that these integrals attain their minimum values when the distribution of stress in the body corresponds to the equilibrium state. This in effect reduces the problem of stress determination to certain standard problems in the calculus of variations that can be solved by direct methods.

The concluding sections of this chapter contain a brief treatment of several numerical methods of solution of problems in elasticity. Some of these are suggested by the variational techniques but do not explicitly depend on them.

106. Variational Problems and Euler's Equations. We shall be concerned with the calculation of the extreme values of functions defined by certain integrals whose integrands contain one or several functions assuming the roles of arguments. As an example, consider the integral

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\, dx, \tag{106.1}$$

where $F(x, y, y')$ is a known real function F of the real arguments x, y and $y' \equiv dy/dx$. The value of the integral (106.1) depends on the choice of $y = y(x)$, hence the notation $I(y)$. We shall use the term *functional* to describe functions defined by integrals whose arguments themselves are functions.

To make the symbol $I(y)$ meaningful, it is clearly necessary to impose some restrictions on the choice of the argument $y(x)$ and on the pre-

scribed function F appearing in the integrand of (106.1). We shall suppose that the admissible arguments $y(x)$ belong to a class C^2 and assume at the end points of the interval (x_0, x_1) the specified values y_0 and y_1. Thus,

$$\begin{cases} y(x_0) = y_0, \\ y(x_1) = y_1, \end{cases} \tag{106.2}$$

where y_0 and y_1 are prescribed in advance. The entire set $\{y(x)\}$ of admissible arguments $y(x)$ can thus be viewed as a family of smooth curves passing through (x_0, y_0) and (x_1, y_1). As regards $F(x, y, y')$, we shall suppose that it is of class C^2 for all values of y' in some specified region of the xy-plane containing the curves $\{y(x)\}$.

For a given curve $y = \bar{y}(x)$ of the set $\{y(x)\}$, the integral (106.1) yields a definite numerical value $I(\bar{y})$, and we pose a problem of determining that particular curve $y(x)$ in the competing set which makes the integral (106.1) a minimum.

If $y(x)$ minimizes this integral, then every function $\bar{y}(x)$ in the neighborhood of $y(x)$ can be represented in the form

$$\bar{y} = y(x) + \epsilon\eta(x), \tag{106.3}$$

where ϵ is a small real parameter. We shall call the difference,

$$\bar{y}(x) - y(x) = \epsilon\eta(x),$$

the *variation of* $y(x)$ and write,

$$\delta y \equiv \epsilon\eta(x).$$

We note that the function $y(x)$ is determined by (106.3) with $\epsilon = 0$. Moreover, every function in the set $\{y(x)\}$ satisfies the end conditions (106.2) and thus

$$\eta(x_0) = \eta(x_1) = 0. \tag{106.4}$$

Since $y(x)$ minimizes (106.1),

$$I(\bar{y}) \geq I(y),$$

or

$$I(y + \epsilon\eta) \geq I(y). \tag{106.5}$$

The left-hand member in the inequality (106.5) is a continuously differentiable function of ϵ, and therefore a necessary condition that $y(x)$ minimize (106.1) is

$$\left.\frac{dI(y + \epsilon\eta)}{d\epsilon}\right|_{\epsilon=0} = 0. \tag{106.6}$$

But,

$$I(y + \epsilon\eta) = \int_{x_0}^{x_1} F(x, y + \epsilon\eta, y' + \epsilon\eta')\, dx,$$

and on differentiating under the integral sign we obtain

$$\frac{dI(y + \epsilon\eta)}{d\epsilon}\bigg|_{\epsilon=0} = \int_{x_0}^{x_1} (\eta F_y + F_{y'}\eta')\, dx = 0. \tag{106.7}$$

As is customary, the subscripts on F indicate partial derivatives of F with respect to the arguments denoted by the subscripts. Integrating the second term in the integral (106.7) by parts, we get

$$\int_{x_0}^{x_1} F_{y'}\eta'\, dx = F_{y'}\eta \bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{dF_{y'}}{dx}\, dx,$$

and since $\eta(x_0) = \eta(x_1) = 0$, we can write (106.7) in the form

$$\int_{x_0}^{x_1} \left(F_y - \frac{dF_{y'}}{dx}\right) \eta(x)\, dx = 0. \tag{106.8}$$

Thus the integral (106.8) vanishes for every $\eta(x)$ of class C^2 satisfying the condition (106.4), and we conclude that[1]

$$F_y - \frac{dF_{y'}}{dx} = 0, \tag{106.9}$$

is a necessary condition that the integral (106.1) be minimized by $y = y(x)$. Equation (106.9) is the *Euler equation* associated with the variational problem $I(y) = \min$. On expanding it we get the second-order ordinary differential equation

$$F_{y'y'} \frac{d^2y}{dx^2} + F_{y'y} \frac{dy}{dx} + F_{y'x} - F_y = 0 \tag{106.10}$$

for the determination of $y(x)$.

We have assumed that the minimizing function is contained in the admissible set $\{y(x)\}$. However, the condition (106.6) is merely a *necessary* condition for the extremum of $I(y)$. Ordinarily, it is important to verify that the solution of the Euler equation indeed satisfies the inequality (106.5) and thus minimizes the integral. If suitable restrictions are placed on the function $F(x, y, y')$, the appropriate solution of (106.9) will in fact minimize the functional $I(y)$. Thus, if

$$F(x, y, y') = p(x)(y')^2 + q(x)y^2 + 2f(x)y,$$

(106.10) yields a self-adjoint second-order differential equation

$$\frac{d}{dx}(py') - qy - f = 0, \tag{106.11}$$

[1] The proof of this lemma, due to Lagrange, is contained in many books. See, for example, R. Courant and D. Hilbert, Methoden der Mathematischen Physik, vol. 1, or I. S. Sokolnikoff, Tensor Analysis, pp. 154–155.

whose solution satisfying the end conditions $y(x_0) = y_0$, $y(x_1) = y_1$ actually minimizes the integral

$$I(y) = \int_{x_0}^{x_1} [py'^2 + qy^2 + 2fy]\, dx, \tag{106.12}$$

whenever[1] $p(x) > 0$ and $q(x) \geq -\dfrac{\pi^2}{x_1 - x_0}\, p_{\min}$ in (x_0, x_1).

A functional of the form (106.12) *arises in the one-dimensional elastostatic problems concerned with the study of deflection of bars and strings.*

Similar calculations performed on the functional

$$I(y) = \int_{x_0}^{x_1} F(x, y, y', y'', \ldots, y^{(n)})\, dx \tag{106.13}$$

yield the Euler equation

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \cdots + (-1)^n \frac{d^n}{dx^n} F_y^{(n)} = 0, \tag{106.14}$$

when certain obvious restrictions on the continuity and differentiability of F and $y(x)$ are imposed.

We consider next the problem of minimizing the double integral

$$I(u) = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy \tag{106.15}$$

on the set $\{u(x, y)\}$ of functions of class C^2, where each $u(x, y)$ in the set takes on the boundary C of the region R specified continuous values $u = \varphi(s)$. We suppose that F, viewed as a function of x, y, u, u_x, u_y is of class C^2 in the appropriate domain of definition of these arguments.

Let us suppose that a certain function $u(x, y)$ in the admissible set actually minimizes (106.15) and that every function $\bar{u}(x, y)$ in this set is included in the formula

$$\bar{u}(x, y) = u(x, y) + \epsilon\eta(x, y),$$

where ϵ is a small parameter. Since $\bar{u} = \varphi(s)$ on the boundary of R, $\eta(x, y) = 0$ on C. We form the integral $I(u + \epsilon\eta)$ and observe that

$$\delta I \equiv \left. \frac{dI(u + \epsilon\eta)}{d\epsilon} \right|_{\epsilon=0} = 0,$$

since $u(x, y)$ minimizes the functional (106.15).

But

$$I(u + \epsilon\eta) = \iint_R F(x, y, u + \epsilon\eta, u_x + \epsilon\eta_x, u_y + \epsilon\eta_y)\, dx\, dy,$$

so that

$$\delta I = \iint_R (F_u\eta + F_{u_x}\eta_x + F_{u_y}\eta_y)\, dx\, dy. \tag{106.16}$$

[1] E. L. Ince, Ordinary Differential Equations, Chap. 10, especially p. 227.

We rewrite (106.16) as

$$\delta I = \iint_R \left(F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y}\right)\eta\, dx\, dy + \iint_R \left[\frac{\partial}{\partial x}(F_{u_x}\eta) + \frac{\partial}{\partial y}(F_{u_y}\eta)\right] dx\, dy$$

and apply Green's Theorem to the second integral. We get

$$(106.17)\quad \delta I = \iint_R \left(F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y}\right)\eta\, dx\, dy + \int_C \eta(F_{u_x}\, dy - F_{u_y}\, dx).$$

But $\eta = 0$ on C, and since δI vanishes for an arbitrary choice of η in R, we conclude from Lagrange's lemma that

$$(106.18)\qquad F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} = 0$$

for the minimizing function $u(x, y)$.

A calculation in every respect similar to the foregoing for the functional

$$(106.19)\qquad I(u) = \iint_R F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dx\, dy,$$

in which the admissible u assume specified continuous values on the boundary C of R, leads to the Euler equation,

$$(106.20)\quad F_u - \frac{\partial}{\partial x}(F_{u_x}) - \frac{\partial}{\partial y}(F_{u_y}) + \frac{\partial^2}{\partial x^2}(F_{u_{xx}}) + \frac{\partial^2}{\partial x\, \partial y}(F_{u_{xy}}) + \frac{\partial^2}{\partial y^2}(F_{u_{yy}}) = 0.$$

A special form of the functional (106.15) is of particular interest in the sequel. It is

$$(106.21)\qquad I(u) = \iint_R [(u_x)^2 + (u_y)^2 + 2f(x, y)u]\, dx\, dy,$$

where the admissible u satisfy the condition

$$(106.22)\qquad u = \varphi(s) \qquad \text{on } C.$$

On substituting $F = (u_x)^2 + (u_y)^2 + 2fu$ in (106.18), we find that the minimizing function satisfies the Poisson equation

$$(106.23)\qquad \nabla^2 u = f(x, y) \qquad \text{in } R.$$

We can easily show that the solution of this equation, satisfying the boundary condition (106.22), makes the functional (106.21) an absolute minimum on the set of all functions of class C^2 taking on C the values $\varphi(s)$. Indeed, let $\bar{u}(x, y)$ be any function in such set, and we define $\eta(x, y)$ by writing

$$\bar{u}(x, y) = u(x, y) + \eta(x, y),$$

where $u(x, y)$ is the solution of (106.23) and $\eta(x, y) = 0$ on C.

But

$$\begin{aligned}\Delta I &\equiv I(\bar{u}) - I(u)\\ &= \iint_R [(u_x + \eta_x)^2 + (u_y + \eta_y)^2 + 2f(u + \eta)]\, dx\, dy\\ &\qquad\qquad - \iint_R [(u_x)^2 + (u_y)^2 + 2fu]\, dx\, dy\\ &= 2\iint_R (u_x\eta_x + u_y\eta_y + f\eta)\, dx\, dy + \iint_R [(\eta_x)^2 + (\eta_y)^2]\, dx\, dy.\end{aligned}$$

From Green's Theorem,

$$\iint_R (u_x\eta_x + u_y\eta_y)\, dx\, dy = -\iint_R \eta\nabla^2 u\, dx\, dy + \int_C \eta \frac{du}{d\nu}\, ds,$$

so that

$$\begin{aligned}\Delta I &= -2\iint_R \eta(\nabla^2 u - f)\, dx\, dy + \iint_R [(\eta_x)^2 + (\eta_y)^2]\, dx\, dy,\\ &= \iint_R [(\eta_x)^2 + (\eta_y)^2]\, dx\, dy,\end{aligned}$$

since $\nabla^2 u = f$ in R and $\eta = 0$ on C.

Thus, $\Delta I \geq 0$, and hence

$$I(\bar{u}) \geq I(u). \tag{106.24}$$

Moreover, the equality sign in (106.24) holds if, and only if, $\bar{u} = u$. For $\Delta I = 0$ if, and only if, $\eta_x = \eta_y = 0$, that is, when $\eta = \text{const.}$ But $\eta = 0$ on C, and hence $\eta \equiv 0$ in R.

107. Theorem of Minimum Potential Energy. In this section we introduce an important functional, called the *potential energy of deformation*, and prove that it attains an absolute minimum when the displacements of the body are those of the equilibrium configuration. This theorem lies at the basis of several direct variational methods of solution of elastostatic problems.

Let a body τ be in equilibrium under the action of specified body and surface forces. The surface forces T_i may be prescribed only over a por-

tion Σ_T of the surface Σ, and we shall suppose that over the remaining part Σ_u the displacements are known. Denote the displacements of the equilibrium state by u_i, and consider a class of arbitrary displacements $u_i + \delta u_i$, consistent with constraints imposed on the body. This means that, over the portion Σ_u of Σ where the displacements are assigned, the functions δu_i vanish, but, over the part Σ_T, the δu_i are arbitrary save for the condition that they belong to class C^3 and are of the order of magnitude of displacements admissible in linear elasticity. We shall term such arbitrary displacements δu_i the *virtual displacements*. The *virtual work* δU performed by the external forces F_i and T_i in a virtual displacement δu_i is defined by the equation

(107.1) $$\delta U = \int_\Sigma T_i\, \delta u_i\, d\sigma + \int_\tau F_i\, \delta u_i\, d\tau,$$

and we recall from Sec. 26 that the strain energy U is given by the formula

[26.8] $$U \equiv \int_\tau W\, d\tau,$$

where[1]

[26.16] $$W = \frac{\lambda}{2}\vartheta^2 + \mu e_{ij}e_{ij}.$$

The strain energy U is equal to the work done by the external forces on the body in the process of bringing the body from the natural state to the equilibrium state characterized by the displacements u_i.

Since the volume τ is fixed and the T_i and F_i do not vary when the arbitrary variations δu_i of the displacements are considered, (107.1) can be written in the form

(107.2) $$\delta U = \delta\left(\int_\Sigma T_i u_i\, d\sigma + \int_\tau F_i u_i\, d\tau\right),$$

where, from (26.8),

(107.3) $$\delta U = \delta \int_\tau W\, d\tau.$$

It follows from (107.2) and (107.3) that

(107.4) $$\delta\left(\int_\tau W\, d\tau - \int_\Sigma T_i u_i\, d\sigma - \int_\tau F_i u_i\, d\tau\right) = 0.$$

This formula states that *the expression in the parentheses has a stationary value in a class of admissible variations δu_i of the displacements u_i of the equilibrium state.*

[1] The considerations of this and following sections need not be restricted to isotropic bodies. If the body is anisotropic, we use $W = \frac{1}{2}c_{ij}e_ie_j$ $(i, j = 1, 2, \ldots, 6)$, or $W = \frac{1}{2}\tau_i e_i$, where $\tau_i = c_{ij}e_j$. The essential point in the proof is that W is a positive definite form.

If we define the *potential energy* V by the formula

$$V = \int_\tau W\,d\tau - \int_\Sigma T_i u_i\,d\sigma - \int_\tau F_i u_i\,d\tau, \tag{107.5}$$

the foregoing equation reads,

$$\delta V = 0. \tag{107.6}$$

We can prove a stronger theorem, namely, *the functional V assumes a minimum value when the displacements u_i are those of the equilibrium state.*

To show this, we demonstrate that the increment ΔV produced in V by replacing the equilibrium displacements u_i by $u_i + \delta u_i$ is positive for all nonvanishing variations δu_i. We first calculate the increment ΔW in W in the right-hand member of (107.5). From (26.16), we have

$$\Delta W = \left(\frac{\lambda}{2}\vartheta^2 + \mu e_{ij}e_{ij}\right)\bigg|_{u+\delta u} - \left(\frac{\lambda}{2}\vartheta^2 + \mu e_{ij}e_{ij}\right)\bigg|_{u},$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. But

$$\begin{aligned} e_{ij}\Big|_{u+\delta u} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}) + \tfrac{1}{2}[(\delta u_i)_{,j} + (\delta u_j)_{,i}] \\ &= e_{ij} + \tfrac{1}{2}(\delta u_i)_{,j} + \tfrac{1}{2}(\delta u_j)_{,i}. \end{aligned}$$

Hence

$$\vartheta\Big|_{u+\delta u} = e_{ii} + (\delta u_i)_{,i},$$

and, therefore,

$$\begin{aligned} \Delta W = &\frac{\lambda}{2}[\vartheta + (\delta u_i)_{,i}][\vartheta + (\delta u_j)_{,j}] \\ &+ \mu[e_{ij} + \tfrac{1}{2}(\delta u_i)_{,j} + \tfrac{1}{2}(\delta u_j)_{,i}][e_{ij} + \tfrac{1}{2}(\delta u_i)_{,j} + \tfrac{1}{2}(\delta u_j)_{,i}] \\ &- \frac{\lambda}{2}\vartheta^2 - \mu e_{ij}e_{ij}. \end{aligned}$$

Upon expanding this, we get

$$\Delta W = \lambda\vartheta(\delta u_i)_{,i} + 2\mu e_{ij}(\delta u_i)_{,j} + P, \tag{107.7}$$

where

$$P \equiv \frac{\lambda}{2}[(\delta u_i)_{,i}]^2 + \frac{\mu}{4}[(\delta u_i)_{,j} + (\delta u_j)_{,i}]^2 \geq 0. \tag{107.8}$$

We note that $P = 0$ only in the trivial case when

$$\delta e_{ij} = \tfrac{1}{2}[(\delta u_i)_{,j} + (\delta u_j)_{,i}] = 0.$$

Since

$$\tau_{ij} = \lambda\vartheta\delta_{ij} + 2\mu e_{ij},$$

Eq. (107.7) can be rewritten in the form

$$\begin{aligned} \Delta W &= (\lambda\vartheta\delta_{ij} + 2\mu e_{ij})(\delta u_i)_{,j} + P \\ &= \tau_{ij}(\delta u_i)_{,j} + P. \end{aligned}$$

Accordingly, the increment ΔU in strain energy U is,

$$(107.9)\qquad \Delta U = \int_\tau \Delta W\, d\tau = \int_\tau \tau_{ij}(\delta u_i)_{,j}\, d\tau + \int_\tau P\, d\tau$$
$$= \int_\tau (\tau_{ij}\,\delta u_i)_{,j}\, d\tau - \int_\tau \tau_{ij,j}\,\delta u_i\, d\tau + Q$$
$$= \int_\Sigma \tau_{ij}\nu_j\,\delta u_i\, d\sigma - \int_\tau \tau_{ij,j}\,\delta u_i\, d\tau + Q,$$

where

$$(107.10)\qquad Q \equiv \int_\tau P\, d\tau \geq 0.$$

But if the body is in equilibrium,

$$\tau_{ij,j} = -F_i \quad \text{in } \tau,$$
$$\tau_{ij}\nu_j = T_i \quad \text{on } \Sigma,$$

and, therefore,

$$(107.11)\qquad \Delta U = \int_\Sigma T_i\,\delta u_i\, d\sigma + \int_\tau F_i\,\delta u_i\, d\tau + Q.$$

Recalling the definition of potential energy V, we get,

$$\Delta V = \Delta U - \int_\Sigma T_i\,\delta u_i\, d\sigma - \int_\tau F_i\,\delta u_i\, d\tau$$

and, inserting in this from (107.11), we have finally $\Delta V = Q$. Since $Q \geq 0$, we have proved the following theorem:

THEOREM OF MINIMUM POTENTIAL ENERGY: *Of all displacements satisfying the given boundary conditions those which satisfy the equilibrium equations make the potential energy an absolute minimum.*

In applications one is usually concerned with the converse of this theorem. We now prove that the converse theorem is true.

Let us suppose that, by some means, we have obtained a set of functions $u_i + \delta u_i$ of class C^3 which satisfy assigned boundary conditions and such that

$$(107.12)\qquad \Delta V = \Delta U - \int_\Sigma T_i\,\delta u_i\, d\sigma - \int_\tau F_i\,\delta u_i\, d\tau \geq 0$$

on this set.

We recall formula (107.9) for ΔU, insert it in (107.12), and obtain

$$-\int_\tau (\tau_{ij,j} + F_i)\,\delta u_i\, d\tau + \int_\Sigma (\tau_{ij}\nu_j - T_i)\,\delta u_i\, d\sigma + Q \geq 0.$$

The contribution from the surface integral in this inequality is zero, since on the part Σ_T of Σ, where the T_i are assigned, $\tau_{ij}\nu_j = T_i$ and over the remaining part, Σ_u, the δu_i vanish. This is so because of our hypothesis concerning the character of admissible functions $u_i + \delta u_i$. Consequently,

$$(107.13)\qquad -\int_\tau (\tau_{ij,j} + F_i)\,\delta u_i\, d\tau + Q \geq 0.$$

Inasmuch as Q is essentially positive and the δu_i are arbitrary, the inequality implies that

$$\tau_{ij,j} + F_i = 0, \tag{107.14}$$

for every interior point in τ. For suppose that one of the equations (107.14) is not satisfied at some point P of τ, and for definiteness let

$$\tau_{1j,j} + F_1 > 0 \qquad \text{at } P. \tag{107.15}$$

Construct a sphere S with center at P and with radius a so small that (107.15) holds throughout this sphere. Choose next the δu_i as follows:

$$\begin{cases} \delta u_1 = k^2(a^2 - r^2)^4, & \text{for } r^2 \le a^2, \qquad (k^2 > 0), \\ \quad\;\; = 0, & \text{for } r^2 \ge a^2, \\ \delta u_2 = \delta u_3 = 0. & \end{cases} \tag{107.16}$$

The functions δu_i clearly are of class C^3 in τ. If we insert from (107.16) in the integral of the inequality (107.13), we get,

$$\int_\tau (\tau_{ij,j} + F_i)\, \delta u_i \, d\tau = k^2 M,$$

where M is independent of k and is positive by virtue of (107.15). Accordingly, the inequality (107.13) demands that

$$-k^2 M + Q \ge 0. \tag{107.17}$$

But Q depends on the squares of the derivatives of δu_i, as is clear from the definitions (107.8) and (107.10), and hence on the fourth power of k, while the first term in the left-hand member of (107.17) depends on the second power of k. Hence, by choosing k small enough, we can make $k^2 M > Q$ and thus violate (107.17). This contradicts our assumption (107.15).

We remark in conclusion that if the displacements u_i are *prescribed over the entire surface* Σ, *and the body forces* F_i *vanish throughout* τ, then the formula (107.12) demands that $\Delta U \ge 0$. In other words, the strain energy U has, in this case, an *absolute minimum on the set of arbitrary functions* u_i *satisfying the boundary conditions and subject only to the differentiability restrictions ensuring the satisfaction of Saint-Venant's compatibility equations.*

PROBLEM

Consider the case in which there are no body forces and the displacements u_i are prescribed over the entire surface Σ. Show from Eq. (107.11) that the increment ΔU in strain energy is positive in this case and that the equilibrium displacements yield a minimum value for the strain energy U.

108. Theorem of Minimum Complementary Energy. We proceed to establish another important minimum theorem which depends on the notion of varied states of stress.

Let a body τ be in equilibrium under the action of body forces F_i and surface forces T_i assigned over a part Σ_T of the surface Σ. On the remaining part Σ_u of Σ the displacements u_i are assumed to be known. In special cases the T_i (or u_i) may be prescribed over the entire surface.

If the τ_{ij} are the stress components of the equilibrium state, then

$$(108.1)\qquad \begin{cases} \tau_{ij,j} + F_i = 0 & \text{in } \tau, \\ \tau_{ij}\nu_j = T_i & \text{on } \Sigma_T, \\ u_i = f_i & \text{on } \Sigma_u. \end{cases}$$

We introduce a set of functions τ'_{ij} of class C^2 in τ, which we shall also write as

$$(108.2)\qquad \tau'_{ij} \equiv \tau_{ij} + \delta\tau_{ij},$$

with the following properties:

1. in the interior of τ, the τ'_{ij} satisfy the equations

$$(108.3)\qquad \tau'_{ij,j} + F_i = 0;$$

2. on the part Σ_T of Σ

$$(108.4)\qquad \tau'_{ij}\nu_j = T_i;$$

3. on the part Σ_u of Σ, the τ'_{ij} are arbitrary.

It follows from these equations that the *variations* $\delta\tau_{ij}$ satisfy the conditions,

$$(108.5)\qquad \begin{cases} (\delta\tau_{ij})_{,j} = 0 & \text{in } \tau, \\ (\delta\tau_{ij})\nu_j = 0 & \text{on } \Sigma_T, \\ \delta\tau_{ij} \text{ are arbitrary on } \Sigma_u. \end{cases}$$

It should be observed that the τ_{ij} are associated with the equilibrium state of the body and hence they satisfy the Beltrami-Michell compatibility equations, but we do not assume that the $\delta\tau_{ij}$ satisfy any such conditions.

We consider now the strain-energy density W in the form (26.17),

$$[26.17]\qquad W = \frac{1+\sigma}{2E}\tau_{ij}\tau_{ij} - \frac{\sigma}{2E}\Theta^2,$$

where $\Theta = \tau_{ii}$, and compute the increment

$$\Delta U = \int_\tau \Delta W \, d\tau,$$

produced by replacing the τ_{ij} in (26.17) by $\tau'_{ij} = \tau_{ij} + \delta\tau_{ij}$. We have

$$W + \Delta W = \frac{1+\sigma}{2E}(\tau_{ij} + \delta\tau_{ij})(\tau_{ij} + \delta\tau_{ij}) - \frac{\sigma}{2E}(\Theta + \delta\Theta)^2,$$

where $\delta\Theta = \delta\tau_{ii}$.

Upon expanding and using (26.17) we get,

$$\Delta W = \frac{1+\sigma}{2E}\left[2\tau_{ij}(\delta\tau_{ij}) + (\delta\tau_{ij})(\delta\tau_{ij})\right] - \frac{\sigma}{2E}\left[2\Theta\,\delta\Theta + (\delta\Theta)^2\right]$$
$$= \frac{1+\sigma}{E}\tau_{ij}(\delta\tau_{ij}) - \frac{\sigma}{E}\Theta\,\delta\Theta + W(\delta\tau_{ij}),$$

where

$$W(\delta\tau_{ij}) = \frac{1+\sigma}{2E}(\delta\tau_{ij})(\delta\tau_{ij}) - \frac{\sigma}{2E}(\delta\Theta)^2 \geq 0,$$

since W is positive definite.[1]

We rewrite the formula for ΔW in the form

(108.6) $$\Delta W = \left(\frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\Theta\delta_{ij}\right)\delta\tau_{ij} + W(\delta\tau_{ij})$$

and note that since

$$\frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\Theta\delta_{ij} = e_{ij}$$

by Hooke's law, we can write (108.6) as

(108.7) $$\begin{aligned}\Delta W &= e_{ij}\,\delta\tau_{ij} + W(\delta\tau_{ij})\\ &= \tfrac{1}{2}(u_{i,j} + u_{j,i})\,\delta\tau_{ij} + W(\delta\tau_{ij})\\ &= u_{i,j}\,\delta\tau_{ij} + W(\delta\tau_{ij})\\ &= (u_i\,\delta\tau_{ij})_{,j} - u_i(\delta\tau_{ij})_{,j} + W(\delta\tau_{ij}).\end{aligned}$$

The displacements u_i figuring in (108.7) are those of the actual equilibrium state of the body, since the τ_{ij} were assumed to satisfy the Beltrami-Michell compatibility equations.

The increment ΔU in the strain energy U is, therefore,

$$\begin{aligned}\Delta U &= \int_\tau \Delta W\,d\tau\\ &= \int_\tau \left[(u_i\,\delta\tau_{ij})_{,j} - u_i(\delta\tau_{ij})_{,j} + W(\delta\tau_{ij})\right]d\tau\\ &= \int_\Sigma u_i(\delta\tau_{ij})\nu_j\,d\sigma - \int_\tau u_i(\delta\tau_{ij})_{,j}\,d\tau + \int_\tau W(\delta\tau_{ij})\,d\tau,\end{aligned}$$

and it follows from (108.5) that

(108.8) $$\Delta U = \int_{\Sigma_u} u_i(\delta\tau_{ij})\nu_j\,d\sigma + P,$$

where

$$P \equiv \int_\tau W(\delta\tau_{ij})\,d\tau \geq 0.$$

The equilibrium equations on the portion Σ_u of the surface Σ can be written as

$$(\delta\tau_{ij})\nu_j = \Delta T_i,$$

[1] See Sec. 26.

so that (108.8) reads,

$$\Delta U = \int_{\Sigma_u} u_i \, \Delta T_i \, d\sigma + P,$$

and since the u_i are assigned on Σ_u, we can write this as

$$\Delta\left(U - \int_{\Sigma_u} u_i T_i \, d\sigma\right) = P \geq 0. \tag{108.9}$$

We define the *complementary energy* V^* by the formula

$$V^* = U - \int_{\Sigma_u} T_i \, u_i \, d\sigma, \tag{108.10}$$

and conclude from (108.9) that the increment ΔV^* in V^* is essentially positive. This result can be stated as a theorem.

THEOREM OF MINIMUM OF COMPLEMENTARY ENERGY: *The complementary energy V^* has an absolute minimum when the stress tensor τ_{ij} is that of the equilibrium state and the varied states of stress fulfill the conditions* (108.5).

In the special case when the surface forces T_i are assigned over the entire surface, $V^* = U$ and we have the following theorem.

THEOREM OF MINIMUM STRAIN ENERGY: *The strain energy U of a body in equilibrium under the action of prescribed surface forces is an absolute minimum on the set of all values of the functional U determined by the solutions of the system*

$$\begin{cases} \tau_{ij,j} + F_i = 0 & \text{in } \tau, \\ \tau_{ij}\nu_j = T_i & \text{on } \Sigma. \end{cases}$$

This theorem is usually associated with the name of Castigliano.

The converse of the Theorem of Minimum Complementary Energy can be proved in a manner analogous to that used in the proof of the converse of the Theorem of Minimum Potential Energy. Since such proof presents no elements of novelty, we dispense with details and merely sketch the procedure

We suppose that by some means we have obtained a tensor τ_{ij} satisfying the equilibrium equations and assigned boundary conditions on Σ. We further suppose that this tensor is such that $\Delta V^* \geq 0$. We use ΔU in the form[1]

$$\Delta U = \int_\tau \left(\frac{1+\sigma}{E} \tau_{ij} - \frac{\sigma}{E} \Theta \delta_{ij}\right) \delta\tau_{ij} \, d\tau + \int_\tau W(\delta\tau_{ij}) \, d\tau.$$

The variations $\delta\tau_{ij}$ in this formula satisfy the equations $(\delta\tau_{ij})_{,j} = 0$, and hence they can be expressed as derivatives of the stress functions F_{ij}. Calculations, analogous to those in Sec. 107, lead to the formula

$$\int_\tau B_{ij} F_{ij} \, d\tau + Q \geq 0,$$

[1] See (108.6).

where Q is essentially positive and the B_{ij} are expressions formed from the derivatives of the τ_{ij}. Next a special choice of the F_{ij} is made which violates this inequality if one supposes that $B_{ij} \neq 0$ at some point of τ. The conclusion is that

$$B_{ij} \equiv \nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} + \frac{\sigma}{1-\sigma}\delta_{ij}F_{k,k} + F_{i,j} + F_{j,i} = 0 \qquad \text{in } \tau.$$

This, however, is precisely the set of Beltrami-Michell's compatibility equations (24.14). Thus the τ_{ij} correspond to the actual solution of the elastostatic problem since they satisfy the compatibility as well as the equilibrium equations.

A proof of the converse of the Theorem of Complementary Energy appears to have been supplied first, in 1936, by R. V. Southwell.[1]

It is worth noting that a similar argument, making use of the increment of strain energy in the form

$$\Delta U = \int_\tau [e_{ij}\,\delta\tau_{ij} + W(\delta\tau_{ij})]\,d\tau,$$

leads to the Saint-Venant compatibility equations (24.5).

We remark, in conclusion, that in this section the Euler equations associated with the problem $V^* = \min$ are the compatibility equations, whereas in the problem $V = \min$, discussed in the preceding section, they are the equilibrium equations.

The principles established in these sections can be extended to dynamical problems of elasticity.[2]

109. Theorems of Work and Reciprocity. We derive now a very general reciprocal expression relating the equilibrium states of a body under different applied loads.

Consider two equilibrium states of an elastic body: *one with displacements u_i due to the body forces F_i and surface forces T_i, and the other with displacements u'_i due to body forces F'_i and surface forces T'_i.* Let us calculate the work that would be done by the unprimed forces, F_i, T_i, if they acted through the primed displacements u'_i. This work can be written, with the help of the equations of equilibrium, as

$$\begin{aligned}\int_\Sigma T_i u'_i\,d\sigma + \int_\tau F_i u'_i\,d\tau &= \int_\Sigma \tau_{ij}\nu_j u'_i\,d\sigma - \int_\tau \tau_{ij,j}u'_i\,d\tau \\ &= \int_\tau (\tau_{ij}u'_i)_{,j}\,d\tau - \int_\tau \tau_{ij,j}u'_i\,d\tau,\end{aligned}$$

[1] See also V. D. Kliushnikov, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 18 (1954), pp. 250–252.

[2] See, for example, E. Reissner, *Journal of Mathematics and Physics*, vol. 27 (1948), pp. 159–160.

where the Divergence Theorem has been used. Carrying out the indicated differentiation leads to

$$(109.1)\quad \int_\Sigma T_i u'_i \, d\sigma + \int_\tau F_i u'_i \, d\tau$$
$$= \int_\tau \tau_{ij} u'_{i,j} \, d\tau$$
$$= \int_\tau (\lambda \delta_{ij} \vartheta + 2\mu e_{ij}) u'_{i,j} \, d\tau \qquad \text{(by Hooke's law)}$$
$$= \int_\tau [\lambda \vartheta u'_{i,i} + \mu(u_{i,j} + u_{j,i}) u'_{i,j}] \, d\tau$$
$$= \int_\tau (\lambda \vartheta \vartheta' + \mu u_{i,j} u'_{i,j} + \mu u_{j,i} u'_{i,j}) \, d\tau.$$

With the exception of the last term, the integrand is obviously symmetric in the primed and unprimed variables. But the last term can be written, by interchange of i, j, as

$$\mu u_{j,i} u'_{i,j} = \mu u_{i,j} u'_{j,i} = \mu u'_{j,i} u_{i,j},$$

and we see that the integral, and hence the original expression, is symmetric with respect to the primed and unprimed states. That is, we can write

$$(109.2)\qquad \int_\Sigma T_i u'_i \, d\sigma + \int_\tau F_i u'_i \, d\tau = \int_\Sigma T'_i u_i \, d\sigma + \int_\tau F'_i u_i \, d\tau.$$

This theorem can be expressed in words.

Reciprocal Theorem of Betti and Rayleigh: *If an elastic body is subjected to two systems of body and surface forces, then the work that would be done by the first system T_i, F_i in acting through the displacements u'_i due to the second system of forces is equal to the work that would be done by the second system T'_i, F'_i in acting through the displacements u_i due to the first system of forces.*

The Reciprocal Theorem can be written in terms of the stresses and strains by modifying Eq. (109.1). We observe that the integrand on the right-hand side of (109.1) can be written as

$$\tau_{ij} u'_{i,j} = \tau_{ij}[\tfrac{1}{2}(u'_{i,j} + u'_{j,i}) + \tfrac{1}{2}(u'_{i,j} - u'_{j,i})]$$
$$= \tau_{ij} e'_{ij} + \tau_{ij} \omega'_{ij}.$$

But the last term vanishes[1] on account of the skew symmetry of the rotation components, $\omega'_{ij} = -\omega'_{ji}$. Hence we have merely

$$\tau_{ij} u'_{i,j} = \tau_{ij} e'_{ij},$$

[1] This can be seen by interchanging i, j. Thus,

$$\tau_{ij}\omega'_{ij} = \tau_{ji}\omega'_{ji} = -\tau_{ij}\omega'_{ij}$$

or

$$2\tau_{ij}\omega'_{ij} = 0.$$

and Eq. (109.1) takes the form

$$\int_\Sigma T_i u_i' \, d\sigma + \int_\tau F_i u_i' \, d\tau = \int_\tau \tau_{ij} e_{ij}' \, d\tau \tag{109.3}$$

with

$$\int_\tau \tau_{ij} e_{ij}' \, d\tau = \int_\tau \tau_{ij}' e_{ij} \, d\tau.$$

Equation (109.3) is an alternative form of the Reciprocal Theorem, which is thus seen to be a generalization of Eq. (27.1).

The Reciprocal Theorem can also be deduced by means of the following argument: First subject an elastic body to the force system I: T_i, F_i. The resulting displacements are denoted by u_i; the work done, by $U_{\text{I,I}}$. On the elastic body thus strained, superpose the force system II: T_i', F_i'. The additional displacements are the same as though force system I were absent. The additional work done consists of two parts, the work $U_{\text{II,II}}$ done by force system II acting through displacements II, and the work $U_{\text{I,II}}$ done by force system I acting through displacements II. The total work is thus given by

$$U_{\text{I,I}} + U_{\text{II,II}} + U_{\text{I,II}}.$$

If the force systems were applied in the reverse order, II and I, then the total work would be

$$U_{\text{II,II}} + U_{\text{I,I}} + U_{\text{II,I}}.$$

But the final state of the body is independent of the order in which the loads are applied.[1] Hence we must have

$$U_{\text{I,I}} + U_{\text{II,II}} + U_{\text{I,II}} = U_{\text{II,II}} + U_{\text{I,I}} + U_{\text{II,I}}$$

or

$$U_{\text{I,II}} = U_{\text{II,I}};$$

that is,

$$\int_\Sigma T_i u_i' \, d\sigma + \int_\tau F_i u_i' \, d\tau = \int_\Sigma T_i' u_i \, d\sigma + \int_\tau F_i' u_i \, d\tau.$$

The value of the Reciprocal Theorem lies in the fact that, for every choice of values for the variables of the primed state T_i', F_i', u_i', one obtains a theorem relating the applied forces T_i, F_i, and displacements u_i in an elastic body. This is exemplified in the problems at the end of this section and in the theorems we now proceed to derive as special cases of the Reciprocal Theorem.

We write out now an important specialization of the Reciprocal Theorem to the case of a body deformed by concentrated forces. For the

[1] It should be noted that the Reciprocal Theorem depends only on the linearity of the equations of equilibrium and hence on the principle of superposition. We assume that the forces F_i, T_i do not depend on the displacements u_i' and that the displacements u_i do not affect the forces F_i', T_i'.

sake of concreteness, we shall speak of a beam bent by point loads. Assume that the body forces vanish, and write the Reciprocal Theorem in the form

$$\int_\Sigma T_i u_i'\, d\sigma = \int_\Sigma T_i' u_i\, d\sigma.$$

Consider two equilibrium states of a beam, one with load $p(x)$ and deflection $y(x)$, and another with load $p'(x)$ and corresponding deflection $y'(x)$. By the Reciprocal Theorem, we have

$$\int_0^l py'\, dx = \int_0^l p'y\, dx. \tag{109.4}$$

Let the beam be loaded only by concentrated forces $P_1, P_2, \ldots$ applied at the points $x_1, x_2, \ldots$, and denote by α_{ij} *the transverse displacement at* x_i *due to a unit transverse force applied at* x_j. We choose for the load system p the concentrated force P_1; then the corresponding displacements y at x_1 and x_2 are

$$y_1 = \alpha_{11}P_1 \text{ at } x_1, \qquad y_2 = \alpha_{21}P_1 \text{ at } x_2.$$

Similarly, for the load p' in (109.4), we take the force P_2; the associated displacements y' at the points x_1, x_2 are

$$y_1' = \alpha_{12}P_2 \text{ at } x_1, \qquad y_2' = \alpha_{22}P_2 \text{ at } x_2.$$

According to the Reciprocal Theorem, we have

$$P_1 y_1' = P_2 y_2,$$

or

$$P_1\alpha_{12}P_2 = P_2\alpha_{21}P_1;$$

that is,

$$\alpha_{12} = \alpha_{21}. \tag{109.5}$$

The quantity α_{ij} is called the *influence coefficient* (designated so by Maxwell) for transverse deflection at x_i due to a force applied at x_j. The symmetry of the influence coefficients

$$\alpha_{ij} = \alpha_{ji}$$

is seen to be a special case of the Reciprocal Theorem.[1]

Consider now the effect of varying a force T_i applied at a point P on the surface of an elastic body. To this end, denote by Σ' a portion of the surface Σ that includes the point P as an inner point. The remainder of the surface Σ will be denoted by $\Sigma - \Sigma'$. In the Reciprocal Theorem,

[1] For a discussion of the Maxwell influence coefficients and of determinants with these coefficients as elements, see C. B. Biezeno and R. Grammel, Technische Dynamik, Chap. II, Secs. 9–11.

we choose

$$F'_i = F_i,$$
$$T'_i = \begin{cases} T_i & \text{on } \Sigma - \Sigma', \\ T_i + \Delta T_i & \text{on } \Sigma', \end{cases}$$

and get[1]

$$\int_\tau F_i u'_i \, d\tau + \int_\Sigma T_i u'_i \, d\sigma = \int_\tau F_i u_i \, d\tau + \int_\Sigma T_i u_i \, d\sigma + \int_{\Sigma'} (\Delta T_i) u_i \, d\sigma. \tag{109.6}$$

Now, by (27.1), the strain energy U associated with the original load system T_i, F_i is

$$2U = \int_\tau F_i u_i \, d\tau + \int_\Sigma T_i u_i \, d\sigma,$$

while that corresponding to the varied state T'_i, F'_i is

$$\begin{aligned} 2U' &= \int_\tau F'_i u'_i \, d\tau + \int_\Sigma T'_i u'_i \, d\sigma \\ &= \int_\tau F_i u'_i \, d\tau + \int_\Sigma T_i u'_i \, d\sigma + \int_{\Sigma'} (\Delta T_i) u'_i \, d\sigma. \end{aligned}$$

Equation (109.6) can thus be written as

$$2U' - \int_{\Sigma'} (\Delta T_i) u'_i \, d\sigma = 2U + \int_{\Sigma'} (\Delta T_i) u_i \, d\sigma,$$

or

$$2(U' - U) \equiv 2\,\Delta U = \int_{\Sigma'} (\Delta T_i)(u_i + u'_i) \, d\sigma.$$

When the region Σ' is small, we have, approximately,

$$2\,\Delta U \doteq (\Delta T_i)(u_i + u'_i)\Sigma'.$$

We denote the increment of force acting on the area Σ' by

$$\Delta \mathfrak{F}_i = (\Delta T_i)\Sigma'.$$

Then

$$\frac{\Delta U}{\Delta \mathfrak{F}_i} \doteq \frac{1}{2}(u_i + u'_i),$$

and, letting $\Delta \mathfrak{F}_i$ approach zero, we get

$$\frac{dU}{d\mathfrak{F}_i} = u_i. \tag{109.7}$$

Equation (109.7) expresses the *Theorem of Castigliano.*[2]

[1] It is assumed that the body is rigidly supported and hence that the supporting forces do no work.

[2] For further discussion of this important theorem, see C. B. Biezeno and R. Grammel, Technische Dynamik, Chap. II, Sec. 8.

PROBLEMS

1. Consider a beam loaded by concentrated forces P_1 at x_1 and P_2 at x_2, and let y_i be the deflection at x_i. Calculate the additional deflections dy_i and the change dU in the strain energy corresponding to a change dP_1 in the force P_1. Show that

$$\alpha_{12} = \frac{\partial^2 U}{\partial P_1\, \partial P_2},$$

and hence that

$$\alpha_{12} = \alpha_{21}.$$

2. Show that the influence coefficients for a cantilever beam are

$$\alpha_{12} = \alpha_{21} = \begin{cases} \dfrac{1}{6EI}\, x_1^2(3x_2 - x_1), & x_1 < x_2, \\ \dfrac{1}{6EI}\, x_2^2(3x_1 - x_2), & x_2 < x_1, \end{cases}$$

where α_{ij} is the deflection at x_i due to a unit load at x_j.

Show that the deflection of a cantilever beam bent by an end load P is given by

$$y(x) = P\alpha_{xl} = \frac{P}{6EI}\, x^2(3l - x),$$

and (see Sec. 110) verify that the strain energy stored in the beam is

$$U = \frac{l^3}{6EI} P^2 = \frac{3EI}{2l^3}\, \delta^2,$$

where δ is the end deflection $y(l)$. Show that

$$\frac{dU(P)}{dP} = \delta, \qquad \frac{dU(\delta)}{d\delta} = P.$$

3. In the Reciprocal Theorem, take $F'_i = 0$, $\tau'_{ij} = \delta_{ij}$. Show that $T'_i = \tau'_{ij}\nu_j = \nu_i$, $\Theta' = \tau'_{ii} = 3$, $e'_{ij} = \dfrac{1 - 2\sigma}{E}\, \delta_{ij}$, and $u'_i = \dfrac{1 - 2\sigma}{E}\, x_i$. Insert these expressions in (109.2), and derive the following expression for the change in volume ΔV_0 in an elastic body under the action of surface forces T_i and body forces F_i:

$$\Delta V_0 = \int_\tau \vartheta\, d\tau = \frac{1 - 2\sigma}{E} \left(\int_\Sigma T_i x_i\, d\sigma + \int_\tau F_i x_i\, d\tau \right).$$

4. Fill in the details of the following direct calculation of the change in volume ΔV_0 of an elastic body τ under the action of surface forces T_i and body forces F_i:

$$\begin{aligned} \Delta V_0 &= \int_\tau \vartheta\, d\tau = \frac{1 - 2\sigma}{E} \int_\tau \tau_{ii}\, d\tau \\ &= \frac{1 - 2\sigma}{E} \int_\tau \tau_{ij} x_{i,j}\, d\tau \\ &= \frac{1 - 2\sigma}{E} \int_\tau (\tau_{ij} x_{i,j} + \tau_{ij,j} x_i + F_i x_i)\, d\tau \\ &= \frac{1 - 2\sigma}{E} \int_\tau [(\tau_{ij} x_i)_{,j} + F_i x_i]\, d\tau \\ &= \frac{1 - 2\sigma}{E} \left(\int_\Sigma T_i x_i\, d\sigma + \int_\tau F_i x_i\, d\tau \right). \end{aligned}$$

5. Show that the average value of a strain component e_{11}, say, throughout an elastic body subjected to surface forces T_i and body forces F_i, can be found from the Reciprocal Theorem in the form (109.3) by putting $\tau'_{11} = 1$, other $\tau_{ij} = 0$. Derive the formula

$$\int_\tau e_{11}\, d\tau = \frac{1}{E}\int_\Sigma (T_1x_1 - \sigma T_2x_2 - \sigma T_3x_3)\, d\sigma + \frac{1}{E}\int_\tau (F_1x_1 - \sigma F_2x_2 - \sigma F_3x_3)\, d\tau.$$

6. Show that the average deflection of a cantilever beam due to a concentrated load P applied at a point x_0 is equal to the deflection at x_0 produced by the load P distributed uniformly over the length of the beam. Neglect the weight of the beam.

7. In the Reciprocal Theorem, take for the primed system of forces and displacements those of the problem of a beam under tension by end forces. That is, derive the expression

$$\iint\limits_{z=0} (\sigma x\tau_{zz} + \sigma y\tau_{zy})\, dx\, dy + \iint\limits_{z=l} (-\sigma x\tau_{zz} - \sigma y\tau_{zy} + l\tau_{zz})\, dx\, dy$$
$$= -E\iint\limits_{z=0} w\, dx\, dy + E\iint\limits_{z=l} w\, dx\, dy$$

which is valid for the stress system in any beam free of body forces and loaded at the ends. Verify the Reciprocal Theorem by taking the longitudinal displacement w and the stresses τ_{ij} to be those of the problem of bending by end couples.

8. Define the strain deviations e'_{ij} by

$$e_{ij} = \tfrac{1}{3}\vartheta\delta_{ij} + e'_{ij};$$

that is,

$$e_{xx} = \tfrac{1}{3}\vartheta + e'_{xx}, \qquad e_{yy} = \tfrac{1}{3}\vartheta + e'_{yy}, \qquad e_{zz} = \tfrac{1}{3}\vartheta + e'_{zz},$$
$$e_{xy} = e'_{xy}, \qquad e_{yz} = e'_{yz}, \qquad e_{zx} = e'_{zx},$$

where $\tfrac{1}{3}\vartheta \equiv \tfrac{1}{3}(e_{xx} + e_{yy} + e_{zz})$ is the mean extension.

Show that the cubical dilatation

$$\vartheta' \equiv e'_{xx} + e'_{yy} + e'_{zz} = 0,$$

and hence that the strain-deviation tensor represents a change in shape without a change in volume.

Show that the principal strains $e_{\rm I}$ and the principal strain deviations $e'_{\rm I}$ are connected by the relations

$$e'_{\rm I} = e_{\rm I} - \tfrac{1}{3}\vartheta, \qquad e'_{\rm II} = e_{\rm II} - \tfrac{1}{3}\vartheta, \qquad e'_{\rm III} = e_{\rm III} - \tfrac{1}{3}\vartheta.$$

Hint: The principal strain deviations are the roots of the determinantal equation $|e'_{ij} - e'\delta_{ij}| = 0$.

9. Define the stress deviation τ'_{ij} by

$$\tau_{ij} = \tfrac{1}{3}\Theta\delta_{ij} + \tau'_{ij},$$

or

$$\tau_{xx} = \tfrac{1}{3}\Theta + \tau'_{xx}, \qquad \tau_{yy} = \tfrac{1}{3}\Theta + \tau'_{yy}, \qquad \tau_{zz} = \tfrac{1}{3}\Theta + \tau'_{zz},$$
$$\tau_{xy} = \tau'_{xy}, \qquad \tau_{yz} = \tau'_{yz}, \qquad \tau_{zx} = \tau'_{zx},$$

where $\tfrac{1}{3}\Theta \equiv \tfrac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz})$ is the mean normal stress. Show that the stress-deviation and strain-deviation tensors are related by

$$\tau'_{ij} = 2\mu e'_{ij},$$

and that

$$\Theta' \equiv \tau'_{xx} + \tau'_{yy} + \tau'_{zz} = 0.$$

Show that the principal stress deviations τ_i' and the principal stresses τ_i are connected by the relations

$$\tau_{\rm I}' = \tau_{\rm I} - \tfrac{1}{3}\Theta, \qquad \tau_{\rm II}' = \tau_{\rm II} - \tfrac{1}{3}\Theta, \qquad \tau_{\rm III}' = \tau_{\rm III} - \tfrac{1}{3}\Theta.$$

10. Verify the identity

$$\begin{aligned} \vartheta^2 &= (e_{xx} + e_{yy} + e_{zz})^2 \\ &= 3(e_{xx}^2 + e_{yy}^2 + e_{zz}^2) - (e_{xx} - e_{yy})^2 - (e_{yy} - e_{zz})^2 - (e_{zz} - e_{xx})^2, \end{aligned}$$

and show that the strain-energy density can be written in the form

$$W = W_1 + W_2$$

where

$$\begin{aligned} W_1 &= \tfrac{1}{2}k\vartheta^2, \\ W_2 &= \tfrac{1}{3}\mu[(e_{xx} - e_{yy})^2 + (e_{yy} - e_{zz})^2 + (e_{zz} - e_{xx})^2 + 6(e_{xy}^2 + e_{yz}^2 + e_{zx}^2)] \\ &= \tfrac{1}{3}\mu[(e_{xx}' - e_{yy}')^2 + (e_{yy}' - e_{zz}')^2 + (e_{zz}' - e_{xx}')^2 + 6(e_{xy}'^2 + e_{yz}'^2 + e_{zx}'^2)], \end{aligned}$$

and where e_{ij}' is the strain-deviation tensor. Show that W_1 depends only on the change of volume, while W_2 is that part of the strain-energy density arising from a change of shape. We call W_1 the strain-energy density of dilatation and W_2 the strain-energy density of distortion.

11. From $\Theta = 3k\vartheta$ and $\tau_{ij}' = 2\mu e_{ij}'$ (see Prob. 9), show that the strain-energy density W can be written as the sum of the strain-energy density of dilatation W_1 and the strain-energy density of distortion W_2, where

$$\begin{aligned} W_1 &= \frac{1}{18k}\Theta^2, \\ W_2 &= \frac{1}{12\mu}[(\tau_{xx} - \tau_{yy})^2 + (\tau_{yy} - \tau_{zz})^2 + (\tau_{zz} - \tau_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)] \\ &= \frac{1}{12\mu}[(\tau_{xx}' - \tau_{yy}')^2 + (\tau_{yy}' - \tau_{zz}')^2 + (\tau_{zz}' - \tau_{xx}')^2 + 6(\tau_{xy}'^2 + \tau_{yz}'^2 + \tau_{zx}'^2)]. \end{aligned}$$

The strain-energy density of distortion has been used as a criterion for failure of the material. See S. Timoshenko, Strength of Materials, vol. 2 (or Theory of Elasticity, p. 149), and A. and L. Föppl, Drang und Zwang, vol. 1, Sec. 6.

12. Consider a beam stretched by a longitudinal stress p uniformly distributed over the end sections. Show that the strain-energy density of dilatation W_1 and that of distortion W_2 are given by

$$W_1 = \frac{p^2}{18k}, \qquad W_2 = \frac{p^2}{6\mu} = \frac{2(1+\sigma)}{1-2\sigma}W_1.$$

Show that in the torsion of a cylindrical shaft we have

$$W_1 = 0, \qquad W_2 = \frac{\tau^2}{2\mu} = \frac{1}{2\mu}(\tau_{zx}^2 + \tau_{zy}^2).$$

13. Show that the strain energy stored in a beam of length l (parallel to the x-axis) bent by end couples M_z can be written as

$$U = \int_\tau W\,d\tau = \int_0^l \frac{M_z^2}{2EI_z}dx = \frac{M_z^2 l}{2EI_z}.$$

110. Illustrative Examples. As an illustration of the use of the minimum principles in deriving the equilibrium and compatibility equations, we consider several specific problems.

a. Deflection of an Elastic String. Let a stretched string, with the end points fixed at (0, 0) and (l, 0), be deflected by a distributed transverse load $f(x)$ per unit length of the string. We suppose that the transverse deflection $y(x)$ is small and the change in the stretching force T produced by deflection is negligible. These are the usual assumptions used in deriving the equation for $y(x)$ from considerations of static equilibrium. We choose to deduce this equation from the Principle of Minimum Potential Energy.

From (107.5), the potential energy V is

$$V = U - \int_0^l f(x)y\,dx,$$

where the strain energy U is equal to the product of the tensile force T by the total stretch e of the string. But

$$e = \int_0^l (ds - dx) = \int_0^l [\sqrt{1 + (y')^2} - 1]\,dx,$$

and since we are dealing with the linear theory, $(y')^2 \ll 1$, and we can write

$$e = \tfrac{1}{2}\int_0^l (y')^2\,dx.$$

Consequently,

$$U = \frac{T}{2}\int_0^l (y')^2\,dx,$$

and, finally,

$$V = \int_0^l [\tfrac{1}{2}T(y')^2 - f(x)y]\,dx.$$

This functional has the form (106.12), and we see from (106.11) that the appropriate Euler's equation is

$$Ty'' + f(x) = 0.$$

This is the familiar equation for the transverse deflection of the string under the load $f(x)$.

b. Deflection of the Central Line of a Beam. Let the axis of a beam of constant cross section coincide with the x-axis, and suppose that the beam is bent by a transverse load $p = f(x)$ estimated per unit length of the beam. As is customary in the technical theory of beams, we suppose that the shearing stresses are negligible in comparison with the tensile stress

$$\tau_{xx} = \frac{My}{I}.$$

The strain e_{xx} is then given by

$$e_{xx} = \frac{\tau_{xx}}{E} = \frac{My}{EI},$$

and thus, from (26.12), the strain-energy density is,

$$W = \frac{1}{2}\tau_{xx}e_{xx} = \frac{M^2y^2}{2EI^2}.$$

The strain energy per unit length of the beam is found by integrating over the cross section of the beam, and we get

$$\int_R W\,d\sigma = \frac{M^2}{2EI^2}\int_R y^2\,d\sigma = \frac{M^2}{2EI}.$$

But, from the Bernoulli-Euler law, $M = -EIy''$, and thus

$$\int_R W\,d\sigma = \tfrac{1}{2}EI(y'')^2.$$

The total strain energy U is got by integrating this expression over the length of the beam, and we find

$$U = \int_0^l \tfrac{1}{2}EI(y'')^2\,dx.$$

We suppose that the ends of the beam are clamped, hinged, or free so that the supporting forces do no work and hence contribute nothing to potential energy V. If we neglect the weight of the beam, the only external load is $p = f(x)$ and (107.5) then yields

$$\begin{aligned} V &= \int_0^l \tfrac{1}{2}EI(y'')^2\,dx - \int_0^l f(x)y\,dx \\ &= \int_0^l [\tfrac{1}{2}EI(y'')^2 - f(x)y]\,dx. \end{aligned}$$

This functional has the form (106.13), and the Euler equation, therefore, is

$$\frac{d^2}{dx^2}(EIy'') - f(x) = 0.$$

c. Deflection of an Elastic Membrane. Let the membrane, with fixed edges, occupy some region in the xy-plane. We suppose that the membrane is stretched so that the tension T is uniform and that T is so great that it is not appreciably altered when the membrane is deflected by a distributed normal load of intensity $f(x,y)$. The situation here is analogous to that considered in the first problem of this section. As in that problem we first compute the strain energy[1] U.

Now, the total stretch e of the surface $z = u(x, y)$ is

$$e = \iint_R (d\sigma - dx\,dy) = \iint_R (\sqrt{u_x^2 + u_y^2 + 1} - 1)\,dx\,dy,$$

[1] Strictly speaking we are calculating the increase in strain energy over the constant energy U_0 produced by the stretching forces T. Since U_0 is constant and we are concerned with the variation of the total energy, we can disregard U_0.

where $d\sigma = \sqrt{u_x^2 + u_y^2 + 1}$ is the element of area of the membrane in the deformed state. If the displacement u and its first derivatives are small, we can write

$$e = \tfrac{1}{2} \iint_R (u_x^2 + u_y^2)\, dx\, dy,$$

and hence

$$U = \frac{T}{2} \iint_R (u_x^2 + u_y^2)\, dx\, dy.$$

Therefore

$$V = \iint_R \left[\frac{T}{2} (u_x^2 + u_y^2) - f(x, y)u\right] dx\, dy.$$

The equilibrium state is characterized by the condition $\delta V = 0$, and we readily find, on referring to (106.23), that

$$T\nabla^2 u + f(x, y) = 0.$$

d. Torsion of Cylinders. As our final illustration in this section we consider the Saint-Venant torsion problem for a cylinder of an arbitrary cross section. We shall use the Principle of Minimum Complementary Energy to deduce the appropriate compatibility equation.

We assume with Saint-Venant that the only nonvanishing stresses are τ_{zx} and τ_{zy} and recall[1] that the displacement components in the cross section are

(110.1) $$u = -\alpha zy, \qquad v = \alpha zx.$$

In the formula for complementary energy,

[108.10] $$V^* = U - \int_{\Sigma_u} T_i u_i \, d\sigma,$$

the surface integral must be evaluated only over the ends of the cylinder, since the external forces are known over its lateral surface. We recall that

$$U = \int_\tau W\, d\tau,$$

where

$$W = \tfrac{1}{2}\tau_{ij}e_{ij} = (\tau_{zx}e_{zx} + \tau_{zy}e_{zy}).$$

Using the stress-strain relations

$$\tau_{zx} = 2\mu e_{zx}, \qquad \tau_{zy} = 2\mu e_{zy}$$

we find

$$W = \frac{1}{2\mu} (\tau_{zx}^2 + \tau_{zy}^2),$$

[1] Sec. 34.

and hence

$$U = \frac{1}{2\mu} \int_\tau (\tau_{zx}^2 + \tau_{zy}^2)\, d\tau.$$

To compute the surface integral in (108.10), we make use of (110.1) and get, for the end $z = 0$,

$$\int_R T_i u_i \, d\sigma = 0,$$

since $u = v = 0$, for $z = 0$.

On the end $z = l$, we have

$$\begin{aligned} \int_R T_i u_i \, d\sigma &= \iint_R (\tau_{zx} u + \tau_{zy} v)\, dx\, dy \\ &= \iint_R (-\alpha l y \tau_{zx} + \alpha l x \tau_{zy})\, dx\, dy. \end{aligned}$$

Thus,

$$(110.2) \qquad V^* = \frac{l}{2\mu} \iint_R (\tau_{zx}^2 + \tau_{zy}^2)\, dx\, dy - \alpha l \iint_R (x\tau_{zy} - y\tau_{zx})\, dx\, dy.$$

In this case the admissible stresses satisfy the equilibrium equation

$$(110.3) \qquad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \qquad \text{in } R,$$

and the boundary condition

$$(110.4) \qquad \tau_{zx} \cos (x, \nu) + \tau_{zy} \cos (y, \nu) = 0 \qquad \text{on } C.$$

Equation (110.3) will clearly be satisfied if we introduce the stress function $\Psi(x, y)$, such that

$$(110.5) \qquad \tau_{zx} = \mu\alpha \frac{\partial \Psi}{\partial y}, \qquad \tau_{zy} = -\mu\alpha \frac{\partial \Psi}{\partial x}.$$

The boundary condition (110.4) then requires that

$$\mu\alpha \left(\frac{\partial \Psi}{\partial y} \frac{dy}{ds} + \frac{\partial \Psi}{\partial x} \frac{dx}{ds} \right) = 0,$$

or

$$\frac{d\Psi}{ds} = 0.$$

Thus, Ψ has a constant value on C.

On substituting from (110.5) in (110.2), we get

$$(110.6) \qquad V^* = \frac{\mu\alpha^2 l}{2} \iint_R [(\Psi_x)^2 + (\Psi_y)^2 + 2(x\Psi_x + y\Psi_y)]\, dx\, dy,$$

which is of the form (106.15). Accordingly, the Euler equation yields

$$\nabla^2\Psi = -2 \qquad \text{in } R,$$

which is precisely the equation for the Prandtl stress function we got by a different method in Sec. 35.

The formula (110.6) can be written in a simpler form which we shall find useful in subsequent considerations.

We note that

$$\iint_R (x\Psi_x + y\Psi_y)\,dx\,dy = \iint_R \left[\frac{\partial(x\Psi)}{\partial x} + \frac{\partial(y\Psi)}{\partial y}\right] dx\,dy - 2\iint_R \Psi\,dx\,dy.$$

But

$$\iint_R \left[\frac{\partial(x\Psi)}{\partial x} + \frac{\partial(y\Psi)}{\partial y}\right] dx\,dy = \int_C \Psi[x\cos(x, \nu) + y\cos(y, \nu)]\,ds,$$

so that

$$V^* = \frac{\mu\alpha^2 l}{2}\iint_R [(\nabla\Psi)^2 - 4\Psi]\,dx\,dy + 2\int_C \Psi[x\cos(x, \nu) + y\cos(y, \nu)]\,ds,$$

where $(\nabla\Psi)^2 \equiv \Psi_x^2 + \Psi_y^2$.

If the region R is simply connected, we can take $\Psi = 0$ on C and, for the determination of Ψ, we have the functional

$$V^* = \frac{\mu\alpha^2 l}{2}\iint_R [(\nabla\Psi)^2 - 4\Psi]\,dx\,dy. \tag{110.7}$$

This functional is to be minimized on the set of all functions of class C^2 vanishing on the boundary of the simply connected region R.

We shall consider some specific uses of this formula in Sec. 116.

111. Variational Problem Related to the Biharmonic Equation. Let us investigate next the variational problem

$$I(u) = \iint_R [(\nabla^2 u)^2 - 2fu]\,dx\,dy = \min, \tag{111.1}$$

where the admissible functions $u(x, y)$ belong to class C^4 and satisfy on the boundary C of R the conditions

$$\begin{cases} u = g(s), \\ \dfrac{du}{d\nu} = h(s). \end{cases} \tag{111.2}$$

We suppose that the set $\{u(x, y)\}$ includes the minimizing function $u(x, y)$ and represent the functions of this set in the form,

$$\bar{u}(x, y) = u(x, y) + \epsilon\eta(x, y).$$

As in Sec. 106, we calculate the variation

$$\delta I = \left. \frac{\partial I(u + \epsilon\eta)}{\partial \epsilon} \right|_{\epsilon=0,}$$

and find

$$\delta I = 2 \iint_R (\nabla^2 u \nabla^2 \eta - f\eta)\, dx\, dy. \tag{111.3}$$

But

$$\iint_R \nabla^2 u \nabla^2 \eta\, dx\, dy = \iint_R \left[\frac{\partial}{\partial x} \left(\nabla^2 u \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nabla^2 u \frac{\partial \eta}{\partial y} \right) \right] dx\, dy$$

$$\iint_R \left[\frac{\partial}{\partial x} \left(\eta \frac{\partial \nabla^2 u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\eta \frac{\partial \nabla^2 u}{\partial y} \right) \right] dx\, dy$$

$$+ \iint_R \left[\frac{\partial^2 (\nabla^2 u)}{\partial x^2} + \frac{\partial^2 (\nabla^2 u)}{\partial y^2} \right] \eta\, dx\, dy.$$

The first two integrals in the right-hand member of this identity are in the form to which the Divergence Theorem is applicable, and we get,

$$\iint_R \nabla^2 u \nabla^2 \eta\, dx\, dy = \int_C \nabla^2 u \frac{\partial \eta}{\partial \nu}\, ds - \int_C \eta \frac{\partial \nabla^2 u}{\partial \nu}\, ds + \iint_R \eta \nabla^2 \nabla^2 u\, dx\, dy.$$

Inserting this result in (111.3), we find

$$\delta I = 2 \left[\iint_R (\nabla^4 u - f)\eta\, dx\, dy + \int_C \nabla^2 u \frac{\partial \eta}{\partial \nu}\, ds - \int_C \eta \frac{\partial \nabla^2 u}{\partial \nu}\, ds \right]. \tag{111.4}$$

But $\delta I = 0$, since u minimizes (111.1), and inasmuch as η and $\frac{\partial \eta}{\partial \nu}$ vanish on C, we conclude that

$$\nabla^4 u = f(x, y) \qquad \text{in } R. \tag{111.5}$$

This is the differential equation we have encountered in Chap. 5, where it was observed that it also arises in the study of the transverse deflections of thin elastic plates.

We have assumed in the foregoing that the admissible functions in the set $\{u(x, y)\}$ satisfy the boundary conditions (111.2). If we consider a larger set S of all functions u belonging to class C^4, then (111.4) must also vanish for every u in this set. But the set S includes functions that

satisfy the boundary conditions (111.2), and thus we must have,

$$\iint_R (\nabla^4 u - f)\eta \, dx \, dy = 0.$$

Since η is arbitrary, it follows that the minimizing function $u(x, y)$ again satisfies Eq. (111.5) and we conclude from formula (111.4) that

$$\int_C \nabla^2 u \frac{\partial \eta}{\partial \nu} ds - \int_C \eta \frac{\partial(\nabla^2 u)}{\partial \nu} ds = 0 \tag{111.6}$$

for every η of class C^4.

Now if we consider first all η such that $\eta = 0$ on C and $\frac{\partial \eta}{\partial \nu} \neq 0$ on C, it follows from (111.6) that

$$\nabla^2 u = 0 \qquad \text{on } C. \tag{111.7}$$

On the other hand, if we consider only those η which do not vanish on C but whose normal derivatives vanish on C, we get the condition,

$$\frac{\partial(\nabla^2 u)}{\partial \nu} = 0 \qquad \text{on } C. \tag{111.8}$$

Hence if the functional in (111.1) is minimized on the set S of all u of class C^4, the minimizing function will be found among those functions of S which satisfy the conditions (111.7) and (111.8) on the boundary of the region.

112. The Ritz Method. One-dimensional Case. It was demonstrated in Secs. 107 and 108 that the determination of functions that minimize the functional (107.5) for the potential energy V, or the expression (108.10) for the complementary energy V^*, is equivalent to obtaining solutions of appropriate Euler's equations. In the variational problem $V = \min$, the Euler equations are the Cauchy equilibrium equations, while, in the problem $V^* = \min$, they are the compatibility equations. In the preceding section we have indicated some uses of minimum principles in the derivation of the differential equations for specific problems. However, a by far more important use of these principles relates to the construction, with the aid of direct methods of calculus of variations, of sequences of functions which converge to desired solutions of Euler's equations. One such direct method was proposed by Lord Rayleigh and, independently and from a more general point of view, by W. Ritz.[1]

[1] Lord Rayleigh, Theory of Sound (1926), vols. 1 and 2; W. Ritz, *Journal für der reine und angewandte Mathematik*, vol. 135 (1908), or Oeuvres Complets de W. Ritz (1911).

We proceed to describe the essence of this method by considering first the functional

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\, dx, \tag{112.1}$$

to be minimized on the set of all functions $y(x)$ of class C^1, such that

$$y(x_0) = y_0, \qquad y(x_1) = y_1. \tag{112.2}$$

Let the exact solution of the problem $I = \min$ be $y = y^*(x)$, and let $I(y^*) = m$ be the minimum of $I(y)$ on this set. If, for some admissible function $\bar{y}(x)$, the value $I(\bar{y}) = \bar{m}$ is close to m, one would suspect that $\bar{y}(x)$ is a good approximation to the minimizing function $y^*(x)$. Moreover, if a convergent sequence of functions $\bar{y}_n(x)$ $(n = 1, 2, \ldots)$ is determined for which

$$\lim_{n\to\infty} I(\bar{y}_n) = m,$$

one would be inclined to guess that

$$\lim_{n\to\infty} \bar{y}_n = y^*(x).$$

Guided by such considerations, Ritz proposed to construct $\bar{y}(x)$ by choosing a family of functions

$$y(x) = \varphi(x, a_1, a_2, \ldots, a_k), \tag{112.3}$$

depending on k real parameters a_i, where φ is such that for all values of the a_i the end conditions (112.2) are satisfied. Then, on inserting (112.3) in (112.1), one obtains $I(a_1, a_2, \ldots, a_k)$, which can be minimized by determining the values of the a_i from the system of equations

$$\frac{\partial I}{\partial a_r} = 0, \qquad (r = 1, 2, \ldots, k).$$

If the solutions of this system minimizing $I(y)$ are $\bar{a}_i$, then the minimizing function $\bar{y}(x)$ is

$$\bar{y}(x) = \varphi(x, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k).$$

It is reasonable to expect that $\bar{y}(x)$ will be a fair approximation to the minimizing function $y^*(x)$ when the number of parameters in (112.3) is made sufficiently large.

We proceed next to construct a sequence $\{\bar{y}_n(x)\}$ of functions $\bar{y}_n(x)$ such that $\lim I(\bar{y}_n) = m$. Consider a sequence of families of functions of the type (112.3), namely

$$\begin{cases} y_1(x) = \varphi_1(x, a_1), \\ y_2(x) = \varphi_2(x, a_1, a_2), \\ \cdots\cdots\cdots\cdots \\ y_n(x) = \varphi_n(x, a_1, a_2, \ldots, a_n), \end{cases} \tag{112.4}$$

where the family $y_k = \varphi_k(x, a_1, a_2, \ldots, a_k)$ includes in it all functions in the families with subscripts less than k. The parameters a_i in each y_k can be determined so as to minimize the integral $I(y)$. We denote them by $\bar{a}_i$ and the minimizing functions by

$$(112.5) \qquad \bar{y}_n(x) = \varphi_n(x, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n), \qquad (n = 1, 2, \ldots).$$

Since each family $y_k(x)$ includes the families $y_{k-1}(x)$ for special values of parameters a_i, the successive minima $I(\bar{y}_k)$ are nonincreasing and we can write

$$I(\bar{y}_1) \geq I(\bar{y}_2) \geq \cdots.$$

Since the infinite sequence $\{I(\bar{y}_n)\}$ is bounded below by m, it is convergent, but it need not converge to the minimum $I(y^*) = m$. In order to ensure the convergence of the sequence $\{I(\bar{y}_n)\}$ to $I(y^*)$, one must impose some restrictions on the choice of functions φ_i in the set (112.4). These restrictions pertain to a special character of approximation of every admissible function $y(x)$ by functions of the set (112.4).

DEFINITION: *Let $y(x)$, of class C^1 in (x_0, x_1) satisfy the end conditions $y(x_0) = y_0$, $y(x_1) = y_1$. If, for every $\epsilon > 0$, there exists in the family* (112.4) *a function $y_n^*(x) = y_n(x, a_1^*, a_2^*, \ldots, a_n^*)$ such that $|y_n^* - y| < \epsilon$ and $|y_n^{*\prime} - y'| < \epsilon$ for all x in (x_0, x_1), then the set of functiohs* (112.4) *is said to be relatively complete.*

We prove next that when the set (112.4) is relatively complete the sequence $\{\bar{y}_n\}$, defined by (112.5), is such that $\lim I(\bar{y}_n) = I(y^*) = m$.

Indeed, when the set (112.4) is relatively complete, there exists a function $y_n^*(x, a_1^*, a_2^*, \ldots, a_n^*)$ that approximates arbitrarily closely *both* the exact solution $y^*(x)$ of the problem $I(y) = \min$ and the derivative of $y^*(x)$. That is,

$$(112.6) \qquad |y_n^* - y^*| < \epsilon, \qquad |y_n^{*\prime} - y^{*\prime}| < \epsilon.$$

But $F(x, y, y')$ is a continuous function of its arguments, and therefore

$$|F(x, y_n^*, y_n^{*\prime}) - F(x, y^*, y^{*\prime})| < \epsilon$$

for $x_0 \leq x \leq x_1$.

Consequently[1]

$$I(y_n^*) - I(y^*) = \int_{x_0}^{x_1} [F(x, y_n^*, y_n^{*\prime}) - F(x, y^*, y^{*\prime})]\, dx < \epsilon',$$

and therefore,

$$(112.7) \qquad I(y_n^*) < I(y^*) + \epsilon'.$$

But y_n^* is a function of the set (112.4), and since $I(\bar{y}_n)$ is a minimum of $I(y)$ on the family y_n, we have

$$(112.8) \qquad I(\bar{y}_n) \leq I(y_n^*).$$

[1] The absolute-value signs are omitted in the integrand since $I(y^*) \leq I(y_n^*)$.

Combining the inequalities (112.7) and (112.8), we get

$$I(y^*) \leq I(\bar{y}_n) \leq I(y_n^*) < I(y^*) + \epsilon',$$

but ϵ' can be made as small as we wish, and hence

$$\lim_{n \to \infty} I(\bar{y}_n) = I(y^*) = m.$$

This completes the proof.

It should be noted, however, that ordinarily the condition of relative completeness does not ensure the convergence of the minimizing sequence $\{\bar{y}_n(x)\}$ to the exact solution $y^*(x)$. If $F(x, y, y')$ has the special quadratic form $F = py'^2 + qy^2 + 2fy$, with $p(x) > 0$ and $q(x) \geq 0$ in (x_0, x_1), then it is possible to prove that the conditions (112.6) ensure that $\lim \bar{y}_n(x) = y^*(x)$; moreover, the $\bar{y}_n(x)$ converge to $y^*(x)$ uniformly.[1]

Among useful, relatively complete sets of functions in the interval $(0, l)$ are trigonometric polynomials $\sum_{k=1}^{n} a_k \sin(k\pi x/l)$ and algebraic polynomials $\sum_{k=1}^{n} a_k x^k (l - x)$. The fact that these polynomials are relatively complete follows almost directly from Weierstrass' Theorem on Approximation of Continuous Functions.[2]

PROBLEMS

1. Show that the system of equations $\frac{\partial I(y_n)}{\partial a_j} = 0$, $j = 1, 2, \ldots, n$, for the coefficients in the approximate solution $y_n(x) = \sum_{i=1}^{n} a_i \varphi_i(x)$ of the variational problem

$$I(y) = \int_0^l (py'^2 + qy^2 + 2fy)\, dx = \min, \qquad y(0) = y(l) = 0,$$

by the Ritz method is

$$\int_0^l (py_n'\varphi_j' + qy_n\varphi_j + f\varphi_j)\, dx = 0, \qquad (j = 1, 2, \ldots, n).$$

[1] The derivatives $y_n'(x)$, however, may not converge to $y^{*\prime}(x)$. See for example, L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis, 4th ed. (1952), pp. 280–285.

[2] See, for example, E. C. Titchmarsh, Theory of Functions, 2d ed. (1939), p. 414. In the problems considered above $y'(x)$ is continuous in $(0, l)$; hence it can be approximated arbitrarily closely by a trigonometric polynomial $P_n'(x) = b_0 + \sum_{k=1}^{n} b_k \cos(k\pi x/l)$. On integrating this, it follows that not only $|P_n'(x) - y'(x)| < \epsilon$, but also $|P_n(x) - y(x)| < \epsilon$. Similar arguments apply to approximations by algebraic polynomials.

2. Apply the Ritz method to the problem

$$\int_0^1 [(y')^2 - y^2 - 2xy]\,dx = \min, \qquad y(0) = y(1) = 0,$$

by considering the approximate solutions in the form

$$y_n = x(1 - x)(a_1 + a_2 x + \cdots + a_n x^{n-1}).$$

Determine the $\bar{a}_n$ in $\bar{y}_1$ and $\bar{y}_2$, and compare the approximate solutions with the exact solution $y = (\sin x/\sin 1) - x$ of the Euler equation.

3. The deflection of a cantilever beam of length l, carrying a uniform load p per unit length of the beam [see example Sec. 110b] is given by

$$EIy(x) = \tfrac{1}{24}p(x^4 - 4lx^3 + 6l^2x^2),$$

provided that $y(0) = y'(0) = 0$, $y''(l) = y'''(l) = 0$. Note that the end condition $y'''(l) = 0$ is satisfied by choosing $EIy'''(x) = c \cos(\pi x/2l)$. Integrate this, and obtain

$$EIy(x) = -\frac{2cl}{\pi}\left[\frac{x^2}{2} - \frac{2l}{\pi}\left(x - \frac{2l}{\pi}\sin\frac{\pi x}{2l}\right)\right].$$

Compute the potential energy V corresponding to this choice of deflection, and determine c by minimizing V. Show that the maximum deflection found from the formula so obtained is $y(l) = 0.12603pl^4/EI$ and the maximum bending moment

$$M(0) = -EIy''(0) = -0.469pl^2.$$

Compare these values with those given by the exact formula.

4. Solve the problem of a cantilever beam under uniform load p per unit length by means of the Theorem of Minimum Potential Energy. For the approximate displacement take $EIy(x) = cx^2(3l - x)$, a function that satisfies all the boundary conditions except the vanishing of the shear force at the free end $x = l$. Find the value of the constant c, the maximum displacement, and the maximum bending moment, and compare with the exact results.

5. Illustrate the theorem of Eq. (27.1) by showing that the strain energy stored in a cantilever beam of length l bent by a uniformly distributed load pl is $p^2l^5/40EI$, which is one-half the numerical value of the potential energy of the external forces.

6. Consider a beam bent by a load $p(x)$, and with potential energy

$$V = \int_0^l [\tfrac{1}{2}EI(y'')^2 - py]\,dx.$$

Introduce the approximate deflection

$$y_n = \sum_{i=1}^{n} c_i f_i(x),$$

where the functions $f_i(x)$ satisfy all the boundary conditions. Show that, if the ends of the beam are clamped, hinged, or free, then the minimizing condition can be written in the form

$$\frac{\partial V_n}{\partial c_i} = \int_0^l [(EIy_n'')'' - p]f_i\,dx = 0.$$

7. In the problem of a cantilever bent by an end load P, take for the assumed deflection curve the function

$$EIy(x) = ax^2 + bx^3,$$

which satisfies the geometrical boundary conditions

$$y(0) = y'(0) = 0.$$

Determine the constants a, b from the Principle of Minimum Potential Energy.

113. The Ritz Method. Two-dimensional Case. The special technique of constructing minimizing sequences, discussed in the preceding section, can be easily extended to apply to functionals in the form

$$I(u) = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy. \tag{113.1}$$

We suppose that the admissible functions in the problem $I(u) = \min$ satisfy on the boundary C of R the condition $u = \varphi(s)$. Let the exact solution of the problem $I(u) = \min$ be $u^*(x, y)$, and let $I(u^*) = m$ be the minimum value of the functional (113.1).

We introduce a sequence of families of admissible functions

$$u_n(x, y) = \varphi_n(x, y, a_1, \ldots, a_n), \qquad (n = 1, 2, \ldots), \tag{113.2}$$

with parameters a_i, and suppose that each family $u_k(x, y)$ includes in it families with subscripts less than k. We further suppose that the set (113.2) is relatively complete, so that for every admissible $u(x, y)$ there exists a function $u_n^*(x, y) = \varphi_n(x, y, a_1^*, \ldots, a_n^*)$ belonging to the set (113.2) such that

$$|u_n^* - u| < \epsilon, \qquad \left|\frac{\partial u_n^*}{\partial x} - \frac{\partial u}{\partial x}\right| < \epsilon, \qquad \left|\frac{\partial u_n^*}{\partial y} - \frac{\partial u}{\partial y}\right| < \epsilon, \tag{113.3}$$

for all (x, y) in R.

If we form $I(u_n)$ with the aid of (113.2) and determine the parameters a_i so that $I(u_n)$ is a minimum, we obtain a sequence of numbers $\{I(\bar{u}_n)\}$, where

$$\bar{u}_n(x, y) = \varphi_n(x, y, \bar{a}_1, \ldots, \bar{a}_n).$$

The constants $\bar{a}_i$ are the values of the a_i in (113.2) such that $I(u_n) = \min$. The sequence $\{I(\bar{u}_n)\}$ converges to $I(u^*) = m$, where $u^*(x, y)$ is the function that minimizes (113.1). The proof of this is so similar to that given in the preceding section that there is no need to write it out.

Since $I(u^*) \le I(u_n)$, the sequence $\{I(\bar{u}_n)\}$ approaches the limit $I(u^*)$ from above so that $I(u_n)$ is an upper bound for m.

As in the one-dimensional problem of Sec. 112, there is no assurance that $\bar{u}_n \to u^*$, unless some further restrictions are imposed on the form of the integrand in (113.1) and on the behavior of the sequence $\{\bar{u}_n\}$. Indeed the sequence $\{\bar{u}_n\}$ need not even converge.[1]

[1] L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis, 4th ed. (1952), pp. 355–373.

L. V. Kantorovich, *Doklady Akademii Nauk SSSR*, vol. 30 (1941), pp. 107 and 579.

The procedure outlined above can be applied to functionals containing derivatives of higher orders. Because of the importance of the functional

$$I(u) = \iint_R [(\nabla^2 u)^2 - 2fu] \, dx \, dy \tag{113.4}$$

in elasticity (see Sec. 111), we sketch the construction of minimizing sequences for this functional in some detail.

The Euler equation associated with the problem $I(u) = \min$, we recall, is

$$\nabla^4 u = f(x, y), \tag{113.5}$$

and we consider the homogeneous boundary conditions

$$u = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } C, \tag{113.6}$$

appearing in the study of transverse deflections of clamped elastic plates.

Let us suppose that we have obtained a system of coordinate functions $\varphi_i(x, y)$ such that $u_n = \sum_{k=1}^{n} a_k \varphi_k$ can be made to approximate arbitrarily closely every admissible $u(x, y)$ *together with its first and second derivatives*.

If we form $I(u_n)$ and determine the a_k so that $\dfrac{\partial I(u_n)}{\partial a_j} = 0$ $(j = 1, 2, \ldots, n)$, we get the system of equations,

$$\begin{aligned} &\iint_R (\nabla^2 u_n \nabla^2 \varphi_j - f\varphi_j) \, dx \, dy \\ &\quad = \iint_R \sum_{i=1}^{n} (a_i \nabla^2 \varphi_i \nabla^2 \varphi_j - f\varphi_j) \, dx \, dy = 0, \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{113.7}$$

On denoting the solution of this system by $\bar{a}_k$, we get

$$\bar{u}_n = \sum_{k=1}^{n} \bar{a}_k \varphi_k(x, y),$$

and from the assumed relative completeness of our set of coordinate functions it is easy to show that $I(\bar{u}_n) \to I(u^*)$, where u^* minimizes (113.4). Moreover, it turns out[1] that in this problem the hypothesis of relative completeness suffices to establish convergence of $\{\bar{u}_n\}$ to u^*. This may seem odd because the corresponding hypothesis does not suffice to establish the convergence of $\{\bar{u}_n\}$ to u^* for the simpler functional $\iint_R (u_x^2 + u_y^2) \, dx \, dy$, associated with the problem of Dirichlet. However,

[1] E. Trefftz, *Mathematische Annalen*, vol. 100 (1928), p. 503.
K. N. Shevchenko, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 2 (1938), p. 219

Courant[1] observed that the convergence of the minimizing sequence $\{\bar{u}_n\}$ and of its derivatives improves when the order of derivatives in the functional is increased. On the other hand, convergence becomes worse when the number of independent variables is increased. Thus, if instead of considering the functional

$$[106.21] \qquad I(u) = \iint_R (u_x^2 + u_y^2 + 2uf)\, dx\, dy,$$

associated with the Poisson equation $\nabla^2 u = f(x, y)$, one considers the functional

$$J(u) = \iint_R [u_x^2 + u_y^2 + 2uf + (\nabla^2 u - f)^2]\, dx\, dy,$$

which is minimized by the same function $u^*(x, y)$ since $\nabla^2 u^* - f = 0$, the minimizing sequence associated with $J(u)$ will converge more rapidly than that for the $I(u)$. The inclusion of additional terms in the functional $J(u)$ complicates the application of the Ritz method, but it yields a sequence $\{\bar{u}_n\}$ which converges uniformly to u^* in every closed region interior to R. This device of Courant was applied by Shevchenko to a three-dimensional elastostatic problem and to the problem of vibration of an elastic plate in its plane.[2]

An extension of the Ritz method to functionals defined over three-dimensional domains is straightforward, but, as just noted, the convergence becomes worse as the number of independent variables is increased. The main difficulty in the application of the Ritz method is in finding suitable systems of coordinate functions φ_i, complete in the domain under consideration.

PROBLEM

Show that the system of equations $\dfrac{\partial I(u_n)}{\partial a_j} = 0$ $(j = 1, 2, \ldots, n)$, for the determination of coefficients in the minimizing function $\bar{u}_n = \sum_{i=1}^{n} a_i\varphi_i(x, y)$ for the problem

$$I(u) = \iint_R (u_x^2 + u_y^2 + 2fu)\, dx\, dy = \min, \qquad u = 0 \qquad \text{on } C,$$

$$\iint_R \left(\frac{\partial u_n}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial u_n}{\partial y}\frac{\partial \varphi_j}{\partial y} + f\varphi_j\right) dx\, dy = 0, \qquad (j = 1, 2, \ldots, n).$$

[1] R. Courant, *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp.1–23.
[2] K. N. Shevchenko, *Prikl. Mat. Mekh. Akademiya Nauk SSSR*, vol. 2 (1938), vol. 6 (1942).

114. Literature on Direct Methods. We have imposed rather severe restrictions on the choice of coordinate functions φ_i in order to ensure a strong type of convergence of the sequence $\{I(u_n)\}$ to the minimum of the functional $I(u)$. In Sec. 112, for example, we chose the smallness of numbers $\epsilon_n = \max |y_n(x) - y(x)|$ and $\max |y_n'(x) - y'(x)|$ for all x in (x_0, x_1) as our criterion of goodness of approximation. This criterion is unduly severe, since in practice it is impossible to measure physical quantities at a single point. Thus, in measuring strains or displacements in an elastic body, the strain gages and similar devices do not record the desired values at a point, but rather over some region about the point. This suggests adopting as a criterion for the goodness of approximation of $y_n(x)$ to $y(x)$ not the value ϵ_n but some function of this quantity. For example, one can take, as a measure of the magnitude of the error, the integral of the square of the deviation from exact values. This is given by

$$\delta_n = \int_{x_0}^{x_1} |y_n(x) - y(x)|^2 \, dx, \tag{114.1}$$

and when $\delta_n \to 0$ as $n \to \infty$, $y_n(x)$ is said to converge to $y(x)$ in the mean. This is a weaker type of convergence than the uniform convergence, based on the criterion $\lim\limits_{n\to\infty} \epsilon_n = 0$.

Practically all recent investigations of convergence of various direct methods in variational calculus and numerical analysis are based on some such criterion as (114.1). The most effective tool in such investigations proved to be the theory of operators in Hilbert spaces.[1]

The difficult problem of estimating errors in successive approximations by the Ritz method was treated quite fully by Krylov[2] in connection with those variational problems whose Euler's equations are linear and ordinary. The corresponding problem, relating to multiple integrals, is vastly more involved. Some important results were obtained by L. V. Kantorovich and are recorded in his *Uspekhi* paper. No recital of references on approximate methods in applied mathematics can fail to include a superb monograph by L. V. Kantorovich and V. I. Krylov,

[1] See, for example, papers by K. Friedrichs in *Mathematische Annalen*, vol. 109 (1934); *American Journal of Mathematics*, vol. 68 (1946); *Annals of Mathematics*, vol. 68 (1946), which contain fundamental contributions to the problem of convergence of the Ritz and related methods. An excellent account of the role assumed by functional analysis in applied mathematics is presented in a long paper, in Russian, by L. V. Kantorovich in vol. 3, No. 6 (28), of *Uspekhi Matematicheskikh Nauk* (1948). Two monographs by S. G. Mikhlin, entitled Direct Methods in Mathematical Physics (1950) and The Problem of the Minimum of a Quadratic Functional (1952) (both in Russian), provide a systematic treatment of the subjects indicated by the titles and include extensive bibliographical references. These books presuppose some familiarity with Lebesgue integration and linear function spaces.

[2] N. M. Krylov, Les Méthodes de solution approachée des problèmes de la physique mathématique, *Mémorial des sciences mathématiques*, Paris, No. 49 (1931).

entitled Approximate Methods of Higher Analysis (1952) (in Russian), which was used in the preparation of several sections of this chapter.

115. The Galerkin Method. In 1915, Galerkin proposed a method of approximate solution of the boundary-value problems in mathematical physics that is of much wider scope than the method of Ritz.[1] We shall see that the Galerkin method, when applied to variational problems with quadratic functionals, reduces to the Ritz method.

The idea of the method is simple. Let it be required to solve a linear[2] differential equation

$$L(u) = 0 \qquad \text{in } R, \tag{115.1}$$

subject to some linear homogeneous boundary conditions. Inasmuch as the operator L is not necessarily homogeneous, the restriction on the homogeneity of the boundary conditions is not essential, since the boundary conditions can always be cast in the desired form by changing[3] the dependent variable u.

We shall assume, for simplicity of exposition, that the domain R is two-dimensional and shall seek an approximate solution of the problem in the form

$$u_n(x, y) = \sum_{j=1}^{n} a_j\varphi_j(x, y), \tag{115.2}$$

where the φ_j are suitable coordinate functions and the a_j are constants. In regard to the φ_j we shall suppose that they satisfy the same boundary conditions as the exact solution $u(x, y)$ and that the set $\{\varphi_j\}$ is complete in the sense that every piecewise continuous function $f(x, y)$ can be approximated in R by the sum $\sum_{j=1}^{N} c_j\varphi_j$ in such a way that

$$\delta_N \equiv \iint_R \left(f - \sum_{j=1}^{N} c_j\varphi_j\right)^2 dx\,dy \tag{115.3}$$

can be made as small as we wish.

The finite sum (115.2) ordinarily will not satisfy Eq. (115.1), and the substitution of u_n will yield

$$L(u_n) = \epsilon_n(x, y), \qquad \epsilon_n(x, y) \neq 0 \qquad \text{in } R.$$

If max $\epsilon_n(x, y)$ is small, $u_n(x, y)$ can be considered a satisfactory approximation to $u(x, y)$. Thus, $\epsilon_n(x, y)$ can be viewed as an error function, and the task then is to select the a_j so as to minimize $\epsilon_n(x, y)$.

[1] B. G. Galerkin, *Vestnik Inzhenerov*, vol. 1 (1915), pp. 897–908.

[2] The Galerkin method can also be applied to nonlinear problems. See D. V. Panov, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 3 (1939), pp. 139–142.

[3] As an illustration see system (116 10).

A reasonable minimization technique is suggested by the following: If one represents $u(x, y)$ by the series $u(x, y) = \sum_{i=1}^{\infty} a_i\varphi_i$, with suitable properties, and considers the nth partial sum $u_n = \sum_{i=1}^{n} a_i\varphi_i$, then the orthogonality condition

$$\iint_R L(u_n)\varphi_i(x, y)\,dx\,dy = 0 \qquad \text{as } n \to \infty \tag{115.4}$$

is equivalent to the statement that[1] $L(u) = 0$.

This led Galerkin to impose on the error function $L(u_n)$ a set of orthogonality conditions

$$\iint_R L(u_n)\varphi_i(x, y)\,dx\,dy = 0, \qquad (i = 1, 2, \ldots, n), \tag{115.5}$$

yielding the set of n equations

$$\iint_R L\left(\sum_{j=1}^{n} a_j\varphi_j\right)\varphi_i\,dx\,dy = 0, \qquad (i = 1, 2, \ldots, n), \tag{115.6}$$

for the determination of the constants a_j in the approximate solution (115.2).

When the differential equation and the boundary conditions are self-adjoint and the corresponding functional $I(u)$ in the problem $I(u) = \min$ is positive definite,[2] the system (115.6) is equivalent to the Ritz system

$$\frac{\partial I(u_n)}{\partial a_j} = 0.$$

We shall verify this for the problems associated with the functionals (106.21) and (113.4). It is important to note that in Galerkin's formulation there is no reference to any connection of Eq. (115.1) with a variational problem. Indeed, the Galerkin method can be applied to a broad class of problems phrased in terms of integral and other types of functional equations.[3]

[1] For if we represent an arbitrary function $\eta(x, y)$, satisfying the boundary conditions of our problem, in the series $\eta(x, y) = \sum_{i=1}^{\infty} c_i\varphi_i$, and suppose that $L(u_n) \to L(u)$ as $n \to \infty$, then Eq. (115.4) demands that $\iint_R L(u)\eta(x, y)\,dx\,dy = 0$. But the application of Lagrange's lemma then yields $L(u) = 0$. The argument depends, of course, on the proper behavior of the series involved.

[2] This is the case with all particular problems thus far considered in this chapter.

[3] See S. G. Mikhlin's monograph entitled The Problem of the Minimum of a Quadratic Functional (1952), pp. 62–66 (in Russian).

We recall[1] that the system of Ritz's equations, associated with the problem

$$I(u) = \iint_R (u_x^2 + u_y^2 + 2fu)\,dx\,dy = \min, \qquad u = 0 \qquad \text{on } C,$$

is

$$(115.7)\qquad \iint_R \left(\frac{\partial u_n}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial u_n}{\partial y}\frac{\partial \varphi_j}{\partial y} + f\varphi_j\right) dx\,dy = 0, \qquad (j = 1, 2, \ldots, n).$$

But

$$\iint_R \left(\frac{\partial u_n}{\partial x}\frac{\partial \varphi_j}{\partial x} + \frac{\partial u_n}{\partial y}\frac{\partial \varphi_j}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}\left(\varphi_j \frac{\partial u_n}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varphi_j \frac{\partial u_n}{\partial y}\right)\right] dx\,dy - \iint_R \nabla^2 u_n \varphi_j\,dx\,dy.$$

By Green's Theorem the first integral in the right-hand member is equal to

$$\int_C \varphi_j \left(\frac{\partial u_n}{\partial x}\,dy - \frac{\partial u_n}{\partial y}\,dx\right) = 0,$$

since $\varphi_j = 0$ on C. Thus, formula (115.7) can be written as

$$\iint_R (\nabla^2 u_n - f)\varphi_j\,dx\,dy = 0, \qquad (j = 1, 2, \ldots, n).$$

But this is precisely in the form (115.5), since the Euler equation associated with this problem is $L(u) \equiv \nabla^2 u - f = 0$.

In a similar way it follows that the system of Ritz's equations

$$[113.7]\qquad \iint_R (\nabla^2 u_n \nabla^2 \varphi_j - f\varphi_j)\,dx\,dy = 0, \qquad (j = 1, 2, \ldots, n),$$

connected with the functional (113.4), can be cast in the Galerkin form[2]

$$\iint_R (\nabla^4 u_n - f)\varphi_j\,dx\,dy = 0, \qquad (j = 1, 2, \ldots, n).$$

The Galerkin method has been used widely in Russia. A bibliography of its numerous applications to problems in elasticity is contained in a synoptic paper by Perelman.[3]

[1] See problem at the end of Sec. 113.

[2] Note the identity,

$$\iint_R \nabla^2 u \nabla^2 v\,dx\,dy = \iint_R v\nabla^4 u\,dx\,dy + \int_C \nabla^2 u \frac{\partial v}{\partial \nu}\,ds - \int_C v\frac{\partial \nabla^2 u}{\partial \nu}.$$

[3] Ya. I. Perelman, *Prikl. Mat. Mekh., Akademiya Nauk SSSR,* vol. 5 (1941), pp. 345–358.

The convergence of the method is discussed in two monographs by Mikhlin, cited in the preceding section. The study of convergence hinges on a careful investigation of the behavior of systems of infinitely many linear algebraic equations. The majority of results obtained so far pertain to convergence in the mean and are deduced with the aid of the theory of a complex Hilbert space.

PROBLEMS

1. Recast the system of equations in Prob. 1, Sec. 112, in the Galerkin form.
2. Solve Prob. 2, Sec. 112, by the Galerkin method.

116. Applications to Torsion of Beams and Deformation of Plates. We saw in Sec. 110 that the determination of Prandtl's stress function Ψ from the system

$$\begin{cases} \nabla^2\Psi = -2 & \text{in } R, \\ \Psi = 0 & \text{on } C, \end{cases} \tag{116.1}$$

is related to the problem of minimizing the complementary energy V^*, or, what is the same thing, to minimizing the functional

$$I(\Psi) = \iint\limits_R [(\nabla\Psi)^2 - 4\Psi]\,dx\,dy. \tag{116.2}$$

If R is a rectangle $|x| \leq A$, $|y| \leq B$ and we take algebraic polynomials as our coordinate functions φ_i, we can write an approximate solution in the form

$$\Psi_n(x, y) = (x^2 - A^2)(y^2 - B^2)(a_1 + a_2x^2 + a_3y^2 + \cdots + a_nx^{2k}y^{2k}). \tag{116.3}$$

This clearly vanishes on the boundary.[1]

The coefficient a_1, in the first approximation

$$\Psi_1 \equiv a_1\varphi_1 = a_1(x^2 - A^2)(y^2 - B^2). \tag{116.4}$$

is determined by Eq. (115.5), which now reads,

$$\int_{-B}^{B}\int_{-A}^{A} (\nabla^2\Psi_1 + 2)\varphi_1\,dx\,dy = 0,$$

$$\int_{-B}^{B}\int_{-A}^{A} [2(y^2 - B^2)a_1 + 2(x^2 - A^2)a_1 + 2](x^2 - A^2)(y^2 - B^2)\,dx\,dy = 0.$$

We find on integration that

$$\tfrac{128}{45}A^3B^3(A^2 + B^2)a_1 - \tfrac{32}{9}A^3B^3 = 0,$$

so that

$$a_1 = \frac{5}{4}\frac{1}{A^2 + B^2}. \tag{116.5}$$

[1] We discarded the odd powers of x and y because $\Psi(x, y)$ is obviously an even function.

Approximate values of the torsional rigidity D and maximum shear stress τ can now be computed with the aid of (116.4). We recall that

$$D = 2\mu \iint_R \Psi \, dx \, dy \qquad [35.10]$$

and the maximum shear stress τ occurs at the mid-points of the longer sides. If $B > A$, then (Sec. 38)

$$\tau = \tau_{yz}\Big|_{\substack{x=A\\y=0}} = -\mu\alpha\left(\frac{\partial \Psi}{\partial x}\right)_{\substack{x=A\\y=0}} \qquad (116.6)$$

On inserting Ψ_1 in these formulas we find

$$D_1 = \frac{5}{18}\mu a^3 b \frac{(b/a)^2}{1 + (b/a)^2},$$

$$\tau_1 = \frac{5}{4}\mu\alpha a \frac{(b/a)^2}{1 + (b/a)^2},$$

where $a = 2A$ and $b = 2B$ in the notation of Sec. 38. A comparison of these approximate values, for two ratios b/a, with exact values is made in the following table.[1]

$\frac{b}{a}$	$\frac{1}{\mu}\frac{D_1}{a^3 b}$	$\frac{1}{\mu}\frac{D}{a^3 b}$	Error	$\frac{\tau_1}{\mu\alpha a}$	$\frac{\tau}{\mu\alpha a}$	Error
1	0.1389	0.1406	− 1.2%	0 625	0.675	−7.4%
10	0.275	0.312	−11.9%	1 238	1.000	23.8%

The approximations τ_1 for the maximum shearing stress deviate appreciably from true values. This is not surprising since τ is proportional to the derivative of Ψ. To get a better approximation, we can take

$$\Psi_3 = (x^2 - A^2)(y^2 - B^2)(a_1 + a_2 x^2 + a_3 y^2)$$

and determine the a_i from (115.5).

We indicate calculations for the square beam. Symmetry demands that, when $A = B$, $a_2 = a_3$, and we get from (115.5) two equations yielding

$$a_1 A^2 = 1295/2216, \qquad a_2 A^4 = 525/4432.$$

Inserting from these in

$$\Psi_3 = (x^2 - A^2)(y^2 - A^2)[a_1 + a_2(x^2 + y^2)],$$

and making use of (35.10) and (116.6), we get $D_3 = 0.1404\mu a^4$ and $\tau_3 = 0.7028\mu\alpha a$. The first of these is nearly equal to the exact value

[1] Exact values are recorded on p. 277 of Timoshenko and Goodier's Theory of Elasticity (1951). They can be computed from (38.12) and (38.14).

$D = 0.1406\mu a^4$, while the second is about 4 per cent higher than

$$\tau = 0.675\mu\alpha a.$$

Instead of using algebraic polynomials we could have taken trigonometric polynomials

$$\sum_{i,j=1,3,5,\ldots} a_{ij} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b}$$

as our approximating functions Ψ_k. Because of the orthogonality of coordinate functions $\cos(i\pi x/a)\cos(j\pi y/b)$ in the rectangle, formulas (115.5) yield the a_{ij} for arbitrary values of i and j. On letting k increase indefinitely in Ψ_k, we get

$$\Psi = \frac{32a^2b^2}{\pi^4} \sum_{i,j=1,3,5,\ldots}^{\infty} (-1)^{\frac{i+j}{2}-1} \frac{\cos(i\pi x/a)\cos(j\pi y/b)}{ij(b^2i^2 + a^2j^2)},$$

which is a known expression for the Prandtl function in the double series. This series, however, converges rather slowly.[1]

We saw that the Ritz method provides an upper bound to the exact minimum $I(\Psi^*) = m$. A lower bound for the functional (116.2) was obtained by Friedrichs,[2] who proposed a device of setting up an auxiliary variational problem $J(w) = \max$ such that $\max J(w) = \min I(\Psi)$. The determination of upper and lower bounds in the solution of the Dirichlet and Neumann problems (for simply and multiply connected domains) was made, among others, by Diaz, Greenberg, and Weinstein.[3]

As another illustration of the application of the Ritz-Galerkin method, we consider the problem of deformation of an elastic plate by a parabolic distribution of tensile forces over its opposite sides.[4]

[1] Tables of numerical values of D_i and τ_i, obtained by using the approximations Ψ_k in the form of trigonometric polynomials, appear on pp. 220–221 of S. G. Mikhlin's Direct Methods of Mathematical Physics (1950).

[2] K. Friedrichs, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen Mathematische-Physikalische Klasse* (1929), pp. 13–20.

[3] J. B. Diaz and A. Weinstein, *American Journal of Mathematics*, vol. 70 (1948), pp. 107–116; H. J. Greenberg, *Journal of Mathematics and Physics*, vol. 27 (1948), pp. 161–182.

A systematic procedure for constructing monotone sequences of upper and lower bounds for quadratic functionals is developed in a memoir by J. B. Diaz, *Seminario Matematico de Barcelona Collectanea Mathematica*, vol. 4 (1951). This memoir contains extensive bibliographical references.

See also H. F. Weinbe er, *Journal of Mathematics and Physics*, vol. 32 (1935), pp. 54–62.

[4] This problem was first considered by S. Timoshenko, *Philosophical Magazine*, vol. 47 (1924), p. 1095. See also pp. 167–170 in Timoshenko and Goodier's Theory of Elasticity (1951). A modified treatment of it, presented here, is contained in L. V. Kantorovich and V. I. Krylov's Approximate Methods of Numerical Analysis (1950), pp. 304–305. This book contains many illustrations of the use of direct methods in

Let a flat plate occupy a region R bounded by the lines $x = \pm a$, $y = \pm b$. We suppose that the sides $y = \pm b$ are free of external forces and the sides $x = \pm a$ are subjected to tensile loads distributed according

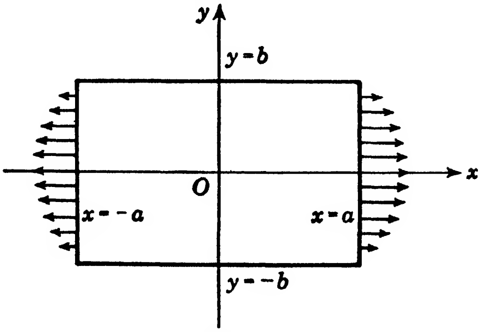

FIG. 61

to the law $\tau_{xx} = S(1 - y^2/b^2)$ (Fig. 61). The state of stress in the plate is determined by the Airy function $U(x, y)$, which satisfies the equation

$$(116.7) \qquad \nabla^4 U = 0.$$

In the notation of Sec. 69,

$$\tau_{xx} = U_{,yy}, \qquad \tau_{xy} = -U_{,xy}, \qquad \tau_{yy} = U_{,xx},$$

so that the assigned distribution of loads on the boundary yields,

$$(116.8) \qquad \begin{cases} U_{,xy} = 0, & U_{,yy} = S\left(\dfrac{b^2 - y^2}{b^2}\right) & \text{on } x = \pm a, \\ U_{,xy} = 0, & U_{,xx} = 0 & \text{on } y = \pm b. \end{cases}$$

It is convenient to reduce these boundary conditions to a homogeneous form by setting

$$U = u_0 + u,$$

where

$$(116.9) \qquad u_0 = \frac{1}{2} S y^2 \left(1 - \frac{1}{6}\frac{y^2}{b^2}\right).$$

physical problems. A number of interesting problems are included in S. G. Mikhlin's Direct Methods of Mathematical Physics (1950). Among these are problems on transverse deflection of clamped plates and membranes, torsion of a circular cylinder with a longitudinal square cavity, radial vibrations of circular cylinders, vibration and stability of plates, critical frequencies of oscillators, and other characteristic value problems. Mikhlin refers to a book by L. S. Leibenson, Variational Methods of Solution of Problems in the Theory of Elasticity (1943), which appears to contain a fund of solutions of concrete problems in elasticity.

A simple calculation then shows that the system (116.7), (116.8) is equivalent to:

$$(116.10)\qquad \begin{cases} \nabla^4 u = \dfrac{2S}{b^2} & \text{for } |x| < a,\ |y| < b, \\ u_{,xy} = u_{,yy} = 0 & \text{on } |x| = a, \\ u_{,xy} = u_{,xx} = 0 & \text{on } |y| = b. \end{cases}$$

In order to fit the problem (116.10) in the pattern of Sec. 113 [Eqs. (113.5), (113.6)], we consider the system

$$(116.11)\qquad \begin{cases} \nabla^4 u = \dfrac{2S}{b^2} \\ u = u_{,x} = 0 & \text{on } |x| = a, \\ u = u_{,y} = 0 & \text{on } |y| = b, \end{cases}$$

and observe that, because of the nature of the region, the function u satisfying the system (116.11) also satisfies the system (116.10).

Accordingly we are led to the problem

$$I(u) = \int_{-b}^{b}\int_{-a}^{a}\left[(\nabla^2 u)^2 - \frac{4S}{b^2}u\right]dx\,dy = \min,$$

where the admissible u satisfy the boundary conditions in (116.11).

On account of symmetry we can discard the odd powers in the approximation

$$u_n = (x^2 - a^2)^2(y^2 - b^2)^2(a_1 + a_2x^2 + a_3y^2 + \cdots),$$

and consider first

$$u_1 = a_1(x^2 - a^2)^2(y^2 - b^2)^2.$$

The coefficient a_1 is determined by

$$\int_{-b}^{b}\int_{-a}^{a} L(u_1)(x^2 - a^2)^2(y^2 - b^2)^2\,dx\,dy = 0,$$

where $L(u) = \nabla^4 u - 2S/b^2$.

A simple calculation yields for the square plate $a_1 = 0.04253S/a^6$ so that the first approximation to U is

$$U_1 = u_0 + \frac{0.04253S(x^2 - a^2)^2(y^2 - a^2)^2}{a^6}$$

with u_0 determined by (116.9). The higher-order approximations can be determined in a similar way.[1]

From remarks made in Sec. 113 it follows that the successive approximations converge to the desired solution since the set of coordinate functions used here is relatively complete.

[1] An expression for U_3 is recorded in the Kantorovich and Krylov monograpn referred to in the preceding footnote.

117. The Method of Kantorovich. An interesting generalization of the Ritz method was proposed in 1932 by Kantorovich.[1] The essence of the method, as we shall presently see, consists in the reduction of integration of partial differential equations to the integration of systems of ordinary differential equations.

In the application of the Ritz method to the problem

$$I(u) = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy = \min, \tag{117.1}$$

we considered approximate solutions in the form

$$u_n = \sum_{k=1}^{n} a_k \varphi_k(x, y), \tag{117.2}$$

where the φ_k satisfy the same boundary conditions as those imposed on $u(x, y)$. We then determined the coefficients a_k so as to minimize $I(u_n)$. If we now suppose that the a_k in (117.2) are no longer constants, but are unknown functions of x, such that the product $a_k(x)\varphi_k(x, y)$ satisfies the same boundary conditions as u, we are led to minimize

$$I(u_n) = I\left[\sum_{k=1}^{n} a_k(x)\varphi_k(x, y)\right].$$

Since the $\varphi_k(x, y)$ are known functions, we can perform integration with respect to y and obtain a functional

$$I(u_n) = \int_{x_0}^{x_1} \mathfrak{F}[a_k(x), a_k'(x), x]\, dx. \tag{117.3}$$

Kantorovich proposed to determine the $a_k(x)$ so that they minimize (117.3). From discussion of the integral (106.1) it is clear that the $a_k(x)$ can be determined by solving the second-order ordinary differential equation of the form (106.10). Moreover, if the functional (117.1) is quadratic, the Euler equation (106.10), associated with (117.3), will be linear.[2]

To fix the ideas let us consider the familiar problem

$$I(u) = \iint_R (u_x^2 + u_y^2 + 2fu)\, dx\, dy = \min, \tag{117.4}$$

where the admissible u satisfy the condition

$$u = \varphi(s) \qquad \text{on } C. \tag{117.5}$$

[1] L. V. Kantorovich, *Izvestiya Akademii Nauk SSSR, Mathematical Series* (1933) pp. 647–652; *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 6 (1942), pp. 31–40. The method is also presented in detail in L. V. Kantorovich and V. I. Krylov, Approximate Methods of Numerical Analysis (1952), pp. 321–344.

[2] We recall that the application of the Ritz-Galerkin method to quadratic functionals invariably leads to a system of linear algebraic equations for the constants a_k.

Let us seek an approximate solution in the form

$$(117.6) \qquad u_n(x, y) = \varphi_0(x, y) + \sum_{k=1}^{n} a_k(x)\varphi_k(x, y),$$

where $\varphi_0(x, y)$ is such that

$$\varphi_0\Big|_C = \varphi(s),$$

and the $\varphi_k(x, y)$ vanish at least over a part of the boundary C. On the part of C where the φ_k do not vanish we shall require that the $a_k(x)$ vanish. Thus,

$$(117.7) \qquad a_k(x)\varphi_k(x, y) = 0 \qquad \text{on } C.$$

Unless the $\varphi_k(x, y)$ vanish over the entire bounaary C, we shall be obliged to impose certain restrictions on the shape of the region R in order to be able to satisfy the boundary conditions imposed on the $a_k(x)$ in (117.7).

We introduce next, in the manner of Sec. 106, a family of varied functions

$$(117.8) \qquad u_n(\epsilon, x, y) \equiv \varphi_0(x, y) + \sum_{k=1}^{n} [a_k(x) + \epsilon_k\eta_k(x)]\varphi_k(x, y),$$

where the ϵ_k are small parameters and the $\eta_k(x)$ are such that

$$(117.9) \qquad \eta_k(x)\varphi_k(x, y) = 0 \qquad \text{on } C.$$

The substitution of (117.8) in (117.4) then yields

$$I(\epsilon_1, \ldots, \epsilon_n) = I\left[\varphi_0 + \sum_{k=1}^{n} (a_k + \epsilon_k\eta_k)\varphi_k\right],$$

and if the $a_k(x)$ are to minimize I, it is necessary that

$$(117.10) \qquad \frac{\partial I}{\partial \epsilon_k}\Big|_{\{\epsilon_k\}=0} = 0, \qquad \text{for } k = 1, 2, \ldots, n.$$

Upon differentiating under the integral we get the system (117.10) in the form

$$(117.11) \qquad 2\iint_R \left[\frac{\partial u_n}{\partial x}\frac{\partial}{\partial x}(\varphi_k\eta_k) + \frac{\partial u_n}{\partial y}\frac{\partial}{\partial y}(\varphi_k\eta_k) + f\varphi_k\eta_k\right] dx\, dy = 0.$$

But

$$\iint_R \left[\frac{\partial u_n}{\partial x}\frac{\partial}{\partial x}(\varphi_k\eta_k) + \frac{\partial u_n}{\partial y}\frac{\partial}{\partial y}(\varphi_k\eta_k)\right] dx\, dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial u_n}{\partial x}\varphi_k\eta_k\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_n}{\partial y}\varphi_k\eta_k\right)\right] dx\, dy - \iint_R \nabla^2 u_n \varphi_k\eta_k\, dx\, dy$$

$$= \int_C \left[\varphi_k\eta_k\left(-\frac{\partial u_n}{\partial y}dx + \frac{\partial u_n}{\partial x}dy\right)\right] - \iint \varphi_k\eta_k\nabla^2 u_n\, dx\, dy.$$

Since the integral over C vanishes by virtue of (117.9), we see that (117.11) can be written as

$$\iint_R (\nabla^2 u_n - f)\varphi_k(x, y)\eta_k(x)\,dx\,dy = 0, \qquad k = 1, 2, \ldots, n. \tag{117.12}$$

Now if the region R has the form shown in Fig. 62, we can write the double

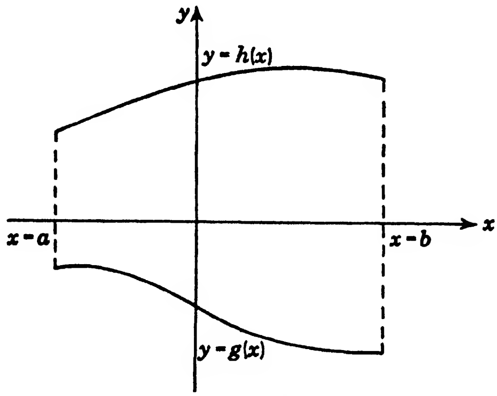

Fig. 62

integral in (117.12) as an iterated integral

$$\int_a^b \eta_k(x)\,dx \int_{y=g(x)}^{y=h(x)} (\nabla^2 u_n - f)\varphi_k(x, y)\,dy = 0,$$

where the meaning of g and h is obvious from the figure. Inasmuch as the η_k are arbitrary, we conclude that

$$\int_{g(x)}^{h(x)} (\nabla^2 u_n - f)\varphi_k(x, y)\,dy = 0, \qquad k = 1, 2, \ldots, n. \tag{117.13}$$

This system of equations,[1] on integration, yields a system of n ordinary differential equations for the determination of the $a_k(x)$.

As a specific illustration of the use of this method, we shall rework the torsion problem treated in the preceding section. In order not to obscure matters, we consider first only one term in the sum (117.6) and take

$$\Psi_1 = (y^2 - B^2)a_1(x). \tag{117.14}$$

Since $\Psi_1 = 0$ on $y = \pm B$, we shall determine $a(x)$ so that

$$a_1(A) = a_1(-A) = 0.$$

[1] These equations could have been deduced more simply by writing out the Euler equations directly, but we wished to emphasize the role played by the variations $\epsilon_k\eta_k$.

Equations (117.13) then yield

$$\int_{-B}^{B} \{\nabla^2[(y^2 - B^2)a_1(x)] + 2\}(y^2 - B^2)\, dy = 0,$$

or

$$\int_{-B}^{B} [(y^2 - B^2)a_1''(x) + 2a_1(x) + 2](y^2 - B^2)\, dy = 0.$$

Integrating, we get the equation

$$a_1''(x) - \frac{5}{2B^2} a_1(x) - \frac{5}{2B^2} = 0,$$

whose solution is

$$a_1(x) = c_1 \cosh \frac{kx}{B} + c_2 \sinh \frac{kx}{B} - 1, \qquad k = \sqrt{\frac{5}{2}}.$$

Since (117.4) must, obviously, be an even function of x, $c_2 = 0$, and since $a_1(A) = 0$, we easily find

$$c_1 = \frac{1}{\cosh (kA/B)}.$$

Thus,

$$\Psi_1 = (y^2 - B^2)\left[\frac{\cosh (kx/B)}{\cosh (kA/B)} - 1\right]. \tag{117.15}$$

It is interesting to compare this first approximation with the approximation

$$\Psi_1 = a_1(y^2 - B^2)(x^2 - A^2), \qquad a_1 = \tfrac{5}{4}(A^2 + B^2) \tag*{[116.4]}$$

got by the Ritz method in Sec. 116. In formula (116.4) the term $a_1(x^2 - A^2)$ replaces the expression in the brackets of (117.15), and it is essentially the first term in the power series expansion of the function in those brackets.

If we use (117.15) to compute an approximate value D_1 with the aid of (35.10), we find

$$D_1 = \frac{1}{3} b^3 a\mu \left(1 - \frac{b}{ak} \tanh \frac{ka}{b}\right),$$

where $a = 2A$ and $b = 2B$. For $b/a = 1$, we get $D_1 = 0.1396\mu a^3 b$, which is within 0.7 per cent of the true value (see table in Sec. 116), and, for $b/a = 10$, $D_1 = 0.303\mu a^3 b$, as compared with the exact value $0.312\mu a^3 b$.

A better approximation can be obtained by the Kantorovich method on taking

$$\Psi_2 = (y^2 - B^2)[a_1(x) + a_2(x)y^2].$$

The reader may find it instructive to take

$$\Psi_n = \sum_{k=1}^{n} a_k(x) \cos \frac{(2k + 1)y\pi}{2B}$$

and obtain the exact solution of the type (38.15) by allowing n to increase indefinitely. In this case computations simplify considerably because of the orthogonality of cosines.

The method of Kantorovich has successfully been applied to numerous problems in elasticity, including some three-dimensional problems.[1]

118. The Trefftz Method. We have noted in Sec. 113 that an application of the Ritz method to the problem $I(u) = \min$, where $I(u)$ is a quadratic functional, yields a minimizing sequence $\{I(u_n)\}$, which approaches the minimum $I(u^*)$ from above. Should we succeed in constructing a sequence $\{I(v_n)\}$ approaching $I(u^*)$ from below, then

$$I(v_n) \leq I(u^*) \leq I(u_n), \tag{118.1}$$

and we may get a good estimate of the minimum $I(u^*)$, if the bounds in (118.1) are sufficiently close.

In 1928 Trefftz[2] suggested a mode of constructing a minimizing sequence of lower bounds $I(v_n)$ (without proving convergence) for the *Dirichlet integral*

$$I(u) = \iint_R (u_x^2 + u_y^2)\, dx\, dy, \tag{118.2}$$

connected with the system

$$\begin{cases} \nabla^2 u = 0 & \text{in } R, \\ u = \varphi(s) & \text{on } C. \end{cases} \tag{118.3}$$

Before proceeding to discuss the special technique proposed by Trefftz, we consider a broader problem of constructing sequences of lower bounds for the Dirichlet integral (118.2). We need the following theorem, formulated by Weinstein,[3] which throws light on the Trefftz method:

THEOREM: *If $v(x, y)$ is any harmonic function in R and $u^*(x, y)$ is the solution of the Dirichlet problem* (118.3), *then $I(v) \leq I(u^*)$ whenever*

$$\int_C (u^* - v) \frac{dv}{d\nu}\, ds = 0.$$

To prove the theorem, we define η by the formula

$$u^* = v + \eta, \tag{118.4}$$

[1] A bibliography of such work will be found on pp. 321–373 of the Kantorovich and Krylov monograph cited in this section. This monograph also includes a careful study of convergence of several direct methods.

[2] E. Trefftz, *Mathematische Annalen*, vol. 100 (1928), pp. 503–521.

[3] See footnote 3 on page 429.

and then

$$(118.5)\quad I(u^*) = \iint_R [(v_x + \eta_x)^2 + (v_y + \eta_y)^2]\,dx\,dy$$
$$= \iint_R (v_x^2 + v_y^2)\,dx\,dy + \iint_R (\eta_x^2 + \eta_y^2)\,dx\,dy$$
$$+ 2\iint_R (v_x\eta_x + v_y\eta_y)\,dx\,dy.$$

But, by Green's Theorem,

$$(118.6)\quad \iint_R (v_x\eta_x + v_y\eta_y)\,dx\,dy = -\iint_R \eta\nabla^2 v\,dx\,dy + \int_C \eta\frac{dv}{d\nu}\,ds.$$

Since v is harmonic, the double integral on the right in (118.6) vanishes, and the line integral vanishes by virtue of our hypothesis

$$(118.7)\quad \int_C (u^* - v)\frac{dv}{d\nu}\,ds = 0.$$

We can thus write (118.5) as

$$I(u^*) = I(v) + I(\eta),$$

and since $I(\eta) \geq 0$, we have the desired result

$$(118.8)\quad I(v) \leq I(u^*).$$

Moreover, the equality sign in (118.8) holds if, and only if, $I(\eta) = 0$, that is, when $\eta = \text{const.}$ In this case $u^* = v + \text{const.}$

We are now in a position to construct a sequence of lower bounds $I(u_n)$ for $I(u^*)$. Let $\{v_n\}$ be a sequence of harmonic functions v_n defined in the closed region R, and construct a sum

$$(118.9)\quad u_n = \sum_{i=1}^{n} a_i v_i,$$

where the a_i are constants. The value of $\eta \equiv u^* - u_n$ on the boundary of R is

$$\eta\Big|_C = \varphi(s) - \sum_{i=1}^{n} a_i v_i\Big|_C,$$

since u^* is the solution of the problem (118.3).

If we choose the constants a_i so that (118.7) is satisfied, that is, so that

$$(118.10)\quad \int_C [\varphi(s) - u_n]\frac{du_n}{d\nu}\,ds = 0,$$

then the theorem just established guarantees that $I(u_n) \leq I(u^*)$. Inserting for u_n in (118.10) from (118.9), we obtain

$$\sum_{j=1}^{n} \int_C a_j[\varphi(s) - u_n] \frac{dv_j}{d\nu}\, ds = 0,$$

and we can surely satisfy this equation by making

$$\int_C [\varphi(s) - u_n] \frac{dv_j}{d\nu}\, ds = 0, \qquad (j = 1, 2, \ldots, n),$$

or

$$\int_C \left[\varphi(s) - \sum_{i=1}^{n} a_i v_i\right] \frac{dv_j}{d\nu}\, ds = 0, \qquad (j = 1, 2, \ldots, n). \tag{118.11}$$

The system of Eqs. (118.11) serves to determine the a_i. We prove next that it has a unique solution whenever the v_i are linearly independent and $\varphi(s) \not\equiv 0$ on C. We write (118.11) in the form

$$\sum_{j=1}^{n} \alpha_{ij} a_j = \beta_i, \qquad (i = 1, 2, \ldots, n), \tag{118.12}$$

where[1]

$$\alpha_{ij} \equiv \int_C v_i \frac{dv_j}{d\nu}\, ds, \qquad \beta_i \equiv \int_C \varphi(s) \frac{dv_i}{d\nu}\, ds. \tag{118.13}$$

The system (118.12) has a unique solution whenever $|\alpha_{ij}| \neq 0$. Suppose the contrary, and assume that $|\alpha_{ij}| = 0$. Then the homogeneous system

$$\sum_{j=1}^{n} \alpha_{ij} a_j = 0, \qquad (i = 1, 2, \ldots, n), \tag{118.14}$$

has solutions $\bar{a}_i$ not all zero, and we can form

$$\bar{u}_n = \sum_{i=1}^{n} \bar{a}_i v_i,$$

[1] Note that $\alpha_{ij} = \alpha_{ji}$, since, by Green's Theorem,

$$\int_C v_i \frac{dv_j}{d\nu}\, ds = \iint_R v_i \nabla^2 v_j\, dx\, dy + \iint_R \left(\frac{\partial v_i}{\partial x}\frac{\partial v_j}{\partial x} + \frac{\partial v_i}{\partial y}\frac{\partial v_j}{\partial y}\right) dx\, dy$$

and

$$\nabla^2 v_j = 0.$$

where $\bar{a}_n \neq$ const, if we reject the trivial case. Recalling the definition of $\alpha_{ij} = \alpha_{ji}$ in (118.13), and inserting it in (118.14), we get

$$\int_C \sum_{j=1}^{n} \bar{a}_j v_j \frac{dv_i}{d\nu}\, ds = 0,$$

or

$$\int_C \bar{u}_n \frac{dv_i}{d\nu}\, ds = 0.$$

On multiplying this by $\bar{a}_i$ and summing for i from 1 to n, we get

$$\int_C \bar{u}_n \frac{d\bar{u}_n}{d\nu}\, ds = 0. \tag{118.15}$$

But $\bar{u}_n$ is harmonic, and hence, by Green's Theorem,

$$\int_C \bar{u}_n \frac{d\bar{u}_n}{d\nu}\, ds = \iint_R (\nabla \bar{u}_n)^2\, dx\, dy.$$

This, however, vanishes, by (118.15), and hence $\bar{u}_n =$ const. But $\bar{u}_n =$ const is the trivial case we rejected in defining $\bar{u}_n$, and we have thus shown that the hypothesis $|\alpha_{ij}| = 0$ is false. Accordingly, the system (118.12) has a unique solution.

The foregoing discussion provides a mode for constructing sequences of lower bounds for $I(u^*)$ but gives no inkling about the behavior of $\{I(u_n)\}$ as regards the convergence to $I(u^*)$. However, one can prove *that, when the sequence of harmonic functions v_i is relatively complete in R, in the sense that the sum $u_n = \sum_{i=1}^{n} a_i v_i$ can be made to approximate* an arbitrary harmonic function, together with its derivatives, arbitrarily closely throughout R, then the a_i can be determined[1] so that $u_n \to u^*$ uniformly in every closed region lying wholly in R.

To determine the coefficients a_i, we proceed (after Trefftz) as follows: We consider the Dirichlet integral (118.2) and construct

$$I(u - u_n) \equiv \iint_R [\nabla(u - u_n)]^2\, dx\, dy \tag{118.16}$$

where $u_n = \sum_{i=1}^{n} a_i v_i$ and $\{v_i\}$ is a complete set of harmonic functions in R.

[1] The corresponding result is established in S. G. Mikhlin's Direct Methods of Mathematical Physics (1950), pp 188–193, under the hypothesis that $I(u - u_n) < \epsilon$ in R for every harmonic u of integrable square and whose first derivatives are also of summable square

We then determine the a_j from the system of equations

$$\frac{\partial I}{\partial a_j} = 0, \qquad (j = 1, 2, \ldots, n).$$

We write out these equations explicitly. On differentiating (118.16) with respect to a_j, we get

$$\begin{aligned}
\frac{\partial I(u - u_n)}{\partial a_j} &= 2 \iint\limits_R \nabla(u - u_n) \cdot \frac{\partial}{\partial a_j} \nabla(u - u_n)\, dx\, dy \\
&= -2 \iint\limits_R \nabla(u - u_n) \cdot \nabla v_j\, dx\, dy \\
&= 2 \iint\limits_R (u - u_n)\nabla^2 v_j\, dx\, dy - 2 \int_C (u - u_n) \frac{dv_j}{d\nu}\, ds \\
&= -2 \int_C (u - u_n) \frac{dv_j}{d\nu}\, ds,
\end{aligned}$$

where we used Green's Theorem and the fact that the v_i are harmonic functions. Thus, for a minimum

$$\int_C (u - u_n) \frac{dv_j}{d\nu}\, ds = 0, \qquad (j = 1, 2, \ldots, n),$$

but these are precisely in the form (118.11).

The dimensionality of R is clearly of no essence in the technique just described. However, a serious shortcoming in the use of the Trefftz method stems from the difficulty of constructing complete systems of harmonic functions. If the region R is plane and simply connected, a set of harmonic polynomials can be shown to be complete. If R is a multiply connected plane domain, there exists a complete system of rational harmonic functions. In three dimensions, however, it is not known whether the system of harmonic polynomials forms a complete set, for example, in an ellipsoid.

Some investigations of the use of the Trefftz method in problems involving the biharmonic equation were made by Weinstein, Diaz, Greenberg, and others.[1]

The use of the Trefftz technique in connection with functionals other than (118.2) is common. Some such uses are followed by unwarranted assertions that the sequences so obtained invariably yield lower bounds for the solution of the corresponding Euler's equations.

[1] A. Weinstein, *Journal of the London Mathematical Society*, vol. 10 (1935), and a monograph by this author in *Mémorial des sciences mathématiques*, No. 88 (1937).

J. B. Diaz and H. J. Greenberg, *Quarterly of Applied Mathematics*, vol. 6 (1948), pp. 326–331; *Journal of Mathematics and Physics*, vol. 27 (1948), pp. 193–201.

P. Cooperman, *Quarterly of Applied Mathematics*, vol. 10 (1953), pp. 359–373.

119. An Application of the Trefftz Method. We illustrate the Trefftz method by calculating an upper bound for the torsional rigidity of a square beam. A sequence of the corresponding lower bounds was computed by the Ritz method in Sec. 116.

If we introduce a function $\psi(x, y)$ related to the Prandtl function Ψ by the formula (see Sec 35)

$$\psi = \Psi + \tfrac{1}{2}(x^2 + y^2), \tag{119.1}$$

the system (116.1) reduces to

$$\begin{cases} \nabla^2\psi = 0 & \text{in } R, \\ \psi = \tfrac{1}{2}(x^2 + y^2) & \text{on } C. \end{cases} \tag{119.2}$$

The associated minimum problem is

$$J(\psi) = \iint\limits_R (\nabla\psi)^2\, dx\, dy = \min, \tag{119.3}$$

which is identical with that treated in the preceding section.

We establish next the connection of this functional with

$$I(\Psi) = \iint\limits_R [(\nabla\Psi)^2 - 4\Psi]\, dx\, dy \tag*{[116.2]}$$

and with torsional rigidity

$$D = 2\mu \iint\limits_R \Psi^*\, dx\, dy, \tag*{[35.10]}$$

where Ψ^* is the solution of the system (116.1) minimizing $I(\Psi)$.

The substitution from (119.1) in (119.3) yields

$$\begin{aligned} J(\psi) &= \iint\limits_R [\nabla\Psi + \tfrac{1}{2}\nabla(x^2 + y^2)]^2\, dx\, dy \\ &= \iint\limits_R (\nabla\Psi)^2\, dx\, dy + \iint\limits_R \nabla\Psi \cdot \nabla(x^2 + y^2)\, dx\, dy + \iint\limits_R (x^2 + y^2)\, dx\, dy \end{aligned}$$

From Green's Theorem

$$\iint\limits_R \nabla\Psi \cdot \nabla(x^2 + y^2)\, dx\, dy = -\iint\limits_R \Psi\nabla^2(x^2 + y^2)\, dx\, dy + \int_C \Psi\, \frac{d(x^2 + y^2)}{d\nu}\, ds,$$

and since Ψ vanishes on C, we have

$$J(\psi) = \iint\limits_R [(\nabla\Psi)^2 - 4\Psi]\, dx\, dy + \iint\limits_R (x^2 + y^2)\, dx\, dy.$$

Thus

$$J(\psi) = I(\Psi) + I_0, \tag{119.4}$$

where $I_0 = \iint_R (x^2 + y^2)\,dx\,dy$ is the polar moment of inertia of the section.

The application of the same form of Green's Theorem also yields

$$\iint_R (\nabla\Psi^*)^2\,dx\,dy = 2\iint_R \Psi^*\,dx\,dy,$$

so that (35.10) can be written as

$$D = \mu \iint_R (\nabla\Psi^*)^2\,dx\,dy. \tag{119.5}$$

But Ψ^* minimizes (116.2), and, on setting $\Psi = \Psi^*$ in (116.2), we conclude that

$$D = -\mu I(\Psi^*).$$

Consequently, the relation (119.4) yields the formula

$$D = \mu \min\,[I_0 - J(\psi)],$$

which shows that the sequence

$$D_n \equiv \mu[I_0 - J(\psi_n)]$$

tends to D from above when $J(\psi)$ is minimized by the Trefftz method.

We proceed to compute D_n by taking

$$\psi_n = \sum_{i=1}^{n} a_i v_i,$$

where the v_i are harmonic polynomials obtained by separating the analytic function $(x + iy)^n$ into real and imaginary parts.

We confine our computations to the square region $|x| \le A$, $|y| \le A$ and note that the first two polynomials satisfying the symmetry requirements are,

$$v_1 = 1, \qquad v_2 = x^4 - 6x^2y^2 + y^4.$$

Accordingly we take

$$\psi_2 = a_1 + a_2(x^4 - 6x^2y^2 + y^4).$$

The constants a_1 and a_2 are determined by formulas (118.11), and we thus get

$$\int_C \left[\frac{1}{2}(x^2 + y^2) - \sum_{i=1}^{2} a_i v_i\right] \frac{dv_j}{d\nu}\,ds = 0, \qquad j = 1, 2. \tag{119.6}$$

Since $v_1 = 1$, we have in fact only one equation[1] and we can obtain another equation by requiring, for example, that the mean error vanish on $|x| = A$, $|y| = A$. This gives

$$\int_C [\tfrac{1}{2}(x^2 + y^2) - \psi_2]\, ds = 0,$$

which on integration yields the equation

$$15a_1 - 12A^4a_2 = 10A^2.$$

The second equation is (119.6) with $j = 2$.

The result of simple calculations shows that

$$a_1 = \frac{53A^2}{90}, \qquad a_2 = -\frac{7}{72A^2},$$

so that an approximate stress function is

$$\begin{aligned}\Psi_2 &\equiv \psi_2 - \tfrac{1}{2}(x^2 + y^2)\\ &= \frac{53}{90}A^2 - \frac{7}{72A^2}(x^4 - 6x^2y^2 + y^4) - \frac{1}{2}(x^2 + y^2).\end{aligned}$$

Thus

$$D_2 \equiv 2\mu \iint_R \Psi_2\, dx\, dy = {}^{19}\!/_{135}\mu a^4 = 0.1407\mu a^4,$$

where $a = 2A$.

The corresponding approximation in the Ritz method gave us a bound $0.1404\mu a^4$. Thus, without knowing the exact value ($D = 0.1406\mu a^4$), one can assert that D lies between $0.1404\mu a^4$ and $0.1407\mu a^4$. Thus the error is, at most, 0.2 per cent.

An approximate value of the maximum shearing stress can be computed from Ψ_2 as was done in Sec. 116. It turns out to be 2.8 per cent higher than the true maximum recorded in the table of Sec. 116.

120. The Rafalson Method for the Biharmonic Equation. The integration of the biharmonic equation

(120.1) $$\nabla^4 u = f(x, y) \qquad \text{in } R,$$

subject to the boundary conditions

(120.2) $$u = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } C,$$

as was shown in Sec. 111, is equivalent to the variational problem

(120.3) $$I(u) = \iint_R [(\nabla^2 u)^2 - 2fu]\, dx\, dy = \min.$$

[1] Recall that in the discussion of the uniqueness of solution of the system (118.11) it is necessary to reject the case v_i = const.

The application of the Ritz method yielding sequences of upper bounds converging to min $I(u) = I(u^*)$ was considered in Sec. 113. We now outline a method of solution of the problem (120.3), proposed by Rafalson,[1] which yields a sequence of lower bounds converging to $I(u^*)$ and an explicit formula for u^*.

The construction depends on a theorem analogous to the one established in Sec. 118.

Theorem: *Let $w(x, y)$ be any function of class C^2 in R which satisfies the boundary conditions* (120.2) *on C and $v(x, y)$ any solution of the Poisson equation*

$$\nabla^2 v = f(x, y) \qquad \text{in } R \tag{120.4}$$

of class C^1 in $R + C$. Then

$$I(w) \geq -\iint_R v^2\, dx\, dy. \tag{120.5}$$

The proof follows from Green's Theorem. We have

$$\begin{aligned}\iint_R wf\, dx\, dy &= \iint_R w\nabla^2 v\, dx\, dy\\ &= \iint_R v\nabla^2 w\, dx\, dy + \int_C \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right) ds\\ &= \iint_R v\nabla^2 w\, dx\, dy,\end{aligned}$$

by virtue of (120.2). But

$$\begin{aligned}I(w) + \iint_R v^2\, dx\, dy &= \iint_R [(\nabla^2 w)^2 - 2fw + v^2]\, dx\, dy\\ &= \iint_R [(\nabla^2 w)^2 - 2v\nabla^2 w + v^2]\, dx\, dy\\ &= \iint_R (\nabla^2 w - v)^2\, dx\, dy \geq 0,\end{aligned}$$

which establishes the theorem.

If in particular we take $w = u^*$, where u^* is the solution of the system (120.1), (120.2), the theorem yields

$$I(u^*) \geq -\iint_R v^2\, dx\, dy. \tag{120.6}$$

Thus, if one obtains any solution of the Poisson equation (120.4), the right-hand member of (120.6) provides a lower bound for $I(u^*)$. To

[1] Z. H. Rafalson, *Doklady Akademii Nauk SSSR*, vol. 64 (1949), p. 779.

obtain a bound that is arbitrarily near $I(u^*)$, Rafalson takes a set of orthonormal *harmonic* functions φ_i of class C^1 in $R + C$, that is, such that

$$\iint_R \varphi_i\varphi_j\,dx\,dy = \delta_{ij}, \tag{120.7}$$

and constructs

$$v_n \equiv v - \sum_{i=1}^{n} a_i\varphi_i, \tag{120.8}$$

where

$$a_i = \iint_R v\varphi_i\,dx\,dy \tag{120.9}$$

are the Fourier coefficients of v with respect to the set $\{\varphi_i\}$.

The function v_n satisfies the condition of the theorem, and hence

$$I(u^*) \geq -\iint_R v_n^2\,dx\,dy.$$

But,

$$\begin{aligned}
-\iint_R v_n^2\,dx\,dy &= -\iint_R \left(v - \sum_{i=1}^{n} a_i\varphi_i\right)^2 dx\,dy \\
&= -\iint_R v^2\,dx\,dy + \iint_R \left(2v\sum_{i=1}^{n} a_i\varphi_i\right) dx\,ay \\
&\quad - \iint_R \left(\sum_{i=1}^{n} a_i\varphi_i\right)^2 dx\,dy.
\end{aligned}$$

On using the relations (120.7) and (120.9), we easily find that

$$\iint_R \left(2v\sum_{i=1}^{n} a_i\varphi_i\right) dx\,dy = 2\sum_{i=1}^{n} a_i^2,$$

$$\iint_R \left(\sum_{i=1}^{n} a_i\varphi_i\right)^2 dx\,dy = \sum_{i=1}^{n} a_i^2,$$

so that

$$I(u^*) \geq -\iint_R v^2\,dx\,dy + \sum_{i=1}^{n} a_i^2. \tag{120.10}$$

It is then possible to prove that

$$I(u^*) = -\iint_R v^2\,dx\,dy + \sum_{i=1}^{\infty} a_i^2 \tag{120.11}$$

and

$$(120.12)\quad u^*(x, y) = -\frac{1}{2\pi}\iint_R \left[v(\xi, \eta) - \sum_{i=1}^{\infty} a_i\varphi_i(\xi, \eta)\right] \log\frac{1}{r}\, d\xi\, d\eta,$$

where $r^2 = (x - \xi)^2 + (y - \eta)^2$.

We see that Rafalson has reduced the problem to the determination of a solution of the Poisson equation (which can be easily done in many special problems) and to the calculation of the Fourier coefficients (120.9) with respect to the set of orthonormal harmonic functions in the region R.

These results can be generalized to three dimensions.

121. The Method of Least Squares. Collocation. In the discussion of the Galerkin method in Sec. 115, the error function in an approximate solution

$$(121.1)\qquad u_n = \sum_{i=1}^{n} a_i\varphi_i$$

of equation $L(u) = 0$ was defined as

$$(121.2)\qquad \epsilon_n(x, y) = L(u_n).$$

The function $u_n(x, y)$ was then minimized by requiring $\epsilon_n(x, y)$ to be orthogonal to every coordinate function φ_i. This has led to a system of equations

$$[115.5]\qquad \iint_R L(u_n)\varphi_i\, dx\, dy = 0, \qquad (i = 1, 2, \ldots, n),$$

for the determination of the a_i in (121.1).

A different definition of the error function, or a different criterion for minimizing u_n, would naturally lead to a different system of equations for the a_i. Thus, in some problems, it may seem desirable to determine the a_i so that they minimize the absolute value of $\epsilon_n(x, y)$ over the region; in others, a simpler criterion, requiring the integral of the square of the error over the region to have the smallest value, may prove adequate. Whatever be the criterion, its choice is determined by the simplicity of required mathematical apparatus and by the type of convergence desired of approximating sequences $\{u_n\}$.

A technique based on the construction of approximating sequences in accordance with the criterion

$$(121.3)\qquad E \equiv \iint_R \epsilon_n^2(x, y)\, dx\, dy = \min,$$

is known as the *method of least squares*. The appropriate equations for

the a_i in this method are thus got by equating to zero $\partial E/\partial a_i$. This leads to the system

$$\iint_R \epsilon_n \frac{\partial \epsilon_n}{\partial a_i} = 0, \qquad (i = 1, 2, \ldots, n). \tag{121.4}$$

To ensure that this system have a solution and the resulting sequence $\{u_n\}$ converge to the solution $u(x, y)$ of $L(u) = 0$, one must impose some restrictions on the operator L and on the choice of coordinate functions φ_i. Such matters have been studied by Krylov[1] and his followers. In constructing the approximate solutions for linear equations, Krylov selects a complete set of coordinate functions, with suitable differentiability properties, such that each φ_i satisfies assigned boundary conditions. Recently, Mikhlin[2] developed a least-squares method in which the coordinate functions φ_i satisfy the equation $L(u) = 0$. The coefficients a_i in (121.1) are then so selected that they minimize the integral of the square of the error $\epsilon_n(x, y)$ in the boundary conditions. This approach, suggestive of the procedure in the Trefftz method, often leads to more rapidly convergent sequences $\{u_n\}$ than in the usual least-squares method.

As an illustration of procedures followed in the standard least-squares method, we sketch the determination of torsional rigidity of a square beam. As our first approximation, we consider the function

$$\Psi_1 = a_1(x^2 - A^2)(y^2 - A^2) \tag{121.5}$$

used in a preceding section. The error function

$$\begin{aligned} \epsilon_1(x, y) &= \nabla^2\Psi_1 + 2 \\ &= 2 + 2a_1[(x^2 - A^2) + (y^2 - A^2)] \end{aligned} \tag{121.6}$$

and, on inserting this in (121.4) with $n = 1$, we easily find that

$$a_1 = \frac{15}{22A^2}.$$

The corresponding approximate values D_1 and τ_1 turn out to be

$$D_1 = 0.1515\mu a_4, \qquad \tau_1 = 0.682\mu\alpha a, \qquad a \equiv 2A,$$

while the exact values recorded in the table of Sec. 116, are,

$$D = 0.1406\mu a^4, \qquad \tau = 0.675\mu\alpha a.$$

Approximations of higher orders can be obtained in the same manner.

[1] N. M. Krylov, *Mémorial des sciences mathématiques*, No. 49 (1931); *Izvestiya Akademii Nauk SSSR*, Mathematical Series (1930), pp. 1089–1114.

[2] S. G. Mikhlin, *Doklady Akademii Nauk SSSR*, vol. 59 (1948); *Uchenye Zapiski, Leningrad State University*, Mathematical Series, No. 111, 16 (1949). See also his monograph Direct Methods in Mathematical Physics (1950), pp. 337–397. This work requires some familiarity with the Hilbert space theory.

Among sundry formal devices proposed for minimizing the error function is the *method of collocation.* This method requires that the error function vanish at n specified points in the region R. In this procedure the error $\epsilon_n(x, y)$ is "collocated," or assigned, at n points of the region, and the n equations for the coefficients a_i in $u_n = \sum_{i=1}^{n} a_i\varphi_i$ are obtained directly without integrations.

The process of collocation will be illustrated by its application to the problem just considered. We can get an equation for a_1 by setting $\epsilon_1(0, 0) = 0$. Then (121.6) yields $a_1 = 1/2A^2$. As a second approximation we take

$$\Psi_2 = (x^2 - A^2)(y^2 - A^2)[a_1 + a_2(x^2 + y^2)],$$

and require that $\epsilon_2(0, 0) = \epsilon_2(A/2, A/2) = 0$. Then

$$a_1 = \frac{25}{42A^2}, \qquad a_2 = \frac{2}{21A^4}.$$

The approximate values of the torsional rigidity and maximum shearing stress, got from Ψ_2, are

$$D_2 = 0.1407\mu a^4, \qquad \tau_2 = 0.690\mu\alpha a, \qquad a \equiv 2A.$$

Another process for finding an approximate solution for this problem is suggested by the analogy between the stress function $\Psi(x, y)$ and the deflection $z(x, y)$ of a membrane, under pressure p and tension T, stretched over a plane simple closed curve C.

For the determination of Ψ we have the system

$$\begin{aligned} \nabla^2\Psi &= -2 \qquad \text{in } R, \\ \Psi &= 0 \qquad \text{on } C, \end{aligned}$$

while the membrane deflection $z(x, y)$ is found from the relations (46.1)

$$\begin{aligned} \nabla^2 z &= -\frac{p}{T} \qquad \text{in } R, \\ z &= 0 \qquad \text{on } C. \end{aligned}$$

We proceed to find an approximate stress function Ψ_n by writing

$$\Psi_n = \sum_{i=1} a_i\varphi_i(x, y),$$

with $\qquad \varphi_i = 0 \qquad \text{on } C.$

The approximate stress function Ψ_n will not, in general, satisfy the

differential equation (46.3), and $\nabla^2\Psi_n$ will equal not -2 but some function[1] $p_n(x, y)$.

The function $p_n(x, y)$ can be interpreted, in terms of the membrane analogy, as the nonuniform pressure necessary to constrain the membrane to take the form defined by the function $\Psi_n(x, y)$.

If the collocation method were applied to this problem, one would demand that the "approximate loading function" $p_n(x, y)$ equal the "true load" of -2 at n points of the region R. Instead of such local conditions, one may impose a different set of conditions. The region R may be divided into n regions R_i, over each of which it is required that the total approximate load equal the total true load. That is, the n coefficients a_i are to be determined from the n conditions

$$\iint\limits_{R_i} \nabla^2\Psi_n \, dx \, dy = \iint\limits_{R_i} \nabla^2\Psi \, dx \, dy, \qquad (i = 1, 2, \ldots, n),$$

or

$$\iint\limits_{R_i} \nabla^2\Psi_n \, dx \, dy = -2 \cdot (\text{area of region } R_i). \tag{121.7}$$

Thus, instead of starting with a prescribed "pressure" -2 and solving for the deflection (or stress function), we have inverted the problem. We start with an assumed stress function Ψ_n, involving n arbitrary coefficients, and calculate the corresponding approximate loading function $p_n(x, y)$. The constants a_i are then so determined that the approximate and true loading functions are equal in the mean over each of n subregions R_i of the section R.

In applying the method outlined above to the torsion of a rectangular beam, choose, as a first approximation,

$$\Psi_1 = a_1(x^2 - A^2)(y^2 - B^2)$$

and take the region R_1 to be the entire region R of the rectangle. Equation (121.7) becomes

$$2a_1 \int_0^B \int_0^A (x^2 - A^2 + y^2 - B^2) \, dx \, dy = -2AB,$$

from which it follows that

$$a_1 = \frac{3}{2(A^2 + B^2)}$$

[1] Since the error function ϵ_n is defined, in this case, by

$$\nabla^2\Psi_n + 2 = \epsilon_n,$$

we see that the approximate loading function p_n is given in terms of the error function by the relation

$$p_n = \epsilon_n - 2.$$
$$\nabla^2\Psi_n = p_n(x, y).$$

and

$$\frac{1}{\mu} D_1 = \frac{1}{3} \frac{(b/a)^2}{1 + (b/a)^2} a^3 b = 0.1667a^4, \qquad \text{(square section)},$$

$$\frac{1}{\mu\alpha} (\tau_{\max})_1 = \frac{3}{2} \frac{(b/a)^2}{1 + (b/a)^2} a = 0.75a, \qquad \text{(square section)},$$

as against the exact values of $0.1406a^4$ and $0.675a$, respectively (for a square section).

As a further approximation, take (for a square section)

$$\Psi_3 = (x^2 - A^2)(y^2 - A^2)[a_1 + a_2(x^2 + y^2) + a_3 x^2 y^2].$$

From considerations of symmetry, it is clear that for the three regions R_{ij} one may take

$$R_{11}: \qquad 0 \le x \le \frac{A}{2}, \qquad 0 \le y \le \frac{A}{2},$$

$$R_{12}: \qquad \frac{A}{2} \le x \le A, \qquad 0 \le y \le \frac{A}{2},$$

$$R_{22}: \qquad \frac{A}{2} \le x \le A, \qquad \frac{A}{2} \le y \le A.$$

The three conditions

$$\iint\limits_{R_{ij}} \nabla^2 \Psi_3 \, dx \, dy = -2 \iint\limits_{R_{ij}} dx \, dy = -\frac{A^2}{2}$$

yield the equations

$$\begin{aligned} 440A^2a_1 - 186A^4a_2 - 17A^6a_3 &= 240, \\ 320A^2a_1 + 564A^4a_2 + 19A^6a_3 &= 240, \\ 200A^2a_1 + 594A^4a_2 + 235A^6a_3 &= 240, \end{aligned}$$

which are solved by

$$A^2a_1 = {}^{149}\!/_{252}, \qquad A^4a_2 = {}^{20}\!/_{252}, \qquad A^6a_3 = {}^{80}\!/_{252}.$$

The torsional rigidity is given approximately by

$$\frac{1}{\mu} D_3 = 2 \iint\limits_{R} \Psi_3 \, dx \, dy = 0.1413a^4$$

and is 0.5 per cent greater than the exact value, $0.1406a^4$. The maximum shearing stress is given approximately by

$$\frac{1}{\mu\alpha} (\tau_{\max})_3 = 0.671a,$$

a value that is 0.6 per cent less than the exact result, $0.675a$.

The method of this section was introduced by Biezeno and Koch,[1] who have applied it to problems of thin plates and elastically supported beams. Biezeno[2] has observed that this procedure may be applied to the general problem of elastic equilibrium.

It is clear that various techniques of obtaining approximate solutions presented above are but special cases of the general procedure of minimizing the error function by making it orthogonal with respect to some function $M(\varphi_i)$, that is, by imposing the conditions

$$(121.8) \qquad \iint\limits_R L(u_n)M(\varphi_i)\,dx\,dy = 0, \qquad (i = 1, 2, \ldots, n).$$

In the Galerkin method, $M(\varphi_i) = \varphi_i$; in the method of least squares, $M(\varphi_i) = \dfrac{\partial\epsilon_n}{\partial a_i}$; in the Biezeno-Koch procedure,

$$M(\varphi_i) = \begin{cases} 1 & \text{in } R_i, \\ 0 & \text{elsewhere in } R. \end{cases}$$

As noted in Sec. 116, when the $M(\varphi_i)$ are such that an arbitrary function $\eta(x, y)$, satisfying the required boundary conditions, can be represented in the series of $M(\varphi_i)$, with suitable properties, then the condition (121.8) may imply that $L(u_n) \to L(u) = 0$ and one can then justify calling $u_n(x, y)$ an "approximate solution."

122. The Function Space Methods. The theory of diverse methods of approximate solution of problems in mathematical physics was presented from a unified point of view by Kantorovich[3] in a paper which shows clearly the growing importance of the Banach and Hilbert space theories in the solution of concrete problems. Not only are the results obtained from a general point of view more complete and incisive, but they also lead to more effective methods of approximate solution and to sharper estimates of errors than the special techniques.

The analytical concepts of functional analysis were cast in a geometric

[1] C. B. Biezeno and J. J. Koch, "Over een nieuwe Methode ter Berekening van vlakke Platen met Toepassing op Enkele voor de Techniek belangrijke belastingsgevallen," *De Ingenieur*, vol. 38 (1923), pp. 25–36.

C. B. Biezeno, "Graphical and Numerical Methods for Solving Stress Problems," *Proceedings of the First International Congress of Applied Mechanics, Delft* (1924), pp. 3–17.

C. B. Biezeno and R. Grammel, Technische Dynamik, Chap. III, Sec. 9.

[2] C. B. Biezeno, "Over een Vereenvoudiging en over een Uitbreiding van de Methode van Ritz," *Christiaan Huygens International Mathematisch Tijdschrift*, vol. 3 (1923), pp. 69–75.

[3] L. V. Kantorovich, "Functional Analysis and Applied Mathematics," *Uspekhi Matematicheskikh Nauk*, vol. 3, No. 6 (1948), pp. 89–185. An English translation of this important paper was prepared for the National Bureau of Standards, *Report* 1509 (1952).

form by Prager and Synge[1] and effectively used by them to construct bounds for exact solutions of elastostatic problems. These authors deal with a real linear function space F, each point of which is a function τ_{ij} representing the state of stress in an elastic body τ. The set of functions τ_{ij} can be thought to define a point P in F or, alternatively, a position vector $\mathbf{T}$ from the origin O ($\tau_{ij} = 0$) to P. The vectors $\mathbf{T}$ are required to satisfy the familiar laws of addition, multiplication by scalars, and scalar multiplication of ordinary vector algebra. The square of length of $\mathbf{T}$ is defined by the formula

$$\mathbf{T} \cdot \mathbf{T} = \iint_R \tau_{ij} e_{ij} \, d\tau, \tag{122.1}$$

where the e_{ij} are defined by Hooke's law,[2]

$$e_{ij} = -\frac{\sigma}{E} \delta_{ij} \Theta + \frac{1+\sigma}{E} \tau_{ij}, \qquad \Theta \equiv \tau_{ii}. \tag{23.10}$$

Formula (122.1) determines metric properties of F, and since $\tau_{ij}e_{ij}$ is twice the strain energy, the metric of F is positive definite. Two vectors $\mathbf{T}$ and $\mathbf{T}'$ associated with a pair of points τ_{ij} and τ'_{ij} have the scalar product

$$\mathbf{T} \cdot \mathbf{T}' = \int_\tau e_{ij} \tau'_{ij} \, d\tau,$$

satisfying the reciprocity relation (Sec. 109) $\mathbf{T} \cdot \mathbf{T}' = \mathbf{T}' \cdot \mathbf{T}$. Consequently, the notions of directions, angle, orthogonality of vectors, etc., can be defined in a manner familiar from Riemannian geometry, and one can use the suggestive language and ideas of elementary geometry as an aid in visualizing relationships among different states of stress.

Thus, if one considers a set S of all functions τ_{ij} of class C^2, this set contains a subset S_1 of those τ_{ij} which satisfy the equilibrium equations

$$\tau_{ij,j} = 0 \qquad \text{in } \tau. \tag{122.2}$$

Also, there is a subset S_2 of S consisting of the τ_{ij} satisfying the compatibility equations

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \qquad \text{in } \tau.$$

Likewise there are subsets S_k containing the τ_{ij} that satisfy various types of boundary conditions on the surface of τ. The exact solution $\bar{\tau}_{ij}$ of a given boundary-value problem consists only of those τ_{ij} which are found

[1] W. Prager and J. L. Synge, *Quarterly of Applied Mathematics*, vol. 5 (1947), pp. 241–269.

J. L. Synge, *Quarterly of Applied Mathematics*, vol. 6 (1948), pp. 15–19; *Proceedings of the Royal Irish Academy* (*A*) vol. 53 (1950), pp. 41–64; *Rendiconti di matematica*, Rome, ser. 5, vol. 10 (1951), pp. 1–21.

[2] Although we write this law for an isotropic body, it is not essential to restrict the theory to isotropic elastic media. Any law of the form $e_{ij} = c_{ijkl}\tau_{kl}$ can be used so long as $\tau_{ij}e_{ij}$ is a positive definite form.

in the intersection of appropriate subsets of S. Prager and Synge obtain "approximate solutions" by isolating those subsets for which one or several conditions determining a unique solution are relaxed.

If the τ_{ij} in S_1, for example, are required to satisfy certain boundary conditions, then the exact solution $\bar{\tau}_{ij}$ is determined by those τ_{ij} which minimize the complementary energy. On the other hand, if one considers the vectors **T** in the subset S_2 which satisfy suitable boundary conditions, it turns out that the true state $\bar{\tau}_{ij}$ maximizes the strain energy. Theorems such as these serve to determine two-sided bounds for the true state.

Prager and Synge have couched their theorems in the language of geometry, and because the true states happen to be located on an intersection of a hyperplane with a hypersphere in the space F, their method has been called the *method of hypercircle*.[1] While it is true that several maximum-minimum principles formulated by these authors are direct consequences of the Schwarz and Bessel inequalities for vectors in a suitable function space,[2] the usefulness of geometric interpretation, as an aid to understanding these inequalities and as a guide in formulating theorems, is quite apparent.

123. The Method of Finite Differences. The various methods for approximate solution of boundary-value problems considered in the foregoing yield analytic expressions for the approximating functions. Although they have been successfully exploited in numerous problems of practical interest, there are serious limitations to their general use. The choice of coordinate functions in the approximate solution is governed by the shape of the region and by the form of the boundary conditions.

When the boundary values are not given by simple analytical expressions, it is difficult to make a judicious choice of coordinate functions, and even when a set of such functions is found, the computations may prove prohibitively heavy. One is then obliged to turn to some numerical or graphical method to obtain an approximate solution of the problem. The most universal of such numerical methods is the *method of finite differences*. In this method the differential equation is replaced by an approximating difference equation and the continuous region R by a set of discrete points. This permits one to reduce the problem to the solution of systems of algebraic equations, which may involve hundreds of unknowns. Ordinarily, some iterative technique has to be devised to solve such systems, and the high-speed electronic computers were devel-

[1] A clear discussion of the uses of the hypercircle method in stress analysis is contained in W. Prager, "The Extremum Principles of the Mathematical Theory of Elasticity and their Use in Stress Analysis," *University of Washington Bulletin* 119 (1951).

[2] See, for example, the Kantorovich *Uspekhi* paper cited in this section, and the Diaz *Collectanea Mathematica* article, mentioned in Sec. 116.

oped principally because of the need for coping with problems of this sort.[1]

The main disadvantage of all numerical techniques is that they give numerical values for unknown functions at a set of discrete points instead of the analytical expressions defined over the initial region R. Of course, when the boundary-value data are determined by measurements at a finite set of points of R, the difference-equations methods may be the best mode of attack on the problem. Any analytic technique would require fitting curves to the discontinuous data.

We proceed to the outline of the general procedure followed in reducing the given analytical boundary-value problem to a problem in difference equations. For definiteness let the region R be bounded by a simple closed curve C, and we seek to determine the function $u(x, y)$ satisfying some differential equation in R. From the definition of partial derivatives it follows that[2]

$$\left\{\begin{aligned} \frac{\partial u}{\partial x} &= \lim_{h\to 0} \frac{u(x+h, y) - u(x, y)}{h}, \\ \frac{\partial^2 u}{\partial x^2} &= \lim_{h\to 0} \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= \lim_{\substack{h\to 0\\ k\to 0}} \frac{u(x+h, y+k) - u(x+h, y) - u(x, y+k) + u(x, y)}{hk}, \end{aligned}\right. \tag{123.1}$$

and so on.

For small values of h and k the partial derivatives are nearly equal to the difference quotients appearing in the right-hand members of formulas (123.1). If one replaces derivatives in the given differential equation by difference quotients, there results a difference equation which is a good approximation to the given equation when h and k are small.

Thus, to Laplace's equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{123.2}$$

there corresponds the difference equation

$$\Delta^2 u \equiv u_{xx} + u_{yy} = 0, \tag{123.3}$$

where

$$u_{xx} \equiv \frac{1}{h^2}[u(x+h, y) - 2u(x, y) + u(x-h, y)],$$

$$u_{yy} \equiv \frac{1}{h^2}[u(x, y+h) - 2u(x, y) + u(x, y-h)].$$

[1] A noniterative technique for numerical solution of boundary-value problems, suitable for high-speed computing machines, was proposed by M. A. Hyman, *Applied Scientific Research*, sec. *B*, vol. 2 (1952), pp. 325–351.

[2] See, for example, E. Goursat, Cours d'analyse (1927), vol. 1, p. 47.

In a difference equation the values of $u(x, y)$ are related at a set of discrete points determined by the choices of h and k. Ordinarily these points are chosen so that they form a square net[1] with specified mesh size h.

The usual procedure is to cover the region R by a net consisting of two sets of mutually orthogonal lines a distance h apart (Fig. 63) and mark off a polygonal contour C' so that it approximates sufficiently closely the

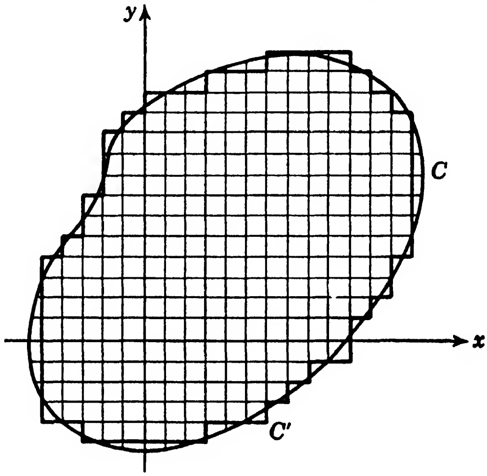

FIG 63

boundary C. The domain R' in which the solution of the difference equation is sought is formed by the lattice points of the net contained within C'. The assigned boundary values on C are then transferred in some manner to the lattice points on C'. When the lattice points on C' do not coincide with points on C, the desired values can be got by interpolation.[2]

One then seeks a solution of the difference equation which satisfies the boundary conditions imposed at the lattice points on C'. Usually, this leads to a consideration of a system of a large number of algebraic equations in many unknowns. An illustration of this procedure is given in the following section in connection with a Dirichlet problem for Laplace's equation.

The literature on finite-difference methods is extensive.[3] The con-

[1] Rectangular, polygonal, and curvilinear nets are also used. See, for example, D. Y. Panov, Handbook on Numerical Solution of Partial Differential Equations, Moscow (1951), which contains a good account of the difference-equations techniques. See also Appendix to S. Timoshenko and J. N. Goodier's Theory of Elasticity (1951).

[2] See, for example, W. E. Milne, Numerical Solution of Differential Equations (1953), or L. M. Milne-Thomson, Calculus of Finite Differences (1933).

[3] See Chap. 10 by T. J. Higgins in L. E. Grinter, editor, Numerical Methods of Analysis in Engineering (1949).

vergence of approximate solutions of the difference equations connected with the second-order partial differential equations was investigated by[1] Lusternik, Courant, Friedrichs, Levy, and Petrovsky. An excellent account of the finite-difference methods, including a proof of convergence, for the case of elliptic equations, is contained in Chap. 3 of the Kantorovich and Krylov monograph[2] cited in Sec. 112.

124. An Illustration of the Method of Finite Differences.[3] The method of finite differences will be illustrated by applying it to the Dirichlet problem of determining the function $\psi(x, y)$ defined by

$$\begin{cases} \nabla^2\psi = 0 & \text{in } R, \\ \psi = g(s) & \text{on } C, \end{cases} \tag{124.1}$$

We lay down a square net over the region R, assign known values to the net points on the boundary, and approximate (guessed) values at interior points. The differential equation of the system (124.1) is replaced by a difference equation, which we proceed to derive.

In the neighborhood of any interior point of R (taken, for the moment, as the origin of coordinates), we can write

$$\begin{aligned}\psi(x, y) &= \psi_0 + \alpha_{10}x + \alpha_{01}y \\ &\quad + \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{11}xy + \cdots \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{ij}x^iy^j.\end{aligned} \tag{124.2}$$

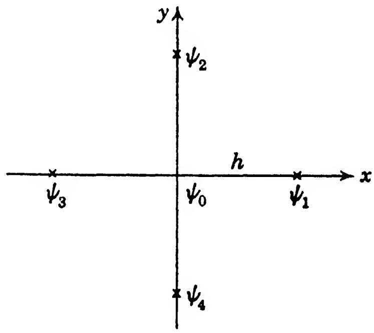

FIG. 64

The value of the function ψ at the origin is precisely

$$\psi(0, 0) = \psi_0 = \alpha_{00},$$

while at the neighboring net points to the right and left one has (Fig. 64)

[1] L. A. Lusternik, "Über einige Anwendungen der direkten Methoden in Variationsrechnung," *Matematicheski Sbornik*, vol. 23 (1926), pp. 173–201.

R. Courant, K. Friedrichs, and H. Levy, *Mathematische Annalen*, vol. 100 (1928), pp. 32–74.

I. G. Petrovsky, *Uspekhi Matematicheskikh Nauk*, No. 8 (1941), p. 161. See also D. Panov, *Matematicheski Sbornik*, vol. 40 (1933), pp. 373–393; *Uspekhi Matematicheskikh Nauk*, No. 4 (1937), pp. 23–33.

S. E. Mikeladze, *Izvestiya Akademii Nauk SSSR*, Mathematical Series, vol. 5 (1941), pp. 57–74.

P. P. Yushkov, *Prikl. Mat. Mekh., Akademiya Nauk SSSR*, vol. 12 (1948), pp. 223–226.

[2] See also Chap. V by W. Feller in J. D. Tamarkin and W. Feller, Partial Differential Equations (1941), pp. 160–196.

[3] This section is taken, virtually without changes, from the first edition of this book. It was prepared for that edition by Dr. R. D. Specht.

$$\psi_1 = \psi(h, 0) = \sum_{i=0}^{\infty} \alpha_{i0}h^i = \psi_0 + \alpha_{10}h + \alpha_{20}h^2 + \cdots,$$

$$\psi_3 = \psi(-h, 0) = \psi_0 - \alpha_{10}h + \alpha_{20}h^2 - \cdots,$$

and

$$\psi_1 + \psi_3 = 2\psi_0 + 2\alpha_{20}h^2 + 2\alpha_{40}h^4 + \cdots.$$

Similarly,

$$\psi_2 + \psi_4 = 2\psi_0 + 2\alpha_{02}h^2 + 2\alpha_{04}h^4 + \cdots.$$

Since the value of the Laplacian at the origin is

$$(\nabla^2\psi)_0 = 2\alpha_{20} + 2\alpha_{02},$$

one can write

$$\frac{\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0}{h^2} = (\nabla^2\psi)_0 + \text{terms in } h^2. \tag{124.3}$$

As the choice of the origin of coordinates is not essential to the argument above, the foregoing expression relates $\nabla^2\psi$ at any point to the value of ψ at that point and to the neighboring values. We drop the terms in h^2 and replace the Laplace differential equation $\nabla^2\psi = 0$ by the Laplace difference equation

$$\psi(x + h, y) + \psi(x - h, y) + \psi(x, y + h) + \psi(x, y - h) - 4\psi(x, y) = 0,$$

or

$$\psi_0 = \frac{\psi_1 + \psi_2 + \psi_3 + \psi_4}{4}. \tag{124.4}$$

Expressed in words, *the value of $\psi(x, y)$ at any point is the mean of its values at the four immediate neighboring points.* This difference equation is equivalent to a set of linear equations for the values of ψ at interior points in terms of the prescribed boundary values. The number of variables is usually so large, however, that direct solution is out of the question. Instead, we may resort to an iterative procedure.

The simplest (but most laborious) way of solving the Laplace difference equation is to guess at the proper values for the interior points of the network; this guess is then corrected by traversing the net, replacing each interior value by the mean of its four immediate neighbors. Repeated traverses of the net will give interior values that converge to the values of the solution function $\psi(x, y)$. The convergence, however, is so slow as to require almost unlimited manpower in order to secure sufficiently accurate results. Fortunately, various procedures are available for improving the rapidity of convergence.

Instead of expressing the value of ψ as the mean of the four immediate

neighbors [Eq. (124.4)], the four diagonal neighbors may be used[1] (Fig. 65). We have from (124.2)

$$\psi_5 + \psi_6 + \psi_7 + \psi_8 = 4\psi_0 + 4(\alpha_{20} + \alpha_{02})h^2 + \text{terms in } h^4,$$

or, neglecting terms in h^2 compared with unity,

$$2(\nabla^2\psi)_0 = \frac{\psi_5 + \psi_6 + \psi_7 + \psi_8 - 4\psi_0}{h^2}.$$

The Laplace difference equation can also be written, then, as

$$\psi_0 = \frac{\psi_5 + \psi_6 + \psi_7 + \psi_8}{4}. \tag{124.5}$$

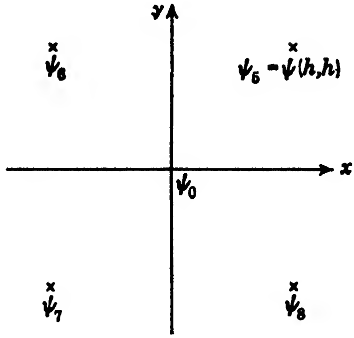

FIG. 65

If both the immediate and the diagonal neighbors are used, then the formula

$$20\psi_0 = 4(\psi_1 + \psi_2 + \psi_3 + \psi_4) + (\psi_5 + \psi_6 + \psi_7 + \psi_8) \tag{124.6}$$

gives the value of ψ_0 for any seventh-order harmonic polynomial.[2]

The slowness of convergence of the original process [Eq. (124.4)] is explained by the fact that, on any one traverse, an interior value is made to depend only on its immediate neighbors, and the effect of the prescribed boundary values moves inland very slowly as successive traverses are made. The boundary values may be made effective at a greater distance by the following procedure, which uses 9 interior points ψ_{ij} and 16 boundary points M_i, D_i, C_i (Fig. 66). The value at the center is first found from[3]

$$\psi_{00} = \tfrac{1}{16}[D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + 2(M_1 + M_2 + M_3 + M_4)]. \tag{124.7}$$

The corner values, such as ψ_{11}, are obtained from the diagonal neighbors, so that

$$\psi_{11} = \frac{\psi_{00} + C_1 + M_1 + M_2}{4},$$

while values such as ψ_{10} make use of the immediate neighbors

$$\psi_{10} = \frac{\psi_{00} + M_1 + \psi_{11} + \psi_{1,-1}}{4}.$$

[1] This follows from invariance with respect to rotation.

[2] G. H. Shortley and R. Weller, *Journal of Applied Physics*, vol. 9 (1938), p. 345.

[3] G. H. Shortley, R. Weller, and B. Fried, "Numerical Solution of Laplace's and Poisson's Equations," *Ohio State University Studies*, Engineering Ser. (1940), p. 11.

This same procedure may be applied, of course, to find the value of ψ on any block of 9 points in terms of the surrounding values M_i, D_i, C_i, whether the latter lie on the boundary of the region R or not.

The method of finite differences will now be applied to the torsion problem for a beam of square cross section with side length $2A$. The conjugate torsion function $\psi(x, y)$ is defined by

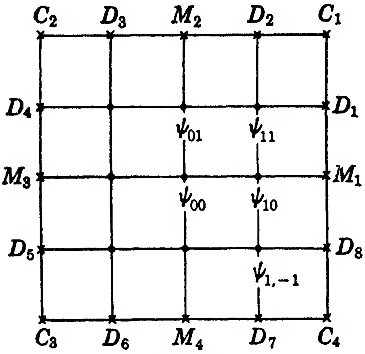

Fig. 66

$$\begin{aligned} \nabla^2\psi &= 0 && \text{in } R, \\ \psi &= \tfrac{1}{2}(x^2 + y^2) && \text{on } C. \end{aligned}$$

We introduce the variables $X = x/A$, $Y = y/A$ and put

$$\Omega = 10^4\left(\frac{2}{A^2}\psi - 1\right). \tag{124.8}$$

Then the function Ω is subject to the conditions

$$\begin{aligned} \nabla^2\Omega &= 0 && \text{in } R, \\ \Omega &= \begin{cases} 10^4Y^2 & \text{on } X = \pm 1, \\ 10^4X^2 & \text{on } Y = \pm 1. \end{cases} \end{aligned}$$

A coarse net is now laid down over the square section (Fig. 67). Equation (124.7) gives $\Omega(0, 0) = 1250$, while $\Omega(\tfrac{1}{2}, \tfrac{1}{2}) = 2812$ is derived from its diagonal neighbors by (124.5) and $\Omega(\tfrac{1}{2}, 0) = 1718$ from its immediate

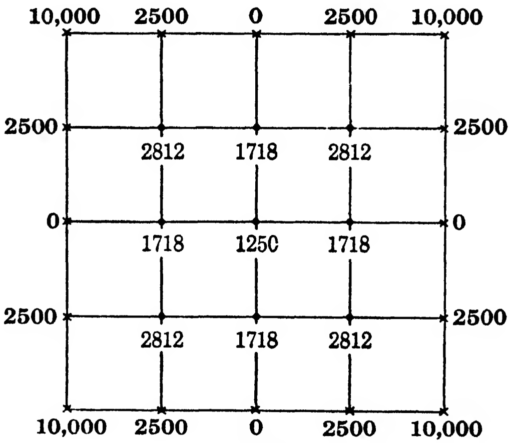

Fig. 67

neighbors by (124.4). From symmetry considerations, it is seen that only one octant of the section need be considered. The values of Ω in this octant are now improved by using Eq. (124.6) to give, in order, $\Omega(\tfrac{1}{2}, \tfrac{1}{2}) = 2250$, $\Omega(\tfrac{1}{2}, 0) = 1572$, and $\Omega(0, 0) = 1708$. Figure 68 shows these values as well as those of the third approximation resulting

from a further application of Eq. (124.6) to the net values in the same order as before.

The values found above can be checked against those given by the

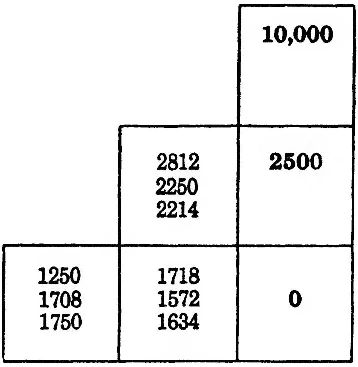

FIG. 68

exact solution of the problem in Sec. 38. From (38.10) and (124.8), it follows that

$$\Omega(X, Y) = 10^4\left[1 + Y^2 - X^2 - 8\sum_{n=0}^{\infty}\frac{(-1)^n}{N^3}\frac{\cosh NY}{\cosh N}\cos NX\right],$$

where $N = (2n + 1)\pi/2$. The exact values at the net points used above are found to be

$$\Omega(0, 0) = 1787.4, \qquad \Omega(0, \tfrac{1}{2}) = 1673.6, \qquad \Omega(\tfrac{1}{2}, \tfrac{1}{2}) = 2245.8,$$

and it is seen that the approximate values are in error at most by 2 per cent.

The torsional rigidity is given by

$$\begin{aligned}\frac{1}{\mu}D &= 2\iint_R \Psi\,dx\,dy = 2\iint_R\left[\psi - \frac{1}{2}(x^2 + y^2)\right]dx\,dy\\ &= A^2\iint_R (1 + 10^{-4}\Omega)\,dx\,dy - \frac{8}{3}A^4,\end{aligned}$$

or

$$\frac{1}{\mu a^4}D = \frac{1}{12} + \frac{10^{-4}}{16}\int_{-1}^{1}\int_{-1}^{1}\Omega\,dX\,dY.$$

The maximum shear stress occurs at a mid-point of the boundary and is given by

$$\frac{1}{2\mu\alpha}\tau_{\max} = -\frac{1}{2}\left(\frac{\partial\psi}{\partial x} - x\right)_{\substack{x=A\\y=0}} = \frac{A}{4}\left(2 - 10^{-4}\frac{\partial\Omega}{\partial X}\right)_{\substack{X=1\\Y=0}}.$$

To find the approximate value of the shear stress at this point, we pass a

parabola through the points $\Omega(1, 0) = 0$, $\Omega(1 - h, 0) \equiv \Omega_1$, and $\Omega(1 - 2h, 0) \equiv \Omega_2$, where h is the mesh of the net in the X, Y coordinates. The slope of the parabola at (1, 0) is given by

$$\left(\frac{\partial \Omega}{\partial X}\right)_{\substack{X=1\\Y=0}} = \frac{1}{2h}(\Omega_2 - 4\Omega_1),$$

and we have, approximately,

$$\frac{1}{2\mu\alpha}\tau_{\max} = \frac{A}{4}\left[2 - \frac{10^{-4}}{2h}(\Omega_2 - 4\Omega_1)\right]. \tag{124.9}$$

The approximate values of Ω given in Fig. 68 yield $\Omega_1 = 1634$,

$$\Omega_2 = 1750,$$

$h = \frac{1}{2}$, and

$$\frac{1}{2\mu\alpha}\tau_{\max} = 0.6196A,$$

which is 8.1 per cent below the exact value $0.675A$. The numerical values of Ω can be obtained by Simpson's rule to give

$$\int_{-1}^{1}\int_{-1}^{1} \Omega\, dX\, dY = \frac{80{,}246}{9},$$

and

$$\frac{1}{\mu a^4} D = 0.1391,$$

which is 1 per cent below the true value 0.1406.

If the exact solution of this problem were not known, it would have been necessary to proceed with the iterative procedure until the net values remained sensibly constant. Before continuing the process, however, we introduce a laborsaving modification.

Denote by $\Omega^{(0)}$ an approximate solution and by Ω the exact solution of the difference equation at a given net point. Then one can write

$$\Omega^{(0)} = \Omega + \epsilon^{(0)},$$

where $\epsilon^{(0)}$ is the error at the net point in the solution of the difference (not the differential) equation. A single traverse of the net yields an improved value $\Omega^{(1)}$ and error $\epsilon^{(1)}$:

$$\Omega^{(1)} = \Omega + \epsilon^{(1)}.$$

We denote by $\delta^{(1)}$ the change in Ω in one traverse; that is,

$$\delta^{(1)} = \Omega^{(1)} - \Omega^{(0)} = \epsilon^{(1)} - \epsilon^{(0)}.$$

Now replace the original boundary-value problem of determining the function Ω with given boundary values by the problem of determining the difference function δ, which vanishes at the boundary net points. In

other words, the differences $\delta^{(1)}$ are improved in the same way [for example, by Eq. (124.4)] as were the original values Ω, and successive traverses yield the improved differences $\delta^{(2)}, \delta^{(3)}, \ldots$. The final values of Ω are obtained by adding to $\Omega^{(1)}$ the improved differences:

$$\Omega^{(n)} = \Omega^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots + \delta^{(n)}.$$

For

$$\delta^{(2)} = \Omega^{(2)} - \Omega^{(1)} = \epsilon^{(2)} - \epsilon^{(1)},$$

and, in general,

$$\delta^{(n)} = \Omega^{(n)} - \Omega^{(n-1)} = \epsilon^{(n)} - \epsilon^{(n-1)}.$$

Then

$$\begin{aligned} \Omega^{(1)} &+ \delta^{(2)} + \delta^{(3)} + \cdots + \delta^{(n)} \\ &= \Omega + \epsilon^{(1)} + (\epsilon^{(2)} - \epsilon^{(1)}) + (\epsilon^{(3)} - \epsilon^{(2)}) + \cdots + (\epsilon^{(n)} - \epsilon^{(n-1)}) \\ &= \Omega + \epsilon^{(n)} = \Omega^{(n)}. \end{aligned}$$

In the torsion problem under discussion, the differences $\delta^{(1)}$ can be found from the values of Ω calculated on the last two traverses (Fig. 68).

Ω	δ	Ω	δ	
				10,000
		2214 2241 2253 2258 (2263) [2263.6]	−36 27 12 5	2500
1750 1776 1788 1794 (1800) [1799.8]	42 26 12 6	1634 1660 1672 1678 (1684) [1683.8]	62 26 12 6	0

FIG. 69

In Fig. 69, these values of Ω are shown in the left-hand columns, while in the columns on the right are entered the differences $\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}$ obtained by applying (124.6) to the differences in the order ($\frac{1}{2}$, $\frac{1}{2}$), ($\frac{1}{2}$, 0), (0, 0). Two advantages of working with the difference function δ rather than with the original function Ω are immediately seen. Both the zero boundary values and the smaller number of significant figures in the differences δ make for easier computation. A third and more powerful advantage is that use of the difference function δ makes possible the estimation of the effect of infinitely many traverses of the network. That is, inspection of the successive differences $\delta^{(2)}, \delta^{(3)}, \delta^{(4)}$ suggests that

these and succeeding values may form a geometric progression of ratio ½. We hazard a guess that this is indeed the case and sum each infinite series of differences, getting, in this case,

$$\delta^{(5)} + \delta^{(6)} + \delta^{(7)} + \cdots = 2\delta^{(5)} = \delta^{(4)}.$$

The sum of the differences $\sum_{2}^{\infty} \delta^{(i)}$ is then added to $\Omega^{(1)}$ at each net point to obtain an estimate of $\Omega^{(\infty)}$ (see the fifth entry in each column on the left in Fig. 69). The assumption that the successive differences $\delta^{(n)}$ form a geometric progression can now be tested by using (124.6) to improve the values of Ω just obtained (see the last entry in the left-hand columns of Fig. 69).

Shortley, Weller, and Fried, in an investigation of the convergence of the method of finite differences,[1] have shown that this extrapolation to the limiting net value by summing the infinite geometric series of differences is possible in general.

The final net values in Fig. 69 satisfy the difference equation (to within one unit) but not the differential equation. This is shown by comparison with the exact ordinates given above, and the disagreement arises from the fact that, in setting up the difference equation, terms of higher order in the net mesh were neglected [see (124.3)]. We proceed, therefore, to decrease the mesh of the net to one-half its original size.

In the process of interpolation leading to the values of Ω at the new net points, the difference equations (124.4) and (124.5) are used to ensure that the interpolated values satisfy, at least approximately, the differential equation $\nabla^2\Omega = 0$. The mean of the diagonal neighbors [Eq. (124.5)] furnishes the values of $\Omega(\frac{3}{4}, \frac{3}{4})$, $\Omega(\frac{3}{4}, \frac{1}{4})$, and $\Omega(\frac{1}{4}, \frac{1}{4})$. The immediate neighbors are used [Eq. (124.4)] to get $\Omega(\frac{1}{4}, 0)$, $\Omega(\frac{3}{4}, 0)$, $\Omega(\frac{1}{2}, \frac{1}{4})$, and $\Omega(\frac{3}{4}, \frac{1}{2})$. The resulting values are shown in Fig. 70. Without any further improvement, these values give $\Omega_1 = 1226$, $\Omega_2 = 1683$ [see Eq. (124.9)], and

$$\frac{1}{2\mu\alpha} \tau_{\max} = 0.661A,$$

which is 2 per cent below the exact value $0.675A$. The integral of Ω is found by Simpson's rule to be approximately 9292, and the approximate torsional rigidity is found to be $D/(\mu a^4) = 0.1414$, a value 0.6 per cent above the exact result of 0.1406.

Instead of traversing the lattice points in a fixed order and extrapolating to the limiting net value, as above, one can correct the lattice values in any way at all. Indeed, all that is required is that one arrive at values Ω for which the difference equation is satisfied—or, alternatively, for

[1] G. H. Shortley, R. Weller, and B. Fried, "Numerical Solution of Laplace's and Poisson's Equations," *Ohio State University Studies*, Engineering Ser. (1940), p. 18.

which the differences δ are zero. In this way, the experience and physical intuition of the computer can be used to good advantage.

Another variation in the finite-difference method consists in replacing the lattice with square mesh, used above, by a lattice formed of regular polygons.[1]

The torsion problem of a beam of square cross section, considered above, is a particularly simple one in that the square cross section

				10,000
			4316	**5625**
		2263	*2672*	**2500**
	1857	*1853*	*1611*	**625**
1799	*1799*	1683	*1226*	**0**

FIG. 70

imposes no special complications at the boundary. When the boundary is curved, the derivatives are replaced by finite-difference expressions involving unequal intervals.[2]

While the finite-difference method has been illustrated by its application to the problem of Dirichlet,

$$\begin{aligned} \nabla^2\Omega &= 0 \qquad \text{in } R, \\ \Omega &\text{ given} \qquad \text{on } C, \end{aligned}$$

it can obviously be extended to a wide variety of problems in engineering and mathematical physics. We mention as examples[3] the plasticity problem of torsion of a shaft strained beyond the elastic limit and the problem of a two-dimensional magnetic field containing a triangular prism of iron.

[1] See, for example, D. G. Christopherson and R. V. Southwell, "Relaxation Methods Applied to Engineering Problems. III. Problems Involving Two Independent Variables," *Proceedings of the Royal Society (London)* (*A*), vol. 168 (1938), pp. 317–350.

[2] For this and other details, both theoretical and practical, relating to finite-difference methods, see G. H. Shortley, R. Weller, and B. Fried, "Numerical Solutions of Laplace's and Poisson's Equations," *Ohio State University Studies*, Engineering Ser. (1940).

[3] D. G. Christopherson and R. V. Southwell, "Relaxation Methods Applied to Engineering Problems. III. Problems Involving Two Independent Variables," *Proceedings of the Royal Society (London)* (*A*), vol. 168 (1938), pp. 317–350.

Tables given by D. Moskovitz[1] can be used to obtain the exact solution of the *difference* equation corresponding to the Poisson equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y),$$

when the region R is assumed to be rectangular.

125. Relaxation Methods. We saw that the application of the method of finite differences may lead to the solution of certain systems of linear algebraic equations. Such systems frequently arise in applications, and a technique for solving them, bearing the name of the *relaxation method*, is the subject of this section.

Let us begin by considering a very simple example.[2] The system

$$(125.1)\qquad \begin{cases} -1 + 2u_1 - u_2 = r_1 = 0 \\ -1 - u_1 + 2u_2 - u_3 = r_2 = 0 \\ -1 - u_2 + 2u_3 - u_4 = r_3 = 0 \\ -1 - u_3 + 2u_4 - u_5 = r_4 = 0 \\ -1 - u_4 + u_5 = r_5 = 0 \end{cases}$$

corresponds to the problem of static equilibrium of 10 equal masses, equally spaced on a light string under a uniform tension (see Fig. 71, in

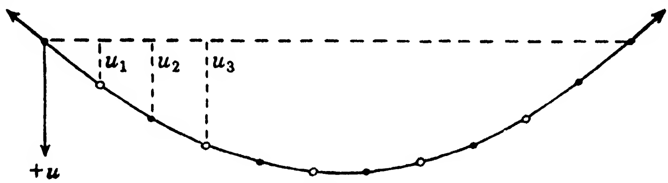

FIG. 71

which the sag is exaggerated). By symmetry we assume $u_6 = u_5$, $u_7 = u_4$, etc.

The desired equilibrium position corresponds to the $u_1, \ldots, u_5$ obtained by solving (125.1) with $r_1 = \cdots = r_5 = 0$. A direct approach to solving the problem would be to solve (125.1) by the systematic elimination of unknowns, in the fashion of high-school algebra. The elimination might alternatively be expressed in terms of determinants. While these direct methods would solve (125.1) fairly readily, they would be very complicated indeed for a larger system like that corresponding to Fig. 72.

An alternate solution of (125.1) has proved very popular among engineers and computers. We make a first guess of the u, say $u_1 = 4$,

[1] "The Numerical Solution of Laplace's and Poisson's Equations," *Quarterly of Applied Mathematics*, vol. 2 (1944), pp. 148–163.

[2] This example, together with other material in this section, is adapted from G. E. Forsythe's chapter in Modern Mathematics for the Engineer, edited by E. F. Beckenbach, (1955). I am indebted to Dr. G. E. Forsythe, Research Mathematician, Numerical Analysis Research Project, Department of Mathematics, University of California, Los Angeles, for the preparation of this section.

$u_2 = 8$, $u_3 = 12$, $u_4 = 14$, $u_5 = 15$. From (125.1) we find that

$$r_1 = r_2 = -1, \; r_3 = 1, \; r_4 = r_5 = 0.$$

Since these *residuals* r_i are not all zero, we shall improve the trial solution by reducing them. We pick one of the numerically largest r_i, say r_3, and bring it to zero by an appropriate change Δu_3 in u_3 alone. It is clear from (125.1) that a unit increase in u_3 ($\Delta u_3 = +1$) would cause changes only in r_2, r_3, and r_4, and these changes would be $\Delta r_2 = -1$, $\Delta r_3 = +2$, $\Delta r_4 = -1$. To make $r_3 = 0$ calls for $\Delta r_3 = -1$, which we bring about by selecting

$$\Delta u_3 = -0.5.$$

As by-products we have $\Delta r_2 = \Delta r_4 = +0.5$. Accumulating the r and Δr, we find the residuals $r_1 = -1.0$, $r_2 = -0.5$, $r_3 = 0.0$, $r_4 = 0.5$, $r_5 = 0.0$. There is now a single numerically largest residual, r_1, and we proceed to "liquidate" it by selecting $\Delta u_1 = +0.5$. Next time $\Delta u_2 = +0.5$, etc. Eight steps of this process are summarized in Table 1. An experienced computer goes very rapidly, calculating mentally and recording a residual only when it changes.

TABLE 1. RELAXATION SOLUTION OF EQS. (125.1)

i		1	2	3	4	5	
First guess of u_i		4.0	8.0	12.0	14.0	15.0	
Residuals r_i		−1.0	−1.0	1.0	0.0	0.0	
i	Δu_i	Residuals					
3	−0.5						
		−1.0	−0.5	0.0	0.5	0.0	
1	+0.5						
		0.0	−1.0	0.0	0.5	0.0	
2	+0.5						
		−0.5	0.0	−0.5	0.5	0.0	
4	−0.3						
		−0.5	0.0	−0.2	−0.1	0.3	
1	+0.3						
		0.1	−0.3	−0.2	−0.1	0.3	
2	+0.2						
		−0.1	0.1	−0.4	−0.1	0.3	
3	+0.2						
		−0.1	−0.1	0.0	−0.3	0.3	
5	−0.3						
		−0.1	−0.1	0.0	0.0	0.0	
Current solution		4.8	8.7	11.7	13.7	14.7	
Residuals		−0.1	−0.1	0.0	0.0	0.0	Check
True solution		5.0	9.0	12.0	14.0	15.0	

Southwell[1] thinks of the r_i as negatives of constraining forces actually applied to the weights to keep the system in equilibrium with the current displacements. Each step of the above calculation is then thought of as a relaxation of one of these external constraints. Hence Southwell's name for the process—relaxation.

At the bottom of Table 1 are cumulated the current values of the u_i. For example, $u_1 = 4 + 0.5 + 0.3 = 4.8$. Recalculation of the residuals then confirms the computation so far. In these eight mental steps the maximum error of the u_i has been reduced from 1.0 to 0.3. Further computing would improve the u_i at a comparable rate, and it would not take long to achieve ordinary engineering accuracy.

There are many tricks used by relaxers. One, illustrated in Table 1, is to work to one significant digit only, and not to complicate the numbers by introducing overprecise corrections like $\Delta u_4 = -0.25$. Thus residuals are liquidated only in the most significant digit. More precision comes automatically in later steps. Other tricks can be used to accelerate the convergence of the u_i to the correct answers. Such acceleration is nearly always essential to solving a problem of any magnitude.

A great timesaver in engineering practice is not to draw up anything like Table 1, but instead to use a working drawing of the model as a computing sheet. The values of Δu_i and r_i can be recorded on the drawing.

Relaxation is really fun for a computer, for several reasons: (1) seeing the partial answer evolve lends a purpose to each step, and combats the usual tedium of day-long computing; (2) one's intelligence is continually challenged by the possibility of improving the speed of convergence; (3) one need never waste much time in erroneous computing, as is possible in elimination.

There are many variations of relaxation methods. They all deal with solving systems of equations, usually linear, and they share these essential properties: (1) for any trial solution there is a measure of the error in each of the equations; (2) for each unsatisfied equation there is a separate formula for improving the trial solution; (3) one calculates at each step with the equation whose error is largest.

The relaxation method was originally devised for pencil-and-paper computing, without a keyboard calculator, and is ideally adapted to such work. It is reasonably adaptable to keyboard calculators, but here it seems to lose some of its relative superiority over other methods. For automatic digital computers, see below.

The relaxation method seems to date from Gauss,[2] who used and

[1] R. V. Southwell, Relaxation Methods in Engineering Science (1940).

[2] C. F. Gauss, Brief an Gerling, 26 December 1823, Werke, vol. 9, pp. 278–281 [translated by G. E. Forsythe, *Mathematical Tables and Other Aids to Computation*, vol. 5 (1951), pp. 255–258].

recommended the basic method and many of the standard tricks. Seidel[1] proved it would converge for linear systems with positive definite matrices. In the thirties Southwell[2] rediscovered Gauss' method and named it. He and his school have developed the method and brought its wide applicability to the attention of engineers and scientists everywhere. The method has proved especially suited to the analysis of complicated redundant pin-jointed frameworks. These have equations like (125.1), with more involved coefficients.

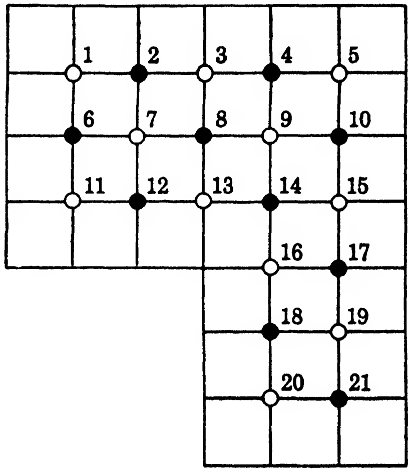

FIG. 72

As a digression it should be noted that one-dimensional problems like that of Fig. 71 should not in practice be solved by relaxation, for they are easily solved by a fast trial-and-recursion scheme, even for variable masses. For example, in (125.1) change the last equation to read

$$-1 - u_4 + 2u_5 - u_6 = r_5 = 0.$$

Fix $r_1 = \cdots = r_5 = 0$. By symmetry we should have $u_6 - u_5 = 0$. Try $u_1^{(1)} = 3$. From the modified equations (125.1), we find successively $u_2^{(1)} = 5$, $u_3^{(1)} = 6$, $u_4^{(1)} = 6$, $u_5^{(1)} = 5$, $u_6^{(1)} = 3$, whence $u_6^{(1)} - u_5^{(1)} = -2$. Now try $u_1^{(2)} = 6$. We find successively $u_2^{(2)} = 11$, $u_3^{(2)} = 15$, $u_4^{(2)} = 18$, $u_5^{(2)} = 20$, $u_6^{(2)} = 21$, $u_6^{(2)} - u_5^{(2)} = +1$. Interpolating linearly between $u_1^{(1)}$ and $u_1^{(2)}$ to make $u_6 - u_5$ zero, we get $u_1 = 5$, whence the true solution is obtained recursively.

The point of Table 1 was to show the technique of relaxation in a simple setting. The practical applications of the method begin with two-dimensional problems—like trusses. Or, in closer relation to Fig. 71, suppose one had an L-shaped network of light strings with 21 weights on it in a horizontal plane (Fig. 72). How might one calculate the equilibrium position of the weights under large tension and under gravity? With the same assumptions as above, we find 21 equations like

$$-1 - u_2 - u_6 + 4u_7 - u_8 - u_{12} = r_7 = 0. \tag{125.2}$$

[1] Ludwig Seidel, "Ueber ein Verfahren, die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, sowie lineäre Gleichungen überhaupt, durch successive Annäherung aufzulösen," *Abhandlungen mathematisch. physischen Klasse, Bayrische Akademie der Wissenschaften (München)*, vol. 11 (1874), pp. 81–108.

[2] R. V. Southwell, Relaxation Methods in Engineering Science (1940) and Relaxation Methods in Theoretical Physics (1946).

It would be a tedious computation at a desk to solve these by elimination, and no simple recursive scheme works here. A practical answer is relaxation, which works numerically about like Table 1, although with slower convergence. The same tricks apply as before.

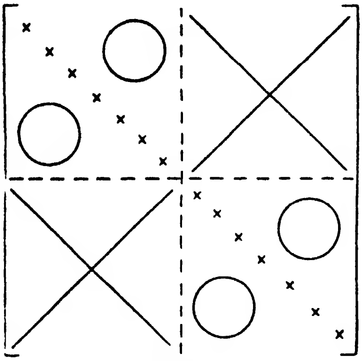

FIG. 73

An essential feature of systems of equations like (125.2) is that most of the coefficients of the u_i are zero. It is this feature of such systems which makes relaxation a possible pencil-and-paper method of solving them.

The matrix A of coefficients a_{ij} of the system (125.1) or of the system of all 21 equations like (125.2) has two properties which will prove very important in our further discussion. First,

(125.3) $\quad A$ is symmetric and positive definite.

The second property, (125.4), concerns the geometry of the connecting strings in Figs. 71 and 72. Note that the weights in the figures have been drawn in two colors: black and white. Note that each string connects weights of opposite color. Hence, in the equations the subscripts i of the unknowns u_i can be divided into two groups B, W (by color), so that

(125.4) $\quad a_{ij} = 0,$

$$\text{for } i \text{ in } B,\, j \text{ in } B\ (i \neq j) \text{ and for } i \text{ in } W,\, j \text{ in } W\ (i \neq j).$$

Another way of expressing (125.4) comes from reordering the unknowns u_i and the corresponding equations so that the "blacks" entirely precede the "whites." Then the matrix takes the schematic form of Fig. 73, where the circles denote zeros, the small crosses denote nonzero numbers, and the large crosses denote submatrices of zero and nonzero elements. Any system of linear equations satisfying (125.4) is said by Young[1] to have property (A).

We note in passing that the first boundary-value problem for any second-order self-adjoint partial differential equation lacking a term in $\partial^2 u/\partial x\, \partial y$ leads to a symmetric linear system with property (A), when difference equations are suitably introduced. If the partial differential equation is elliptic, then (125.3) holds.

Before we discuss methods suitable for electronic computers, it will be convenient to introduce another method for solving a linear system.

[1] David Young, "Iterative Methods for Solving Partial Difference Equations of Elliptic Type," *Transactions of the American Mathematical Society*, vol. 76 (1954), pp. 92–111 (condensation of his 1950 Harvard thesis).

The most general system of n linear algebraic equations in n unknowns can be written in the form

$$\sum_{j=1}^{n} a_{ij}u_j + b_i = r_i = 0 \qquad (i = 1, \ldots, n). \tag{125.5}$$

For the moment we do not assume that the matrix satisfies (125.3) or (125.4), but it is essential that no $a_{ii} = 0$. Iterative methods for solving (125.5) have been popular since Gauss' time, if not longer. One process, called the *Seidel* or *Gauss-Seidel* method, is the following: One solves the first equation (125.5) for u_1, using the current values of $u_2, \ldots, u_n$. Then the second equation is solved for u_2, using the latest known values of $u_1, u_3, u_4, \ldots, u_n$. And so on. All the equations (125.5) are solved in cyclic order for $u_1, \ldots, u_n$, always with the latest values of the other unknowns. In other words, suppose $u_1^{(k)}, \ldots, u_n^{(k)}$ are known. One gets $u_1^{(k+1)}, \ldots, u_n^{(k+1)}$ by successively solving these n equations:

$$\sum_{j=1}^{i} a_{ij}u_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij}u_j^{(k)} + b_i = 0, \qquad (i = 1, 2, \ldots, n). \tag{125.6}$$

One hopes that the $u_i^{(k)}$ will converge as $k \to \infty$ to the u_i which solve the system (125.5).

The reader will note that the Seidel process is closely related to the relaxation process described earlier in this section. The difference is the order in which Eqs. (125.5) are solved. In the relaxation method we do the equations in an order determined by the size of the r_i. In the Seidel process the order is fixed and cyclic.

Let us analyze the behavior of the Seidel process. In our applications the following theorem, apparently first completely proved by Schmeidler[1] in 1949, is essential:

THEOREM: *If the matrix A is symmetric and positive definite, then in the Seidel process the $u_i^{(k)}$ converge as $k \to \infty$ to limits u_i $(i = 1, \ldots, n)$, solving the system* (125.5).

What happens when A is not positive definite? Or, in any case, how does the vector $u^{(k)}$ approach u? If the convergence is slow, how can it be speeded up? The answers to these three questions can be obtained through some use of matrix theory, as follows:

Let the matrix of coefficients A in (125.5), assumed nonsingular, be written as the sum of three matrices: $A = D + E + F$. Here D has the main diagonal of A but is zero elsewhere; E has the below-diagonal elements of A but is zero elsewhere; and F has the above-diagonal ele-

[1] Werner Schmeidler, Vorträge uber Determinanten und Matrizen mit Anwendungen in Physik und Technik, Akademie-Verlag, Berlin (1949). Professor A. Ostrowski has traced incomplete proofs back to P. Pizzetti, *Atti della reale accademia dei Lincei, Rendiconti* (4), vol. 32 (1887), pp. 230 235, 288 293.

ments of A but is zero elsewhere. Thus, schematically,

$$A = \begin{bmatrix} \diagdown & & F \\ & D & \\ E & & \diagdown \end{bmatrix}.$$

Equations (125.6) can be written in the following matrix-vector form:

$$(D + E)u^{(k+1)} + Fu^{(k)} + b = 0. \tag{125.7}$$

If u denotes the unique vector solving (125.5), we have

$$(D + E)u + Fu + b = 0. \tag{125.8}$$

Subtracting (125.8) from (125.7), and letting $e^{(k)} = u^{(k)} - u$ denote the *error*, of $u^{(k)}$, we have

$$(D + E)e^{(k+1)} + Fe^{(k)} = 0. \tag{125.9}$$

Now since no $a_{ii} = 0$, the matrix $D + E$ has an inverse $(D + E)^{-1}$. Letting H denote the matrix $-(D + E)^{-1}F$, we find from (125.9) that

$$e^{(k+1)} = He^{(k)}, \tag{125.10}$$

whence

$$e^{(k)} = H^k e^{(0)}. \tag{125.11}$$

Equation (125.10) shows the linear behavior of the Seidel iteration process. A representation of the error $e^{(k)}$ in terms of the initial error is given by (125.11), on which a complete error analysis can be based. The Gauss-Southwell relaxation process is theoretically more complicated just because it has no simple analog to (125.11).

From the theory of linear transformations we know when and how $e^{(k)}$ goes to zero. The two determinantal equations

$$|H - \mu I| = 0 \quad \text{and} \quad |(D + E)\mu + F| = 0 \tag{125.12}$$

have the same n real or complex roots $\mu_1, \ldots, \mu_n$. If all $|\mu_i| < 1$, then in the Seidel process $u^{(k)} \to u$. If any $|\mu_i| \geq 1$, the Seidel process diverges. In principle this settles the question of convergence.

For most matrices A, to each of the roots μ_i of (125.12) there corresponds a unique vector $y^{(i)}$ such that $Hy^{(i)} = \mu_i y^{(i)}$. That is, the transformation H leaves the vector $y^{(i)}$ unchanged in direction but stretches it ($|\mu_i| > 1$) or shrinks it ($|\mu_i| < 1$) to the fraction μ_i of its previous length. All these vectors $y^{(i)}$ form an oblique coordinate system, in terms of which we can resolve the initial error vector $e^{(0)}$:

$$e^{(0)} = \sum_{i=1}^{n} c_i y^{(i)}.$$

Assume $|\mu_1| > |\mu_2| \geq \cdots \geq |\mu_n|$. After repeated multiplications by H, the resulting vector $H^k e^{(0)}$ is approximately moved into the direction corresponding to the root μ_1 of largest absolute value. Hence we find that

$$e^{(k)} = H^k e^{(0)} \doteq c_1 \mu_1{}^k y^{(1)}, \tag{125.13}$$

and we know how fast $e^{(k)} \to 0$. If $|\mu_1| < 1$, ultimately each step reduces the length of $e^{(k)}$ to the fraction $|\mu_1|$ of itself.

If $|\mu_1| < 1$, $e^{(k)} \to 0$ along one direction, that of $y^{(1)}$. Hence $u^{(k)} \to u$ along the direction of $y^{(1)}$. Cases where more than one $|\mu_i|$ dominate are more complicated but can be treated with similar tools.

Knowing the geometric character of the convergence, it is not difficult to design *acceleration processes* to speed up the convergence of $u^{(k)}$ to u.

As an example of the Seidel process and its convergence, we use it to solve the system (125.1) with the same start as in Table 1. We have the following iteration:

$$\begin{cases} u_1^{(k+1)} = \tfrac{1}{2}(1 + u_2^{(k)}), \\ u_2^{(k+1)} = \tfrac{1}{2}(1 + u_1^{(k+1)} + u_3^{(k)}), \\ u_3^{(k+1)} = \tfrac{1}{2}(1 + u_2^{(k+1)} + u_4^{(k)}), \\ u_4^{(k+1)} = \tfrac{1}{2}(1 + u_3^{(k+1)} + u_5^{(k)}), \\ u_5^{(k+1)} = 1 + u_4^{(k+1)}. \end{cases}$$

Several rounds of this are shown in Table 2. The residuals are not shown, and one line of the table amounts to a full cycle of the above algorithm.

TABLE 2. SEIDEL SOLUTION OF EQS. (125.1)

Ratio of worst errors	u	. . .	5	9	12	14	15
	$u^{(0)}$	. . .	4	8	12	14	15
0.50							
	$u^{(1)}$	. . .	4.50	8.75	11 88	13.94	14.94
0.24							
	$u^{(2)}$	. . .	4.88	8.88	11 88	13.91	14.91
0.75							
	$u^{(3)}$	. . .	4.94	8.91	11.91	13.91	14.91
0.96							
	$u^{(4)}$	. . .	4.955	8.932	11.921	13.916	14.916
0.92							
	$u^{(5)}$	. . .	4.966	8.944	11.930	13.923	14.923
0.909							
	$u^{(6)}$	. . .	4.972	8.951	11.937	13.930	14.930
0.906							
	$u^{(7)}$	. . .	4.9755	8.9562	11.9431	13 9366	14.9366

In the first column of Table 2 is given the ratio of the worst error of the u_i in the preceding and following rows of the table. By (125.13) this ratio converges to μ_1. [A solution of either of Eqs. (125.12) gives $\mu_1 = 0.9045$, $\mu_2 = 0.3455$, $\mu_3 = \mu_4 = 0$.] The approach of $u^{(k)}$ to u is one-sided and very regular. It will take about 22 cycles to gain one decimal point in accuracy. Making educated guesses at u in such problems is easy in desk work, if one knows (125.13) and its analog when $|\mu_1| = |\mu_2|$.

The present availability of electronic digital computing machines makes it possible to solve much larger problems than have been previously feasible. Such machines carry out arithmetic operations at an effective speed on the order of 10^4 times faster than a human being with a desk calculator. Something like 10^3 numbers of desk calculator precision can be held in a fast-access "memory" and made available as rapidly as the arithmetic organ can operate. Something like 10^4 more numbers can be held in an intermediate storage and transferred to the high-speed memory in a few milliseconds. Moreover, current developments will probably have made the figures given here obsolete before this book is published.

Because of the speed and capacity of such computers, many persons want to solve their problems on them. It is pertinent to ask: What methods will prove most feasible for the computers? While definitive answers must await investigations as yet not made, certain indications are now possible.

A first observation is that for large problems of the type of (125.1) or (125.2), iterative methods are relatively attractive, for much the same reasons as for pencil-and-paper calculation. But the relaxation method as outlined in connection with (125.1) has one considerable disadvantage. The scanning of all the residuals r_i in a search for $\max_i |r_i|$ is comparatively time-consuming. In fact, while computing r_i it would take almost no extra time to solve the ith equation for u_i. But if one solves the ith equation for u_i $(i = 1, \ldots, n)$, one is actually carrying out the Seidel process, which is accordingly preferred in machine calculation to conventional relaxation.

A second observation is that solving a large system (125.5) by the Seidel method is likely to be slow. To speed up the solution, acceleration methods are needed, as indicated above. But accelerations involve new routines, new coding, and the mundane but important problems of storing or reading in new codes. It is important with machines to reduce coding and operating to the utmost in simplicity.

It is the remarkable discovery of Young[1] that for certain problems a

[1] David Young, "Iterative Methods for Solving Partial Difference Equations of Elliptic Type," *Transactions of the American Mathematical Society*, vol. 76 (1954), pp. 92–111. The same suggestion was put forward a little earlier in a special case by

modification of the Seidel process will vastly speed up the convergence, while scarcely complicating the coding at all. Where applicable, it thus eliminates the necessity for special acceleration routines.

It has long been observed by relaxers that the Gauss-Southwell process usually goes faster if one "overrelaxes" a little at each step. Young was therefore convinced of the value of analyzing overrelaxation carefully in conjunction with the Seidel process. He confines himself to positive definite symmetric matrices with property (A) [see (125.4)]. In each step of the Seidel process (125.6) or (125.7), Young suggests that one first compute the Seidel value call it $v_i^{(k)}$ and then compute

$$u_i^{(k+1)} = u_i^{(k)} + \beta(v_i^{(k)} - u_i^{(k)}).$$

This amounts to an overrelaxation of $100(\beta - 1)$ per cent. Young asks which choice of β ($1 < \beta < 2$) is best.

The analysis proceeds much as in the Seidel process, since this *systematic overrelaxation* process is also a linear one. We have

$$Eu^{(k+1)} + Dv^{(k)} + Fu^{(k)} + b = 0 \tag{125.14}$$

and

$$u^{(k+1)} = u^{(k)} + \beta(v^{(k)} - u^{(k)}) = (1 - \beta)u^{(k)} + \beta v^{(k)}.$$

We now eliminate $v^{(k)}$ from (125.14). Since

$$Du^{(k+1)} = (1 - \beta)\, Du^{(k)} + \beta\, Dv^{(k)},$$

we have

$$Eu^{(k+1)} + \frac{1}{\beta}\, Du^{(k+1)} + \left(1 - \frac{1}{\beta}\right) Du^{(k)} + Fu^{(k)} + b = 0. \tag{125.15}$$

Equation (125.15) describes systematic overrelaxation, just as Eq. (125.7) describes the Seidel process. [Note that (125.15) reduces to (125.7) for $\beta = 1$.] The speed of the convergence of Young's process is measured by the largest in modulus of the roots σ_i of the following determinantal equation, analogous to the second part of (125.12):

$$\left| \sigma E + \frac{\sigma}{\beta} D + \left(1 - \frac{1}{\beta}\right) D + F \right| = 0. \tag{125.16}$$

For matrices satisfying (125.3) and (125.4) and for a certain ordering of the equations, it can be shown that the maximum of the $|\sigma_i|$ is least when we choose $\beta = 2(1 + \sqrt{1 - \mu_1})^{-1}$, where μ_1 is the largest root of (125.12). Hence this β defines the optimal amount to overrelax. Moreover, for this β all $|\sigma_i|$ are equal.

Stanley P. Frankel, "Convergence Rates of Iterative Treatments of Partial Differential Equations," *Mathematical Tables and Other Aids to Computation*, vol. 4 (1950), pp. 65–75.

To illustrate the method, we show in Table 3 the result of solving (125.1) by systematic overrelaxation. Corresponding to $\mu_1 = 0.9045$, we take $\beta = 1.528$. The same ratio of worst errors is given as in Table 2. It is irregular in Table 3 but will converge to $\beta - 1 = 0.528$. With this value, one will require only 3.6 cycles per decimal point. Thus the value $\beta = 1.528$ is much superior to the value $\beta = 1$ of the Seidel method.

TABLE 3. SOLUTION OF (125.1) BY SYSTEMATIC OVERRELAXATION

Ratio of worst errors \ u		5	9	12	14	15
	$u^{(0)}$	4	8	12	14	15
0.34						
	$u^{(1)}$	4 76	9.34	12.26	14.20	15 31
0.51						
	$u^{(2)}$	5.39	9.32	12.26	14.33	15.34
0 54						
	$u^{(3)}$	5.039	9.060	12.161	14.209	15 140
0.76						
	$u^{(4)}$	5.025	9.110	12.159	14 119	15.108
0.74						
	$u^{(5)}$	5.071	9.118	12.097	14.093	15 085
0 52						
	$u^{(6)}$	5.053	9.052	12.059	14.061	15.048
0.59						
	$u^{(7)}$	5 012	9.027	12 036	14.032	15.024
0 67						
	$u^{(8)}$	5.014	9.024	12.024	14.020	15.018
0.58						
	$u^{(9)}$	5.011	9.014	12.013	14.013	15 010
0.56						
	$u^{(10)}$	5.0049	9.0063	12.0079	14.0068	15.0051

G. E. Forsythe has coded for SWAC[1] the Young method for a simple difference equation,

$$u(x + h, y) + u(x, y + h) + u(x - h, y) + u(x, y - h) - 4u(x, y) = 0, \tag{125.17}$$

corresponding to the Laplace differential equation. The code will accommodate a network as large as 32 by 128 points. The shape of the boundary is immaterial, except that the boundary points must lie on the nodes of the network. The following example may illustrate the usefulness of the method:

For a rectangle of $30 \times 68 = 2040$ interior unknown points, each cycle

[1] National Bureau of Standards Western Automatic Computer, located in the Department of Mathematics, University of California, Los Angeles.

of relaxation takes 8.5 sec. For the Seidel process the dominant eigenvalue $\mu_1 = 0.99416$. To reduce the error to 10^{-6} times its initial value by the Seidel process ($\beta = 1$) would require about 2300 cycles, because approximately one has $(0.99416)^{2300} = 10^{-6}$. This would take over 5 hr on SWAC. If, however, β is taken at its optimal value of

$$2[1 + (1 - 0.99416)^{1/2}]^{-1} \doteq 1.85802,$$

then the dominant eigenvalue is $\sigma_1 = 0.85802$. For this σ_1, it requires about 90 iterations, accomplished in only 13 min, to reduce the error from 1 to 10^{-6}.

Some practice with SWAC convinces people it is not difficult to estimate β well enough in a few minutes. Actual running time to reduce the error by a factor of 10^{-6} is on the order of 20 to 30 min, including the time necessary to determine β approximately. Similar experience is reported by Young and Lerch.[1]

On SWAC the progress of the calculation can be monitored by observing the value of $E^{(k)} = \sum_{i=1}^{n} |r_i^{(k)}|$ as the calculation proceeds. When $\beta = 1$, $E^{(k)}$ decreases monotonically and smoothly. When β is around the optimal value, $E^{(k)}$ has comparatively wild fluctuations, probably because of "beats" (something of this is seen in Table 3) between the various complex frequencies σ_i of equal magnitude. It is perhaps also because the operator carrying $E^{(k)}$ into $E^{(k+1)}$ has a nonlinear elementary divisor.

For recent research on Young's and similar processes, see Riley[2] and Sheldon.[3] There is evidence[4] that systematic overrelaxation is also useful for some matrices not having property (A). The Seidel method, Young's modification, and the SWAC codes are adaptable to obtaining the fundamental eigenvalue of a matrix with property (A).

[1] David Young and Francis Lerch, "The Numerical Solution of Laplace's Equation on ORDVAC," *Ballistics Research Laboratories Memorandum Report* 708 (July, 1953).

[2] Riley, J. D. "Iteration Procedures for the Dirichlet Difference Problem," *Mathematical Tables and Other Aids to Computation*, vol. 8 (1954), pp. 125–131.

[3] J. W. Sheldon, "On the Numerical Solution of Elliptic Difference Equations," *Mathematical Tables and Other Aids to Computation*, vol. 9 (1955).

[4] J. G. Charney, and N. A. Phillips, Numerical Integration of the Quasi-geostrophic Equations for Barotropic and Simple Baroclinic Flows, *Journal of Meterology*, vol. 10 (1953), pp. 71–99.

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